Suggested Homework Problems #6

MATH 516-01 Fall 2001

Please do **all** of the following exercises:

• Let X be a topological space. Show that the following statements are equivalent:

(1) X is *connected* (recall that X is *connected* if there exists no disjoint nonempty open subsets of X whose union is X)

(2) The only subsets of X that are both open and closed in X are $\emptyset$ and X

(3) There exists no disjoint nonempty closed subsets of X whose union is X

(4) There exists no separation of X (recall that a *separation* of X is a pair of disjoint nonempty subsets A and B whose union is X such that neither of which contains a limit point of the other, i.e. $\bar{A} \cap B = \emptyset = A \cap \bar{B}$)

(5) Every *continuous* function $f : X \to \mathbb{Z}_2$ is constant, where $\mathbb{Z}_2$ denotes $\{-1, 1\}$ given the *discrete topology*

• Is $\mathbb{R}_\ell$ (i.e. $\mathbb{R}$ with *lower limit topology*) connected? Justify your answer.

• **(Cut Point Theory):** (a) Show that no two of the spaces $(0, 1)$, $[0, 1)$, and $[0, 1]$ are homeomorphic. (b) Show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ if $n > 1$. (**Remark:** It however requires surprisingly more efforts to show $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ iff $n = m$)

• **(More on Cut Point Theory):** Let $A_1 = [0, 1]$. Let $A_2 = S^1$ be the unit circle in the plane. Let $A_3 = S^2$ be the unit sphere in $\mathbb{R}^3$. Let $A_4 = S^1 \vee S^1$ be the figure "8" in the plane. Show that A_i is not homeomorphic to A_j if $i \neq j$.