Some remarks on my research interests Fall 2007
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My research interests are in algebra and combinatorics, but particularly in interactions between these two areas. I like to use combinatorial methods to solve algebraic problems, and vice-versa. This specialty is sometimes identified as algebraic combinatorics, and it is a fertile and growing area of mathematics.

For a starting point, consider the following simple combinatorics questions. These have somewhat simple answers — but these answers are not necessarily easy to come by.

1. It takes one year for a baby elf to mature into an adult elf. Each year an adult elf will "spontaneously reproduce" a baby elf. Elves have an extremely long lifespan. How many elves will there be after n years, if we start with one baby elf in year zero? ANS: The answer is the nth Fibonacci number. An explicit formula for this number is: $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

2. Suppose n people attend a Christmas party. Each guest brings a gift. The gifts are placed in a bin. Each guest blindly picks one gift from the bin. What is the probability that no guest takes home the gift that he/she brought to the party? ANS: The answer is, surprisingly, $\frac{1}{e}$ — or, rather, the nth partial sum in a sequence of partial sums that converges very quickly to $\frac{1}{e}$.

3. How many different six-bead bracelets can one make if each bead can be either red, black, or green? ANS: 92. This is the same as asking: How many different benzene molecules are there, where six carbon atoms arranged in a hexagon are each bonded to one of three different "radicals"?

4. Find a set of n positive integers which has the most subsets adding up to the same number. ANS: The set $\{1, 2, \ldots, n\}$. For example, when $n = 6$, the set $\{1, 2, 3, 4, 5, 6\}$ has five subsets ($\{4, 6\}$, $\{1, 4, 5\}$, $\{1, 3, 6\}$, $\{2, 3, 5\}$, $\{1, 2, 3, 4\}$) whose entries sum up to the same number (10, in this case). This problem was originally posed by Paul Erdős.

As in these problems, *combinatorics* studies relations and properties of discrete and sometimes finite collections. It is noted for its wide applicability (in mathematics as well as other disciplines, like computer science, biology, and process engineering) and its accessibility. With a minimum of terminology and machinery, many problems can be simply stated, and examples of phenomena are often easy to come by. The combinatorics problems that arise in my work are enumerative (where the goal is to count something – like questions 1, 2, and 3 above) and extremal (where the goal is to find those objects that optimize some condition – like question 4 above). The combinatorics problems I study most often have a connection with algebra and involve discrete structures like graphs and finite partially ordered sets.

Algebraic structures sometimes serve as models for certain physical phenomena (group representations in the study of crystals is one example), and sometimes algebraic properties of certain mathematical objects can yield insights into the behavior of these objects. Nonetheless, because of their abstract nature, algebraic structures can be difficult to grasp concretely. On the other hand, some algebraic structures admit curious combinatorial descriptions that are easy to understand and which can lead to new insights and results, both algebraic and combinatorial. This is especially true in representation theory, which studies how algebraic structures like *groups* and *Lie algebras* can be presented as collections of matrices or as permutations of a finite set. A representation of the "dihedral group" D_6 as permutations of the vertices of a regular hexagon can be used together with an algebra result known as "Burnside's Lemma" to find the answer to question 3 above. For another example, all of the information needed to understand a particular matrix representation of a certain "simple" Lie (pronounced "Lee") algebra is encoded in the graph on the next page. In fact, the relationship between this Lie algebra and a family of graphs (which includes the one shown on the next page) was exploited in order to answer question 4 above.

Summary. With the preceding comments in mind, I might summarize my interests as follows: The overarching goal of my research is to explore connections between combinatorics and representation theory and to enjoy the results, both combinatorial and algebraic, that issue from this program.

A graph associated to a representation of a simple Lie algebra (and related to question 4)