The numbers game
and
classification theorems

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A preliminary note: Mathematics is the often abstract study of the way the universe organizes information. That information is encoded in physical, chemical, biological, geological, social, and cognitive structures. It is therefore the job of a mathematician to investigate the mathematical reality of the universe we inhabit, to understand that reality as best he/she can, and then to explain it. This is the spirit in which this study has been undertaken.

Our Motivation

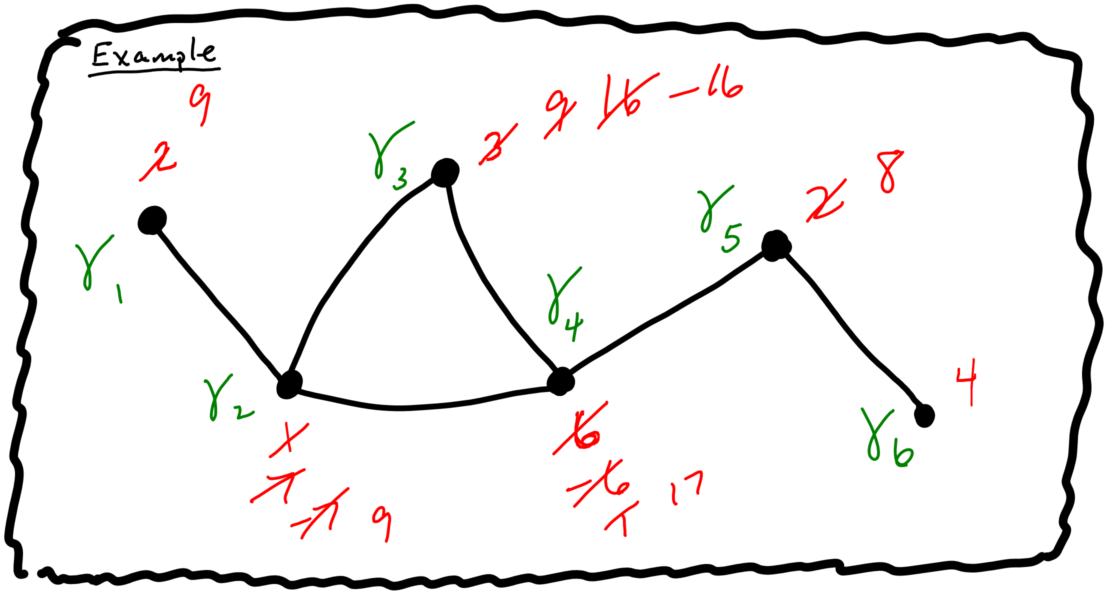
- Study naturally occurring algebraic and geometric finiteness phenomena

- Utilize and advance methods in the study of symmetry, enumeration, and algebra

- Satisfy a human taxonomic impulse

These are three central and millenia-old mathematical themes.

The numbers game



Rules for game play

0. Choose a graph and assign numbers
1. Fire a node with a positive number, else terminate.
2. Return to step 1.

In this example ...

- Node names are in green
- Numbers are in red

  The initial position for the game is $(2, 1, 3, 6, 2, 4)$
- Nodes are fired in the following sequence: $(\gamma_4, \gamma_2, \gamma_3)$

  This is not a complete game, since there are still nodes we could fire.

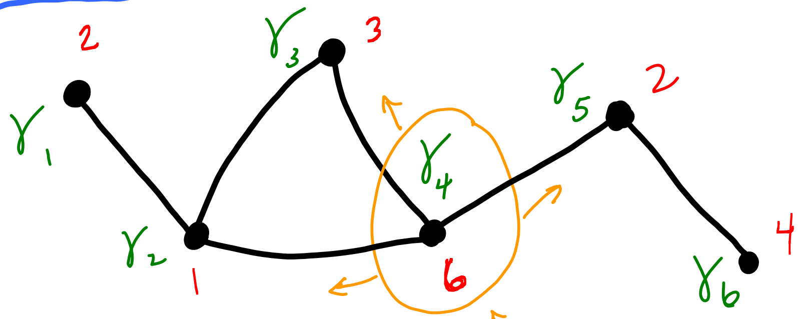
NOTE:

While this game might not be compelling as a diversion or seem to be particularly well-motivated as an object of study, it is undoubtedly an elemental occurrence of basic arithmetic in a simple geometric setting.

The numbers game

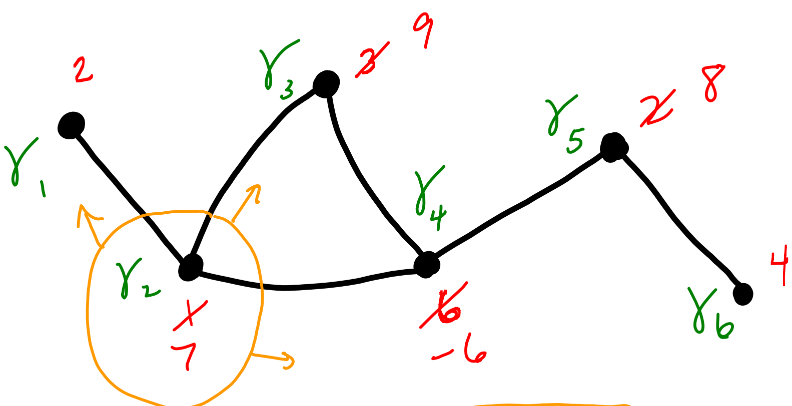
A breakdown of the moves illustrated on the previous slide

Start here



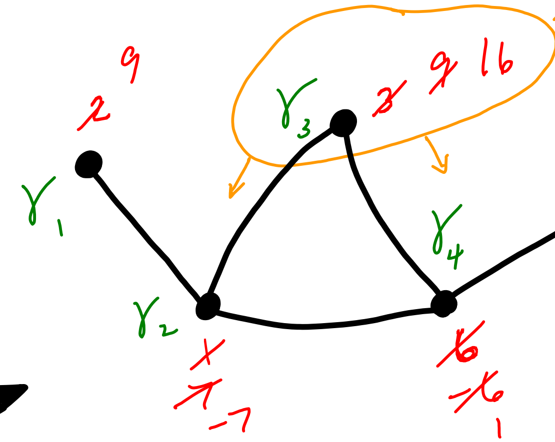
We choose to fire this node first

1st, fire γ_4



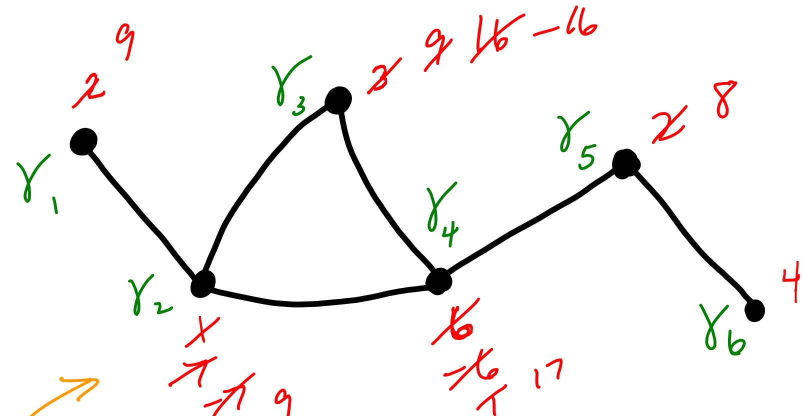
We choose to fire this node second

2nd, fire γ_2



We choose to fire this node third

3rd, fire γ_3



This is the result of the firing sequence $(\gamma_4, \gamma_2, \gamma_3)$. Obviously the game has not yet reached a terminal state...

Questions

1. Does termination depend on one's node-firing strategy?
2. Does it depend on the initial assignment of numbers?
3. Does it depend on the graph?

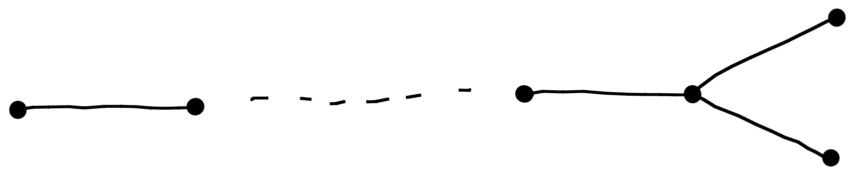
Main Question

For which connected such graphs is there a nontrivial assignment of non-negative numbers for which there is a terminating game?

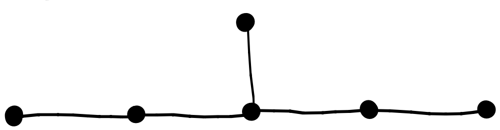
Theorem (D.) The answer to the main question is...

A_n

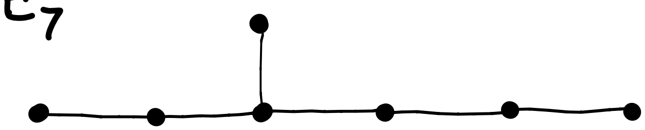
D_n



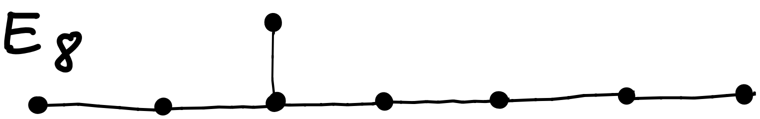
E_6



E_7



E_8



NOTE: Notice that "E_9" is not in the list. It makes one wonder what it is about arithmetic and perhaps even the structure of the universe that causes this exclusion. Is it conceivable that in some universe E_9 is part of the classification?

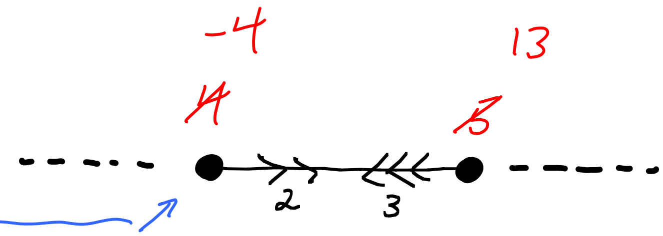
NOTE: In fact, for any starting position, all numbers games on these graphs will terminate.

That is, termination of a numbers game depends only on the graph, not the initial choice of numbers or the choice of which nodes to fire.

Variations

Variation #1

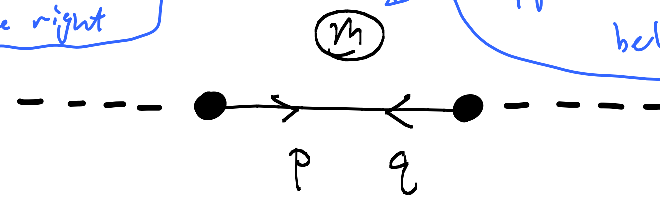
Integer weights (or "multipliers") on edges



Here we fired the node on the left, which meant that 2×4 was added to the node on the right

Variation #2

Real number weights on edges



Same as the m that appears in $4\cos^2(\pi/m)$ below.

"Strong convergence" $\iff$ $pq = 4\cos^2(\pi/m)$, $m \geq 3$

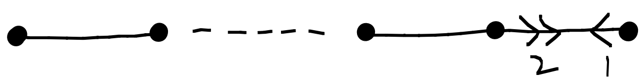
or

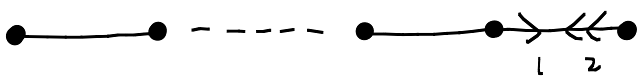
$pq \geq 4$

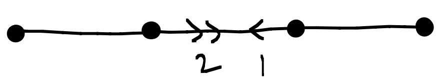
NOTE: A graph is strongly convergent if, from any initial position, all numbers games either diverge or else all terminate at the same position and in the same number of moves. Kimmo Eriksson showed that a necessary and sufficient condition for strong convergence is for all edge products pq to meet the above numerical requirements.

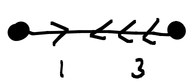
Theorem (D.) For variations, the answers to the main question are ...

Integer A_n; D_n; $E_{6,7,8}$; and

B_n 

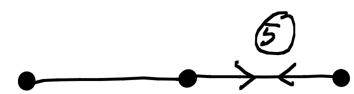
C_n 

F_4 

G_2 

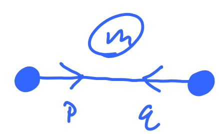
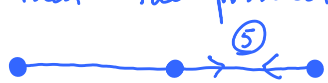
Real A_n — G_2 and

H_2 

H_3 

H_4 

$I_2(m)$  $(m \geq 7)$

<u>NOTE</u>: For the Real cases, the notation  means only that the product pq of the multipliers p and q on this edge must be $4\cos^2(\pi/m)$. The notation  by itself means that the product of the multipliers is $4\cos^2(\pi/3) = 1$. So, for example, the notation H_3  actually represents a <u>family</u> of edge-weighted graphs.

Connection to other classifications

Hypertetrahedron — Hyperoctahedron, Hypercube — 24-cell — Hexagon — Pentagon — Icosahedron, Dodecahedron — 120-cell, 600-cell — Regular polygons

$A_n \quad D_n \quad E_{6,7,8} \quad B_n \quad C_n \quad F_4 \quad G_2 \quad H_2 \quad H_3 \quad H_4 \quad I_2(m)$

"Simply laced" finite Weyl groups & finite-dimensional simple Lie algebras

All finite Weyl groups & finite-dimensional simple Lie algebras

All finite reflection groups

All regular polytopes (= analogs of regular polygons and platonic solids)

Challenge Find a mathematical classification problem/theorem whose answer consists of some infinite families and some sporadic, exceptional cases.

Thesis Such results are almost always interesting.