

Root systems for asymmetric geometric representations of Coxeter groups

Robert G. Donnelly¹

Department of Mathematics and Statistics, Murray State University, Murray, KY 42071

Abstract

Results are obtained concerning root systems for asymmetric geometric representations of Coxeter groups. These representations were independently introduced by Vinberg and Eriksson, and generalize the standard geometric representation of a Coxeter group in such a way as to include all Kac–Moody Weyl groups. In particular, a characterization of when a non-trivial multiple of a root may also be a root is given in the general context. Characterizations of when the number of such multiples of a root is finite and when the number of positive roots sent to negative roots by a group element is finite are also given. These characterizations are stated in terms of combinatorial conditions on a graph closely related to the Coxeter graph for the group.

Mathematics Subject Classification: 20F55 (05E99)

Keywords: Coxeter group, geometric representation, root system, Tits cone, Kac–Moody algebra, numbers game

§1 Introduction. A certain natural symmetric bilinear form is used to define the familiar geometric representation of a given Coxeter group, often called the “standard” geometric representation. See [Bour] Ch. 5, [Hum] Ch. 5, or [BB] §4.4. These representations are well understood and are useful for studying Coxeter groups and their applications in many different contexts. See for example [Gun] and references therein. Following work of Vinberg and Eriksson, when considering geometric representations of Coxeter groups in Chapter 4 of the book [BB], Björner and Brenti initially do not require that the bilinear form be symmetric. The purpose here is to further study the root systems associated to such representations. Much of what we record here generalizes the standard theory as presented for example in [Hum] §5.3, 5.4, 5.6, and 5.13. Since the form is no longer required to be symmetric, all statements here may be applied to the sets of real roots of Kac–Moody algebras. This yields new proofs of standard Kac–Moody results (one direction of the first statement in Corollary 3.7, one direction of the second statement in Corollary 3.10).

These asymmetric geometric realizations of Coxeter groups were introduced by Vinberg in [Vin], for geometric reasons. A main focus of Vinberg’s study is the behavior of the “fundamental chamber” (a convex polyhedral cone) under the group action. In a different context, Lusztig used such asymmetric forms when constructing certain irreducible representations of Hecke algebras [Lus]. Eriksson applied asymmetric geometric representations of Coxeter groups in [Erik1] (§4.3, §6.9, Ch. 8) and [Erik2] (§3, 4) in connection with the combinatorial numbers game of Mozes [Moz]. While the numbers game is of combinatorial interest in its own right, it is also helpful for facilitating computations with Coxeter groups and their geometric representations (e.g. computing orbits, solving the word problem, or finding reduced decompositions) and for obtaining combinatorial models of Coxeter groups. See for example §4.3 of [BB]. The results of this paper are needed for our further study of the numbers game in [Don]. There we further investigate connections between moves of the game and reduced decompositions for group elements, characterize “full commutativity” of group elements in terms of the game, characterize when all positive roots can be obtained from game play, and obtain a new Dynkin diagram classification theorem whose answer consists of versions of Coxeter graphs for finite Coxeter groups.

¹Email: rob.donnelly@murraystate.edu, Fax: 1-270-809-2314

The possible asymmetry of the bilinear forms here leads to some curious differences with the standard case. In Exercise 4.9 of [BB], the authors point out that without symmetry of the bilinear forms, some important properties of root systems would not be true. However, we will see that these properties do not fail too badly, at least not all of the time. In particular, we determine precisely when non-trivial scalar multiples of roots can also be roots (Theorem 3.3), and we relate the finiteness of this set of root multiples to a combinatorial condition on a graph closely related to the Coxeter graph for the group (Theorem 3.6). Further, we determine when the number of positive roots sent to negative roots by a given group element is finite, and we say how this quantity is related to the length of the given group element (Theorem 3.9). In Theorem 4.3 and Proposition 4.4 we relate the size of the Coxeter group (infinite or finite) to certain conditions on the “asymmetric” version of the Tits cone.

The original version of this paper was written with only the numbers game motivations above in mind. Recently, for unrelated reasons Proctor decided to relate the treatment of Weyl groups in [Kac] and [Kum] to the study of asymmetric geometric representations of Coxeter groups in [BB]. This led to the definition of ‘real Weyl groups’ in [Pro] and his realization that our Theorem 3.3 would play a key role in those notes. Quoting from an earlier draft of [Pro]: “There are many statements concerning Weyl groups and the ‘real’ roots of Kac–Moody algebras which can at least be conjectured in the general context of real Weyl groups. If still true, it would seem that each of these statements should be provable without any reference to Lie brackets or to root spaces, if one could formulate suitable sufficient conditions for them in terms of real Weyl group concepts. One example of such a statement is “no ‘non-trivial’ real multiple of a real root is also a root”. Within the general context, two successive restricting assumptions (which are both automatically satisfied by Weyl groups) guarantee [via our Theorem 3.3] that this example statement holds true in a context which is still much more general than that of Weyl groups or of Section 4.4 of [BB].”

At the end of Section 2 we observe that any Kac–Moody Weyl group arises as one of our representing groups $\sigma(W) \subset GL(V)$. Hence all of our results pertain to the special case consisting of arbitrary Kac–Moody Weyl groups. Our complete characterizations of the “no non-trivial multiple of a (real) root is also a root” (Corollary 3.7) and the “set of positive (real) roots sent to negative is finite” (Corollary 3.10) properties are given proofs which are naturally set in a general environment which encompasses both the standard geometric representations of Coxeter groups and Kac–Moody Weyl groups. Only combinatorial positivity arguments are used in these proofs; no references to Lie brackets or root spaces are needed.

§2 Definitions and preliminaries. In this section we present the main objects of interest for this paper. The crucial information identifying an asymmetric geometric representation of a Coxeter group is a certain real matrix analog of a generalized Cartan matrix. We take this matrix as our starting point. Fix a positive integer n and a totally ordered set I_n with n elements (usually $I_n := \{1 < \dots < n\}$). An *E-generalized Cartan matrix* (*E-GCM*)² is an $n \times n$ matrix $A = (a_{ij})_{i,j \in I_n}$ with real entries satisfying the requirements that each main diagonal matrix entry is 2, that all

²Motivation for terminology: E-GCM’s with integer entries are generalizations of ‘generalized’ Cartan matrices (GCM’s), which are the starting point for the study of Kac–Moody algebras. Here we use the modifier “E” because of the relationship between these matrices and the combinatorics of Eriksson’s E-games [Erik1], [Erik2].

other matrix entries are nonpositive, that if a_{ij} is nonzero then a_{ji} is also nonzero, and that for $i \neq j$ either $a_{ij}a_{ji} \geq 4$ or $a_{ij}a_{ji} = 4 \cos^2(\pi/k_{ij})$ for some integer $k_{ij} \geq 2$. The peculiar quantities $4 \cos^2(\pi/k)$ appear in the developments of [Bour], [Hum] as the products of transpose entries of a symmetric matrix for the defining bilinear form of the standard geometric representation of a Coxeter group. To an $n \times n$ E-generalized Cartan matrix $A = (a_{ij})_{i,j \in I_n}$ we associate a finite graph Γ as follows: The nodes $(\gamma_i)_{i \in I_n}$ of Γ are indexed by the set I_n , and an edge is placed between nodes γ_i and γ_j if and only if $i \neq j$ and the matrix entries a_{ij} and a_{ji} are nonzero. We display this edge as $\gamma_i \xrightarrow{p} \gamma_j \xleftarrow{q}$, where $p = -a_{ij}$ and $q = -a_{ji}$. We call the pair (Γ, A) an *E-GCM graph*. See Figure 3.1 for a six-node example.

Define the associated Coxeter group $W(\Gamma, A)$ to be the Coxeter group with identity ε , generators $\{s_i\}_{i \in I_n}$, and defining relations $s_i^2 = \varepsilon$ for $i \in I_n$ and $(s_i s_j)^{m_{ij}} = \varepsilon$ for all $i \neq j$, where the m_{ij} are determined by:

$$m_{ij} = \begin{cases} k_{ij} & \text{if } a_{ij}a_{ji} = 4 \cos^2(\pi/k_{ij}) \text{ for some integer } k_{ij} \geq 2 \\ \infty & \text{if } a_{ij}a_{ji} \geq 4 \end{cases}$$

(Conventionally, $m_{ij} = \infty$ means there is no relation between generators s_i and s_j .) When A is a generalized Cartan matrix or GCM (i.e. an E-GCM with integer entries), then m_{ij} is finite only for the pairs $\{-a_{ij}, -a_{ji}\} = \{1, 1\}, \{1, 2\}, \{1, 3\}$; the corresponding values of such m_{ij} are 3, 4, 6. One can think of the E-GCM graph as a refinement of the information from the Coxeter graph for the associated Coxeter group. One can use the standard representation to see that any Coxeter group on a finite set of generators is isomorphic to $W(\Gamma, A)$ for some E-GCM graph (Γ, A) . We let ℓ denote the length function for $W = W(\Gamma, A)$. An expression $s_{i_1} s_{i_2} \cdots s_{i_p}$ for an element of W is *reduced* if $\ell(s_{i_1} s_{i_2} \cdots s_{i_p}) = p$. For $J \subseteq I_n$, let W_J be the subgroup generated by $\{s_i\}_{i \in J}$, a *parabolic* subgroup, and $W^J := \{w \in W \mid \ell(ws_j) > \ell(w) \text{ for all } j \in J\}$ is the set of *minimal coset representatives*. If $J = \{i, j\}$, then W_J is a dihedral group of order $2m_{ij}$.

From here on, fix an arbitrary E-GCM graph (Γ, A) with index set I_n and associated Coxeter group $W = W(\Gamma, A)$. We now define the representations of W which are of interest to us here, cf. §4.2 of [BB]. To fix notation that will help set up some subsequent arguments, we present some of the details here. Let V be a real n -dimensional vector space freely generated by $(\alpha_i)_{i \in I_n}$. (Elements of this ordered basis are *simple roots*.) Equip V with a possibly asymmetric bilinear form $B : V \times V \rightarrow \mathbb{R}$ defined on the basis $(\alpha_i)_{i \in I_n}$ by $B(\alpha_i, \alpha_j) := \frac{1}{2}a_{ij}$. For each $i \in I_n$ define an operator $S_i : V \rightarrow V$ by the rule $S_i(v) := v - 2B(\alpha_i, v)\alpha_i$ for each $v \in V$. One can check that S_i^2 is the identity transformation, so $S_i \in GL(V)$. Fix $i \neq j$ and set $V_{i,j} := \text{span}_{\mathbb{R}}\{\alpha_i, \alpha_j\}$. Observe that $S_k(V_{i,j}) \subseteq V_{i,j}$ for $k = i, j$. Let \mathfrak{B} be the ordered basis (α_i, α_j) for $V_{i,j}$, and for any linear mapping $T : V_{i,j} \rightarrow V_{i,j}$ let $[T]_{\mathfrak{B}}$ be the matrix for T relative to \mathfrak{B} . Then

$$[S_i|_{V_{i,j}}]_{\mathfrak{B}} = \begin{pmatrix} -1 & -a_{ij} \\ 0 & 1 \end{pmatrix}, [S_j|_{V_{i,j}}]_{\mathfrak{B}} = \begin{pmatrix} 1 & 0 \\ -a_{ji} & -1 \end{pmatrix}, [S_i S_j|_{V_{i,j}}]_{\mathfrak{B}} = \begin{pmatrix} a_{ij}a_{ji} - 1 & a_{ij} \\ -a_{ji} & -1 \end{pmatrix}$$

Analysis of the eigenvalues for $X_{i,j} := [S_i S_j|_{V_{i,j}}]_{\mathfrak{B}}$ as in the proofs of Proposition 3.13 of [Kac] and Proposition 1.3.21 of [Kum] shows that $X_{i,j}$ has infinite order when $a_{ij}a_{ji} \geq 4$, and hence $S_i S_j$ has infinite order as an element of $GL(V)$. When $0 < a_{ij}a_{ji} < 4$, write $a_{ij}a_{ji} = 4 \cos^2 \theta$ for

$\theta := \pi/m_{ij}$. In this case check that $X_{i,j}$ has two distinct complex eigenvalues ($e^{2i\theta}$ and $e^{-2i\theta}$). It follows that $X_{i,j}$ has finite order m_{ij} . When $a_{ij}a_{ji} = 0$, then $X_{i,j} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which clearly has order $m_{ij} = 2$. Now assume $0 \leq a_{ij}a_{ji} < 4$, and set $V'_{i,j} := \{v \in V \mid B(\alpha_i, v) = 0 = B(\alpha_j, v)\}$. One can easily check that $V_{i,j} \cap V'_{i,j} = \{0\}$. The facts that $\dim V_{i,j} = 2$, $\dim V'_{i,j} \geq n - 2$, and $V_{i,j} \cap V'_{i,j} = \{0\}$ together imply that $\dim V'_{i,j} = n - 2$ and $V = V_{i,j} \oplus V'_{i,j}$. Since $S_i S_j$ acts as the identity on $V'_{i,j}$, it follows that $S_i S_j$ has order m_{ij} as an element of $GL(V)$. Then there is a unique homomorphism $\sigma_A : W \rightarrow GL(V)$ for which $\sigma_A(s_i) = S_i$. With the dependence on A understood, we set $\sigma := \sigma_A$. We now have W acting on V , and for all $w \in W$ and $v \in V$ we write $w.v$ for $\sigma(w)(v)$. We call σ a *geometric representation* of W . If A is symmetric such that $a_{kl}a_{lk} \geq 4 \Rightarrow a_{kl} = a_{lk} = -2$ for all $k \neq l$, then σ is the *standard* geometric representation. The *root system* for σ is $\Phi := \Phi_A := \{w.\alpha_i\}_{i \in I_n, w \in W}$. For each $w \in W$, $\sigma(w)$ permutes Φ , so σ induces an action of W on Φ . Evidently, $\Phi = -\Phi$. Elements of Φ are *roots* and are necessarily nonzero. If $\alpha = \sum c_i \alpha_i$ is a root with all c_i nonnegative (respectively nonpositive), then say α is a *positive* (resp. *negative*) root. Let Φ^+ and Φ^- denote the collections of positive and negative roots respectively. Clearly $\Phi^+ \cap \Phi^- = \emptyset$.

Kac–Moody Weyl groups are subsumed into this paper as follows: Let A be a generalized Cartan matrix. We identify our simple roots $\{\alpha_1, \dots, \alpha_n\}$ with the simple roots in $\mathfrak{h}_{\mathbb{R}}^*$ of [Kac], which is the dual of a real vector space $\mathfrak{h}_{\mathbb{R}}$ of dimension $n + l$, where $l = \text{nullity}(A)$. The simple “coroots” of [Kac] are a linearly independent set $\{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}_{\mathbb{R}}$ for which $\alpha_j(\alpha_i^\vee) = a_{ij}$. Now for $1 \leq i \leq n$, a mapping $R_i : \mathfrak{h}_{\mathbb{R}}^* \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ is defined in [Kac] by $R_i(v) = v - v(\alpha_i^\vee)\alpha_i$. The associated Kac–Moody Weyl group is the subgroup of $GL(\mathfrak{h}_{\mathbb{R}}^*)$ generated by $\{R_i\}_{i=1}^n$. If we identify our V with $\text{span}_{\mathbb{R}}\{\alpha_1, \dots, \alpha_n\} \subseteq \mathfrak{h}_{\mathbb{R}}^*$ and restrict each R_i to V , then the homomorphism $W \rightarrow GL(V)$ determined by $s_i \mapsto R_i|_V$ is the representation σ . The real roots of Kac–Moody theory are the roots $\Phi \subset V$ obtained here from this geometric representation of W .

§3 Root system results. Asymmetry of the bilinear form leads to crucial differences with the symmetric case. Most notably, $\sigma(W)$ preserves the form B if and only if A is symmetric. From this fact for symmetric A it readily follows that if $K\alpha_x \in \Phi$ for some $x \in I_n$ and real number K , then $K = \pm 1$. (See equation 4.27 of [BB].) However, when A is asymmetric sometimes $K\alpha_x$ is a root for $K \neq \pm 1$, as can be seen in Exercise 4.9 of [BB] and Example 3.12 below.³ To understand how such a W -action can generate scalar multiples of roots in Φ , we first analyze how s_i and s_j act in tandem on $V_{i,j}$. Our next result strengthens Lemma 4.2.4 of [BB] and provides a different proof. It also answers Exercise 4.6 of [BB].

Lemma 3.1 *Fix $i \neq j$ in I_n , and let k be a positive integer. If $m_{ij} = \infty$, then $(s_i s_j)^k . \alpha_i = a\alpha_i + b\alpha_j$ and $s_j (s_i s_j)^k . \alpha_i = c\alpha_i + d\alpha_j$ for positive coefficients a, b, c , and d . Now suppose $m_{ij} < \infty$. If $2k < m_{ij}$, then $(s_i s_j)^k . \alpha_i = a\alpha_i + b\alpha_j$ with $a \geq 0$ and $b > 0$. In this case, $a = 0$ if and only if m_{ij} is odd and $k = (m_{ij} - 1)/2$, and consequently $(s_i s_j)^k . \alpha_i = \frac{-a_{ji}}{2 \cos(\pi/m_{ij})} \alpha_j$. Similarly, if $2k < m_{ij} - 1$,*

³In Proposition 6.9 of [Erik1] and in [Erik2] just prior to Proposition 4.4, it is asserted that $s_x(\Phi^+ \setminus \{\alpha_x\}) = \Phi^+ \setminus \{\alpha_x\}$ for all $x \in I_n$. However, this will not be the case if $K\alpha_x$ is a root for some $K \neq \pm 1$. Only Theorem 6.9 of [Erik1] and Proposition 4.4 of [Erik2] are affected by this misstatement. (See Lemma 3.8 below.)

then $s_j(s_i s_j)^k \cdot \alpha_i = c\alpha_i + d\alpha_j$ with $c > 0$ and $d \geq 0$. In this case, $d = 0$ if and only if m_{ij} is even and $k = (m_{ij} - 2)/2$, and consequently $s_j(s_i s_j)^k \cdot \alpha_i = \alpha_i$.

Proof. Let \mathfrak{B} and $X_{i,j}$ be as above, and set $X_i := [S_i|_{V_{i,j}}]_{\mathfrak{B}}$ and $X_j := [S_j|_{V_{i,j}}]_{\mathfrak{B}}$. To understand $(s_i s_j)^k \cdot \alpha_i$ and $s_j(s_i s_j)^k \cdot \alpha_i$ we compute $X_{i,j}^k$ and $X_j X_{i,j}^k$. Set $p := -a_{ij}$ and $q := -a_{ji}$.

For $m_{ij} = \infty$, first take $pq = 4$. We can write $X_{i,j} = PYP^{-1}$ for nonsingular P and upper triangular Y as follows:

$$X_{i,j} = \frac{1}{p} \begin{pmatrix} p & p \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & p \\ 2 & -p \end{pmatrix}.$$

Then for any positive integer k we obtain $X_{i,j}^k = \begin{pmatrix} 2k+1 & -kp \\ kq & -2k+1 \end{pmatrix}$. It follows that $(s_i s_j)^k \cdot \alpha_i = (2k+1)\alpha_i + kq\alpha_j$, with both coefficients of the linear combination positive. From the first column of the matrix $X_j X_{i,j}^k$ we see that $s_j(s_i s_j)^k \cdot \alpha_i = (2k+1)\alpha_i + (2k+1)q\alpha_j$, with both coefficients of the linear combination positive. Next take $pq > 4$. In this case we get distinct eigenvalues $\lambda = \frac{1}{2}(pq - 2 + \sqrt{pq(pq - 4)}) > 1$ and $\mu = \frac{1}{2}(pq - 2 - \sqrt{pq(pq - 4)}) < 1$ for $X_{i,j}$ (here we have $\lambda\mu = 1$). Similar to the above, we may write $X_{i,j} = PDP^{-1}$ for the diagonal matrix $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and a nonsingular matrix P , from which we obtain

$$X_{i,j}^k = \frac{1}{p(\lambda - \mu)} \begin{pmatrix} p & p \\ \mu' & \lambda' \end{pmatrix} \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix} \begin{pmatrix} \lambda' & -p \\ -\mu' & p \end{pmatrix},$$

for any positive integer k , where $\lambda' := \lambda + 1$ and $\mu' := \mu + 1$. This (eventually) simplifies to

$$X_{i,j}^k = \frac{1}{\lambda - \mu} \begin{pmatrix} \lambda' \lambda^k - \mu' \mu^k & -p(\lambda^k - \mu^k) \\ q(\lambda^k - \mu^k) & \lambda' \mu^k - \mu' \lambda^k \end{pmatrix}.$$

From this we also get

$$X_j X_{i,j}^k = \frac{1}{\lambda - \mu} \begin{pmatrix} \lambda' \lambda^k - \mu' \mu^k & -p(\lambda^k - \mu^k) \\ q(\lambda^{k+1} - \mu^{k+1}) & \mu' \mu^k - \lambda' \lambda^k \end{pmatrix}.$$

The factor $\frac{1}{\lambda - \mu}$ is positive, and for both matrices $X_{i,j}^k$ and $X_j X_{i,j}^k$, the first column entries are positive. So, $(s_i s_j)^k \cdot \alpha_i = a\alpha_i + b\alpha_j$ with both a and b positive, and $s_j(s_i s_j)^k \cdot \alpha_i = c\alpha_i + d\alpha_j$ with c and d both positive.

For the $m_{ij} < \infty$ case, set $\theta := \pi/m_{ij}$. Check that $X_{i,j}$ can be written as $X_{i,j} = PDP^{-1}$ for a nonsingular matrix P and diagonal matrix D in the following way:

$$\frac{1}{q(e^{2i\theta} - e^{-2i\theta})} \begin{pmatrix} e^{2i\theta} + 1 & e^{-2i\theta} + 1 \\ q & q \end{pmatrix} \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix} \begin{pmatrix} q & -e^{-2i\theta} - 1 \\ -q & e^{2i\theta} + 1 \end{pmatrix}.$$

Then for any positive integer k we have

$$X_{i,j}^k = PD^k P^{-1} = \frac{1}{\sin(2\theta)} \begin{pmatrix} \sin(2(k+1)\theta) + \sin(2k\theta) & -p \sin(2k\theta) \\ q \sin(2k\theta) & -\sin(2k\theta) - \sin(2(k-1)\theta) \end{pmatrix}$$

and

$$X_j X_{i,j}^k = \frac{1}{\sin(2\theta)} \begin{pmatrix} \sin(2(k+1)\theta) + \sin(2k\theta) & -p \sin(2k\theta) \\ q \sin(2(k+1)\theta) & (1-pq) \sin(2k\theta) + \sin(2(k-1)\theta) \end{pmatrix}$$

Use the first column of $X_{i,j}^k$ and $X_j X_{i,j}^k$ to see that $(s_i s_j)^k \cdot \alpha_i = \frac{1}{\sin(2\theta)} [\sin(2(k+1)\theta) + \sin(2k\theta)] \alpha_i + \frac{q}{\sin(2\theta)} \sin(2k\theta) \alpha_j$ and that $s_j (s_i s_j)^k \cdot \alpha_i = \frac{1}{\sin(2\theta)} [\sin(2(k+1)\theta) + \sin(2k\theta)] \alpha_i + \frac{q}{\sin(2\theta)} \sin(2(k+1)\theta) \alpha_j$. As long as $2(k+1) < m_{ij}$, then all the coefficients of these linear combinations will be positive. So now suppose $2(k+1) \geq m_{ij}$. First we consider $(s_i s_j)^k = a\alpha_i + b\alpha_j$ for some positive k with $2k < m_{ij}$. There are two possibilities now: $2(k+1) = m_{ij}$ or $2(k+1) = m_{ij} + 1$. In the former case both a and b are positive. In the latter case we have m_{ij} odd, $a = \frac{1}{\sin(2\theta)} [\sin(2(k+1)\theta) + \sin(2k\theta)] = 0$, and $b = \frac{q \sin \theta}{\sin(2\theta)} = \frac{q}{2 \cos \theta}$. Second we consider $s_j (s_i s_j)^k = c\alpha_i + d\alpha_j$ for some positive k with $2k < m_{ij} - 1$. Now the fact that $2(k+1) \geq m_{ij}$ implies we have $2(k+1) = m_{ij}$. In particular, m_{ij} is even. With $k = (m_{ij} - 2)/2$ now, one can check that $d = 0$ and $c = 1$. \square

The next statement is Proposition 4.2.5 of [BB] and appears in a somewhat different form as Corollary 4.3 in [Erik1].

Proposition 3.2 *Let $w \in W$ and $i \in I_n$. If $\ell(ws_i) > \ell(w)$, then $w \cdot \alpha_i \in \Phi^+$. If $\ell(ws_i) < \ell(w)$, then $w \cdot \alpha_i \in \Phi^-$.*

This result analogizes Theorem 5.4 of [Hum], which handles the standard case. As with Corollary 5.4 of [Hum], it is a consequence of Proposition 3.2 that the representation σ is faithful. (See [BB] Theorem 4.2.7.) It also follows that $\Phi = \Phi^+ \cup \Phi^-$. This is Equation 4.24 of [BB], which actually could have been derived at the end of Section 4.2.

We say two adjacent nodes γ_i and γ_j in (Γ, A) are *odd-neighborly* if m_{ij} is odd. When m_{ij} is odd and $a_{ij} \neq a_{ji}$, we say that the adjacent nodes γ_i and γ_j form an *odd asymmetry*. In view of Lemma 3.1, for odd m_{ij} let v_{ji} be the element $(s_i s_j)^{(m_{ij}-1)/2}$ of W , and set $K_{ji} := \frac{-a_{ji}}{2 \cos(\pi/m_{ij})}$, which is positive. Then $v_{ji} \cdot \alpha_i = K_{ji} \alpha_j$. Observe that $K_{ij} K_{ji} = 1$ and that $v_{ij} = v_{ji}^{-1}$. A *path of odd neighbors* (or *ON-path*, for short) in (Γ, A) is a sequence $\mathcal{P} := [\gamma_{i_0}, \gamma_{i_1}, \dots, \gamma_{i_p}]$ of nodes from Γ for which consecutive pairs are odd-neighborly. This ON-path has length p , and we allow ON-paths to have length zero. We say γ_{i_0} and γ_{i_p} are the *start* and *end* nodes of the ON-path, respectively. If ON-path $\mathcal{Q} = [\gamma_{j_0}, \gamma_{j_1}, \dots, \gamma_{j_q}]$ has the same start node as the end node of \mathcal{P} , then their *concatenation* $\mathcal{P} \natural \mathcal{Q}$ is the ON-path $[\gamma_{i_0}, \gamma_{i_1}, \dots, \gamma_{i_p} = \gamma_{j_0}, \dots, \gamma_{j_q}]$. Let $w_{\mathcal{P}} \in W$ be the Coxeter group element $v_{i_p i_{p-1}} \cdots v_{i_2 i_1} v_{i_1 i_0}$, and let $\Pi_{\mathcal{P}} := K_{i_p i_{p-1}} \cdots K_{i_2 i_1} K_{i_1 i_0}$, where $w_{\mathcal{P}} = \varepsilon$ with $\Pi_{\mathcal{P}} = 1$ when \mathcal{P} has length zero. Note that $w_{\mathcal{P}} \cdot \alpha_{i_0} = \Pi_{\mathcal{P}} \alpha_{i_0}$ and that $w_{\mathcal{P} \natural \mathcal{Q}} = w_{\mathcal{Q}} w_{\mathcal{P}}$.

From Lemma 3.1 and the discussion above it follows that *if (Γ, A) has an odd asymmetry, then there exists a root which is a non-trivial multiple of a simple root*. For a more general statement that includes the converse, see Corollary 3.7 below. When A is an integer matrix, odd neighbors γ_i and γ_j must have $\{-a_{ij}, -a_{ji}\} = \{1, 1\}$. These are not asymmetric. Therefore the matrices A defining Weyl groups have no odd asymmetries. In this integer matrix setting, Kac ([Kac] Proposition 5.1.b) and Kumar ([Kum] Corollary 1.3.6.a) show that for a “real” root α and real number K , $K\alpha$ is also a root if and only if $K = \pm 1$. Their proofs use Lie brackets and root space reasoning. But alternatively, this result is also a very special case of the following:

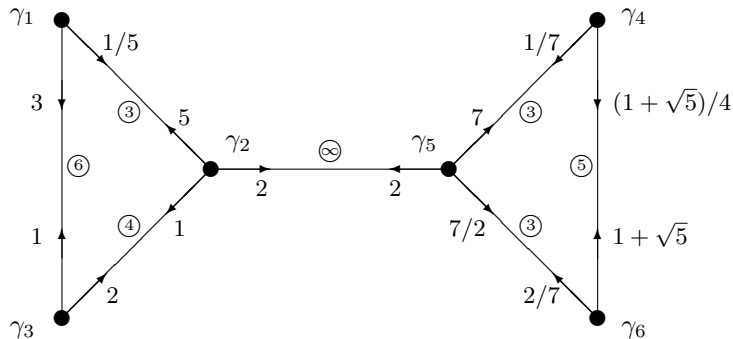
Theorem 3.3 Let $w \in W$ and $i \in I_n$. (1) Then $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and some $K > 0$ if and only if $w.\alpha_i = w_{\mathcal{P}}.\alpha_i$ for some ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \dots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$, in which case $K = \Pi_{\mathcal{P}}$. (2) Similarly $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and some $K < 0$ if and only if $w.\alpha_i = (w_{\mathcal{P}}s_i).\alpha_i$ for some ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \dots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$, in which case $K = -\Pi_{\mathcal{P}}$.

Proof. Note that (2) follows from (1). For (1), the “if” direction is handled by the definitions paragraph preceding the theorem statement. For the “only if” direction, we induct on $\ell(w)$. The case $\ell(w) = 0$ follows from the definitions. Now suppose $\ell(w) > 0$. Take any $j \in I_n$ for which $\ell(ws_j) = \ell(w) - 1$. Since $\ell(ws_i) > \ell(w)$, then $i \neq j$. Let $J := \{i, j\}$, and let v^J be the unique element in W^J and v_J the unique element in W_J for which $w = v^J v_J$. Then $\ell(w) = \ell(v^J) + \ell(v_J)$ by Proposition 2.4.4 of [BB]. Write $v_J.\alpha_i = a\alpha_i + b\alpha_j$. Since $\ell(ws_i) > \ell(w)$, then $\ell(v_J s_i) > \ell(v_J)$, and hence $v_J.\alpha_i \in \Phi^+$ (Proposition 3.2). So $a \geq 0$ and $b \geq 0$. Suppose $a > 0$ and $b > 0$. Now $v^J \in W^J$ implies that $\ell(v^J s_i) > \ell(v^J)$ and $\ell(v^J s_j) > \ell(v^J)$, and hence $v^J.\alpha_i \in \Phi^+$ and $v^J.\alpha_j \in \Phi^+$ (Proposition 3.2). Write $v^J.\alpha_i = \sum_{y \in I_n} c_y \alpha_y$ ($c_y \geq 0$) and $v^J.\alpha_j = \sum_{y \in I_n} d_y \alpha_y$ ($d_y \geq 0$). Then $K\alpha_x = w.\alpha_i = v^J.(a\alpha_i + b\alpha_j) = \sum_{y \in I_n} (ac_y + bd_y)\alpha_y$ implies that for all $y \neq x$, $ac_y + bd_y = 0$ and hence $c_y = d_y = 0$. Then $v^J.\alpha_i$ and $v^J.\alpha_j$ are both multiples of α_x . But then $(v^J)^{-1}.\alpha_x$ is a scalar multiple of α_i and of α_j , which is absurd. So we must have $a = 0$ or $b = 0$. If $v_J = s_j$, then $s_j.\alpha_i = \alpha_i$ and therefore $w.\alpha_i = v^J.\alpha_i$. The induction hypothesis now applies to v^J to obtain the desired result. So now suppose $\ell(v_J) > 1$. Then $v_J = (s_i s_j)^k$ or $s_j (s_i s_j)^k$ for some positive integer k . Lemma 3.1 implies that m_{ij} must be finite. In this case the longest element in W_J has length m_{ij} and can be written in two ways, one ending in s_i . Therefore $\ell(v_J) < m_{ij}$, so in particular $m_{ij} \geq 3$ and γ_i and γ_j are adjacent. Apply Lemma 3.1 now. If $v_J.\alpha_i = \alpha_i$, then $w.\alpha_i = v^J.\alpha_i$, and we can apply the induction hypothesis to v^J . If $v_J.\alpha_i \neq \alpha_i$, then we see that m_{ij} is odd and $v_J.\alpha_i = v_{j_i}.\alpha_i = K_{j_i}\alpha_j$. Again apply the induction hypothesis to v^J , where now $v^J.\alpha_j = \frac{K}{K_{j_i}}\alpha_x = w_{\mathcal{Q}}.\alpha_j$ for some ON-path $\mathcal{Q} = [\gamma_{j_0=j}, \gamma_{j_1}, \dots, \gamma_{j_q=x}]$. Let $\mathcal{P} := [\gamma_i, \gamma_j] \natural \mathcal{Q}$. Then $w.\alpha_i = K\alpha_x = w_{\mathcal{P}}.\alpha_i$. \square

For any $\alpha \in \Phi$, set $\mathfrak{S}(\alpha) := \mathfrak{S}_A(\alpha) := \{K\alpha\}_{K \in \mathbb{R}} \cap \Phi^+$. Our analysis of the sets $\mathfrak{S}(\alpha)$ requires some additional notation. For ON-paths \mathcal{P} and \mathcal{Q} , write $\mathcal{P} \sim \mathcal{Q}$ and say \mathcal{P} and \mathcal{Q} are Π -equivalent if these ON-paths have the same start and end nodes and $\Pi_{\mathcal{P}} = \Pi_{\mathcal{Q}}$. This is an equivalence relation on the set of all ON-paths. An ON-path \mathcal{P} is *simple* if it has no repeated nodes with the possible

Figure 3.1: A unital ON-cyclic E-GCM graph for Example 3.12.

The notation \textcircled{m} (resp. $\textcircled{\infty}$) on an edge $\bullet \xrightarrow{p} \bullet \xleftarrow{q} \bullet$ indicates that $pq = 4 \cos^2(\pi/m)$ (resp. $pq \geq 4$).



exception that the start and end nodes may coincide. Two ON paths \mathcal{P} and \mathcal{Q} are *scalar-distinct* if $\Pi_{\mathcal{P}} \neq \Pi_{\mathcal{Q}}$. An ON-path $\mathcal{P} = [\gamma_{i_0}, \dots, \gamma_{i_p}]$ is an *ON-cycle* if $\gamma_{i_p} = \gamma_{i_0}$. It is a *unital ON-cycle* if $\Pi_{\mathcal{P}} = 1$. We say (Γ, A) is *unital ON-cyclic* if and only if $\Pi_{\mathcal{C}} = 1$ for all ON-cycles \mathcal{C} . See Figure 3.1. From the definitions it follows that (Γ, A) is unital ON-cyclic if it has no odd asymmetries. So if A is a GCM, then (Γ, A) is unital ON-cyclic. If A is a symmetrizable E-GCM, then by applying Exercise 2.1 of [Kac] or Exercise 1.5.E.1 of [Kum] to the environment of E-GCM's, one sees that (Γ, A) is unital ON-cyclic. However, a unital ON-cyclic E-GCM graph need not have a symmetrizable matrix A , as Example 3.12 shows. To check if an E-GCM graph is unital ON-cyclic, it is enough to check that each simple ON-cycle is unital. An E-GCM graph is *ON-connected* if any two nodes can be joined by an ON-path. An *ON-connected component* of (Γ, A) is an E-GCM subgraph (Γ', A') whose nodes form a maximal collection of nodes in (Γ, A) which can be pairwise joined by ON-paths.

Lemma 3.4 *Let α and β be roots in Φ . Suppose an element of $\mathfrak{S}(\alpha)$ is in the same orbit as an element of $\mathfrak{S}(\beta)$ under the action of W on Φ . Then there is a one-to-one correspondence between the sets $\mathfrak{S}(\alpha)$ and $\mathfrak{S}(\beta)$. If γ_i and γ_j are nodes in the same ON-connected component of (Γ, A) , then there is a one-to-one correspondence between the sets $\mathfrak{S}(\alpha_i)$ and $\mathfrak{S}(\alpha_j)$.*

Proof. Since $\mathfrak{S}(\alpha) = \mathfrak{S}(K\alpha)$ for all $K\alpha \in \mathfrak{S}(\alpha)$, it suffices to assume that α and β are in the same W -orbit, i.e. $\beta = w.\alpha$ for some $w \in W$. It is easy to see that the mapping $\mathfrak{S}(\alpha) \rightarrow \mathfrak{S}(\beta)$ given by $\sigma(w)|_{\mathfrak{S}(\alpha)}$ gives the desired one-to-one correspondence. If γ_i and γ_j are in the same ON-connected component, then by Theorem 3.3, some positive scalar multiple of α_j is in the W -orbit of α_i . Thus there is a one-to-one correspondence between the sets $\mathfrak{S}(\alpha_i)$ and $\mathfrak{S}(\alpha_j)$. \square

The proof of the following lemma is a routine verification, so it is omitted.

Lemma 3.5 *Suppose (Γ, A) is unital ON-cyclic. Then for any ON-path \mathcal{P} there is a simple ON-path which is Π -equivalent to \mathcal{P} .*

Although Theorem 3.6 and Corollary 3.7 ask readers to look at a subgraph (Γ', A') of (Γ, A) , the conclusions pertain to the action of $W = W(\Gamma, A)$ on Φ .

Theorem 3.6 *Choose any ON-connected component (Γ', A') of (Γ, A) , and let $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$. Then the following are equivalent:*

- (1) (Γ', A') is unital ON-cyclic.
- (2) $|\mathfrak{S}(w.\alpha_x)| < \infty$ for some $x \in J$ and $w \in W$.
- (3) $|\mathfrak{S}(w.\alpha_x)| < \infty$ for all $x \in J$ and $w \in W$.

In these cases for all $x, y \in J$ and $w \in W$, $|\mathfrak{S}(\alpha_x)| = |\mathfrak{S}(w.\alpha_y)|$. This common quantity is equal to the largest number of pairwise scalar-distinct simple ON-paths in (Γ, A) with end node γ_x .

Proof. We show (2) \Rightarrow (1) \Rightarrow (3), the implication (3) \Rightarrow (2) being obvious. For (1) \Rightarrow (3), let $x \in J$. Observe that if $K\alpha_x \in \Phi^+$, then by Theorem 3.3 we must have $K = \Pi_{\mathcal{P}}$ for some ON-path \mathcal{P} with end node γ_x . Therefore \mathcal{P} is in (Γ', A') . By Lemma 3.5, we may take a simple ON-path \mathcal{Q} Π -equivalent to \mathcal{P} (all ON-paths Π -equivalent to \mathcal{P} must be in (Γ', A')), so that $K = \Pi_{\mathcal{Q}}$. Since there can be at most a finite number of simple ON-paths, then there can be at most finitely many positive roots that are scalar multiples of a given α_x . That $|\mathfrak{S}(w.\alpha_x)| = |\mathfrak{S}(\alpha_x)|$ for all $w \in W$

follows from Lemma 3.4. For (2) \Rightarrow (1), we show the contrapositive. Let $\mathcal{C} = [\gamma_x, \dots, \gamma_x]$ be a non-unital ON-cycle with start/end node γ_x for an $x \in J$. So necessarily \mathcal{C} has nonzero length. Note that $w_{\mathcal{C}} \cdot \alpha_x = \Pi_{\mathcal{C}} \alpha_x$. Next, for $y \in J$ (and possibly $y = x$) take any ON-path \mathcal{P} with start node γ_x and end node γ_y . Since $w_{\mathcal{P}} \cdot \alpha_x = \Pi_{\mathcal{P}} \alpha_y$, it follows that $w_{\mathcal{P}} w_{\mathcal{C}}^k \cdot \alpha_x = \Pi_{\mathcal{P}} \Pi_{\mathcal{C}}^k \alpha_y$ for any integer k . In particular, for all $y \in J$, we have $|\mathfrak{S}(\alpha_y)| = \infty$. So by Lemma 3.4 $|\mathfrak{S}(w \cdot \alpha_y)| = \infty$ for all $y \in J, w \in W$. The next-to-last claim of the theorem statement follows from Lemma 3.4. The final claim follows from our proof above of the (1) \Rightarrow (3) part of the theorem statement. \square

Corollary 3.7 *We have $|\mathfrak{S}(\alpha)| = 1$ for all $\alpha \in \Phi$ if and only if (Γ, A) has no odd asymmetries. More generally, choose any ON-connected component (Γ', A') of (Γ, A) , and let $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$. Then $|\mathfrak{S}(w \cdot \alpha_x)| = 1$ for some $x \in J$ and $w \in W$ if and only if $|\mathfrak{S}(w \cdot \alpha_x)| = 1$ for all $x \in J$ and $w \in W$ if and only if (Γ', A') has no odd asymmetries.*

Proof. Follows from Theorem 3.6 and Theorem 3.3. \square

Analogizing [BB] and [HRT], for any $w \in W$ set $N(w) := N_A(w) := \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^-\}$. (For the matrices A considered in [Kum] Ch. 1, this set is notated $\Phi_{w^{-1}}$.)

Lemma 3.8 *For any $i \in I_n$, $s_i(\Phi^+ \setminus \mathfrak{S}(\alpha_i)) = \Phi^+ \setminus \mathfrak{S}(\alpha_i)$. Now let $w \in W$. If $w \cdot \alpha_i \in \Phi^+$, then $N(ws_i) = s_i(N(w)) \cup \mathfrak{S}(\alpha_i)$, a disjoint union. If $w \cdot \alpha_i \in \Phi^-$, then $N(ws_i) = s_i(N(w) \setminus \mathfrak{S}(\alpha_i))$.*

Proof. Using Proposition 3.2, the proof of Proposition 5.6.(a) from [Hum] is easily adjusted to prove the first claim. Proofs for the remaining claims involve routine set inclusion arguments. \square

When (Γ, A) is ON-connected and unital ON-cyclic, set $f_{\Gamma, A} := |\mathfrak{S}(\alpha_x)|$ for any given $x \in I_n$. At this point, Proposition 3.2, Theorem 3.3, Lemma 3.8, and Theorem 3.6 allow us to modify the proof of Proposition 5.6 of [Hum] to obtain the result that for all $w \in W$, $|N(w)| = f_{\Gamma, A} \ell(w)$. Theorem 3.9 below generalizes this statement for arbitrary E-GCM graphs. For $J \subseteq I_n$, let $\mathfrak{C}(J)$ denote the set of all ON-connected components of (Γ, A) containing some node from the set $\{\gamma_x\}_{x \in J}$.

Theorem 3.9 *Let $w \in W$ with $p = \ell(w) > 0$. (1) Then $N(w)$ is finite if and only if w has a reduced expression $s_{i_1} \cdots s_{i_p}$ for which $\mathfrak{S}(\alpha_{i_q})$ is finite for all $1 \leq q \leq p$ if and only if every reduced expression $s_{i_1} \cdots s_{i_p}$ for w has $\mathfrak{S}(\alpha_{i_q})$ finite for all $1 \leq q \leq p$. (2) Now suppose $w = s_{i_1} \cdots s_{i_p}$ and $N(w)$ is finite. Let $J := \{i_1, \dots, i_p\}$. In view of (1), let f_1 be the min and f_2 the max of all integers in the set $\{f_{\Gamma', A'} \mid (\Gamma', A') \in \mathfrak{C}(J)\}$. Then $f_1 \ell(w) \leq |N(w)| \leq f_2 \ell(w)$.*

Proof. (1) follows from Lemma 3.8. For (2), induct on $\ell(w)$. Take $w' := s_{i_1} \cdots s_{i_{p-1}}$ with $w = w' s_{i_p}$. Now γ_{i_p} is in an ON-connected component (Γ', A') of (Γ, A) . Then by Lemma 3.8, $|N(w)| = |N(w')| + f_{\Gamma', A'}$. Since $f_1 \ell(w') \leq |N(w')| \leq f_2 \ell(w')$, the result follows. \square

Apply Theorems 3.6 and 3.9 to get:

Corollary 3.10 *We have $N(w)$ finite for all $w \in W$ if and only if (Γ, A) is unital ON-cyclic. Moreover $|N(w)| = \ell(w)$ for all $w \in W$ if and only if (Γ, A) has no odd asymmetries.* \square

When W is infinite, the length function must take arbitrarily large values. Then by Theorem 3.9, Φ is infinite as well. If W is finite, then Φ is finite as well, so $|\mathfrak{S}(\alpha_x)| < \infty$ for all $x \in I_n$. In this case let w_0 be the longest element of W (cf. Exercise 5.6.2 of [Hum]). It is easily seen that if $w_0 = s_{i_1} \cdots s_{i_l}$ is reduced, then $\{i_1, \dots, i_l\} = I_n$.

Corollary 3.11 *Suppose W is finite. Let Φ_{std} denote the root system for the standard geometric representation. Then $f_1|\Phi_{\text{std}}^+| \leq |\Phi^+| \leq f_2|\Phi_{\text{std}}^+|$, where f_1 is the min and f_2 is the max of all integers in the set $\{f_{\Gamma', A'} \mid (\Gamma', A') \in \mathfrak{C}(I_n)\}$.*

Proof. Apply Proposition 3.2 to see that $N(w_0) = \Phi^+$. By Theorem 3.9, $f_1\ell(w_0) \leq |\Phi^+| \leq f_2\ell(w_0)$. To see that $\ell(w_0) = |\Phi_{\text{std}}^+|$, apply the previous reasoning in the standard case. \square

Example 3.12 In Figure 3.1 is depicted a connected, unital ON-cyclic E-GCM graph (Γ, A) with three ON-connected components: (Γ_1, A_1) is the E-GCM subgraph with nodes γ_1 and γ_2 ; (Γ_2, A_2) has nodes γ_4, γ_5 , and γ_6 ; and (Γ_3, A_3) has only the node γ_3 . The matrix A is not symmetrizable by Exercise 2.1 of [Kac] or Exercise 1.5.E.1 of [Kum]. Pertaining to the pair (γ_4, γ_6) , we have $4\cos^2(\pi/5) = \frac{3+\sqrt{5}}{2}$ and $2\cos(\pi/5) = \frac{1+\sqrt{5}}{2}$. Since $a_{46} = -\frac{1+\sqrt{5}}{4}$ and $a_{64} = -(1+\sqrt{5})$, then $K_{46} = \frac{-a_{46}}{2\cos(\pi/5)} = \frac{1}{2}$ and $K_{64} = \frac{-a_{64}}{2\cos(\pi/5)} = 2$. For all other odd neighbors (γ_i, γ_j) in this graph, $m_{ij} = 3$, so $K_{ij} = -a_{ij}$ and $K_{ji} = -a_{ji}$. By the last statement of Theorem 3.6, $f_{\Gamma_1, A_1} = 2$ and $f_{\Gamma_2, A_2} = 3$. For example, $\mathfrak{S}(\alpha_2) = \{\alpha_2, \frac{1}{5}\alpha_2\} = N(s_2)$ and $\mathfrak{S}(\alpha_5) = \{\alpha_5, \frac{1}{7}\alpha_5, \frac{2}{7}\alpha_5\} = N(s_5)$. By Theorem 3.9, we can see that $f_{\Gamma_1, A_1}\ell(s_5s_2) = 4 \leq |N(s_5s_2)| \leq 6 = f_{\Gamma_2, A_2}\ell(s_5s_2)$. More precisely, by Lemma 3.8 we get $N(s_5s_2) = s_2(N(s_5)) \cup \mathfrak{S}(\alpha_2)$, whence $|N(s_5s_2)| = 5$. \square

§4 The Tits cone. We close with two results which relate the size of a Coxeter group W and the behavior of a “fundamental domain” for the “contragredient” W -action. We continue to consider $\sigma : W \rightarrow GL(V)$. We have the natural pairing $\langle \lambda, v \rangle := \lambda(v)$ for elements λ in the dual space V^* and vectors v in V . The contragredient representation $\sigma^* := \sigma_A^* : W \rightarrow GL(V^*)$ is determined by $\langle \sigma^*(w)(\lambda), v \rangle = \langle \lambda, \sigma(w^{-1})(v) \rangle$. When $w \in W$ and $\lambda \in V^*$, we write $w.\lambda$ for $\sigma^*(w)(\lambda)$. Let $D := \{\lambda \in V^* \mid \langle \lambda, \alpha_i \rangle \geq 0 \text{ for all } i \in I_n\}$. Following [Erik1] and [Erik2], the *Tits cone* is $U := U_A := \cup_{w \in W} wD$. This generalizes the standard case of [Hum]. In view of Proposition 3.2, the results of [Hum] §5.13 hold here. So D is the aforementioned fundamental domain, and U is a convex cone. Let \bar{U} denote the closure of U . See the lecture notes of Howlett [How] for further discussion of properties of the Tits cone for the standard geometric representation σ , and in particular an investigation of phenomena in $\bar{U} \setminus U$. If T is any convex cone, let T_0 denote the maximal subspace contained in T . It is not hard to see that $T_0 = T \cap (-T)$.

Our Tits cone results below concern U_0 . These results both use/produce consequences from/for the numbers game. Elements of V^* will be referred to as *positions* for (Γ, A) . We define a process of acting on positions in V^* with certain sequences of Coxeter group generators that is equivalent to Eriksson’s numbers game as presented in §4.3 of [BB]. For a positive integer p we say a sequence $(\gamma_{i_1}, \dots, \gamma_{i_p})$ from (Γ, A) is *legal* from a given position λ if $\langle s_{i_{q-1}} \cdots s_{i_1}.\lambda, \alpha_{i_q} \rangle > 0$ for all $1 \leq q \leq p$. Repeated application of Proposition 3.2 implies that in this case, $s_{i_p} \cdots s_{i_1}$ is reduced. Call this the *Reduced Word Result*. Next, say a position λ is *good* if $\lambda \in -D$ or there exists a legal sequence $(\gamma_{i_1}, \dots, \gamma_{i_p})$ from λ such that $s_{i_p} \cdots s_{i_1}.\lambda \in -D$. In the latter case say $(\gamma_{i_1}, \dots, \gamma_{i_p})$ is a *terminated* legal sequence. Think of a good position as a position from which there is a (possibly empty) terminated legal sequence. Eriksson’s Strong Convergence Theorem (see Theorem 2.2 of [Erik2]) shows that all legal sequences of maximal length from a good λ terminate at the same “terminal position” in the same finite number of steps. Lemma 5.13 of [Hum] is the basis for an argument

in §4 of [Erik2] showing that if $\lambda = w.\mu$ for $\mu \in -D$, then μ can be reached from λ by a legal sequence. Then we get the following characterization of the set of good positions:

Proposition 4.1 (Eriksson) *The set of good positions for (Γ, A) is precisely $-U$.*

Our next result generalizes Remark 4.4 of [Deo] to our current setting. This is needed for Theorem 4.3. For $J \subseteq I_n$, let $\Phi^J := \{\alpha \in \Phi^+ \mid \alpha \notin \text{span}_{\mathbb{R}}\{\alpha_j\}_{j \in J}\}$.

Lemma 4.2 *If (Γ, A) is connected, Φ is infinite, and $J \subset I_n$ (proper), then Φ^J is infinite.*

Proof: In the “(ix) \Rightarrow (ii)” part of the proof of Proposition 4.2 in [Deo], assume $|\Phi^J| < \infty$ and begin reading at line -8 of page 620. \square

Proposition 3.2 of [HRT] states that if (Γ, A) is connected, σ is standard, and W is infinite, then $U_0 = \{0\}$. In view of Corollary 3.10 and Lemma 4.2, we can use the proof of Proposition 3.2 of [HRT] verbatim to get the generalization of that result stated as Theorem 4.3 below. One can see that that proof will work if it is known that all $N(w)$ are finite; by Corollary 3.10 this is guaranteed by our hypothesis in the statement of Theorem 4.3 requiring that (Γ, A) is unital ON-cyclic. (Lemma 15 of [Vin] uses different hypotheses in a closely related overall context to obtain $(\overline{U})_0 = \{0\}$.)

Theorem 4.3 *Suppose (Γ, A) is connected and unital ON-cyclic and W is infinite. Then $U_0 = \{0\}$, i.e. U is a “strictly convex” cone. \square*

Remark A Tits cone is similarly defined in the context of Kac–Moody theory e.g. [Kac] §3.12, [Kum] §1.4. Let A be a GCM. Here we follow Kac [Kac] and the end of §2 above. The Kac–Moody Tits cone is the set $C := C_A := \{w.\lambda \mid w \in W, \lambda \in \mathfrak{h}_{\mathbb{R}} \text{ such that } \alpha_i(\lambda) \geq 0 \text{ for } 1 \leq i \leq n\} \subseteq \mathfrak{h}_{\mathbb{R}}$. When A is nondegenerate ($\text{nullity}(A) = 0$), then $V = \mathfrak{h}_{\mathbb{R}}^*$ and hence C and U coincide. Now suppose (Γ, A) is connected and W is infinite. We have that the GCM graph (Γ, A) is unital ON-cyclic. Thus if A is nondegenerate, the result $C_0 = \{0\}$ holds by Theorem 4.3. Allowing $\text{nullity}(A) \geq 0$, Kumar (personal communication) has supplied the following description of C_0 : $C_0 = \{v \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(v) = 0 \text{ for } 1 \leq i \leq n\}$. Here $\dim(C_0) = \text{nullity}(A)$. He notes that this statement may be deduced from Part (c) of Proposition 3.12 of [Kac]. Note that the topological interior of the Kac–Moody Tits cone can never intersect its negative. This follows from [Kac] Exercise 3.15 (see also [Kum] Exercise 1.4.E.1).

When (Γ, A) is connected and unital ON-cyclic, if a nonzero $\lambda \in D$ is good, then by Proposition 4.1 and Theorem 4.3 W must be finite. This observation is used in the proof of a Dynkin diagram classification result for E-GCM graphs given in Theorem 6.1 of [Don]. That result shows that for (Γ, A) connected, there is a nonzero good position $\lambda \in D$ if and only if W is finite. That argument also uses the classification of finite Coxeter groups. See [DE] for a proof that does not require the classification of finite Coxeter groups.

It is an open question whether the conclusion of Theorem 4.3 holds if we relax the hypothesis that (Γ, A) is unital ON-cyclic. In contrast, for finite W the overlap $U \cap (-U)$ is all of V^* :

Proposition 4.4 *If W is finite, then $U = V^* = -U$, so $U_0 = V^*$.*

Proof. Since W is finite, then by the Reduced Word Result it follows that the set of good positions is all of V^* . Proposition 4.1 now implies that $V^* = -U$, hence $U = V^*$ also. \square

Acknowledgments We thank Kimmo Eriksson for providing us with a copy of his thesis and for many helpful conversations during the preparation of this paper. We thank Bob Proctor for his helpful feedback, which included the remarks concerning Weyl groups, and for sharing with us in advance some of the results of [Pro]. We thank Shrawan Kumar for helpful comments concerning Kac–Moody Tits cones.

References

- [BB] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Springer, New York, 2005.
- [Bour] N. Bourbaki, *Groupes et Algèbres de Lie*, Hermann, Paris, 1968.
- [Deo] V. V. Deodhar, “On the root system of a Coxeter group,” *Comm. Algebra* **10** (1982), 611–630.
- [Don] R. G. Donnelly, “Eriksson’s numbers game and finite Coxeter groups,” *European J. Combin.* **29** (2008), 1764–1781.
- [DE] R. G. Donnelly and K. Eriksson, “The numbers game and Dynkin diagram classification results,” preprint.
- [Erik1] K. Eriksson, “Strongly Convergent Games and Coxeter Groups,” Ph.D. thesis, KTH, Stockholm, 1993.
- [Erik2] K. Eriksson, “The numbers game and Coxeter groups,” *Discrete Math.* **139** (1995), 155–166.
- [Gun] P. E. Gunnells, “Cells in Coxeter groups,” *Notices Amer. Math. Soc.* **53** (2006), 528–535.
- [How] R. B. Howlett, “Introduction to Coxeter groups,” preprint, 1997. Available online as Algebra Research Report 1997-06 from the Mathematics and Statistics Department at the University of Sydney, <http://www.maths.usyd.edu.au/res/Algebra/How/1997-6/html>.
- [HRT] R. B. Howlett, P. J. Rowley, and D. E. Taylor, “On outer automorphism groups of Coxeter groups,” *Manuscripta Math.* **93** (1997), 499–513.
- [Hum] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.
- [Kac] V. G. Kac, *Infinite-dimensional Lie Algebras*, 3rd edition, Cambridge University Press, Cambridge, 1990.
- [Kum] S. Kumar, *Kac-Moody Groups, Their Flag Varieties and Representation Theory*, Birkhäuser Boston Inc, Boston, MA, 2002.
- [Lus] G. Lusztig, “Some examples of square integrable representations of semisimple p -adic groups,” *Trans. Amer. Math. Soc.* **277** (1983), 623–653.
- [Moz] S. Mozes, “Reflection processes on graphs and Weyl groups,” *J. Combin. Theory Ser. A* **53** (1990), 128–142.
- [Pro] R. A. Proctor, “General Weyl groups,” in preparation.
- [Vin] E. B. Vinberg, “Discrete linear groups generated by reflections,” *Math. USSR-Izvestiya* **5** (1971), 1083–1119.