Secret Santa

and Other Enumerative Diversions

Fall meeting of ΠME

November 10, 2003

Four Counting Problems

1. How many different license plates with 6 "characters" can we make, if there are 36 choices for each character (A through Z, 0 through 9)?
2. How many different ways are there to form a line with 6 people, if the 6 people are to be chosen from this room?
3. How many different ways are there to put 6 people from this room into an elevator?
4. How many different bowls of alphabet soup are possible if there are 100 letters in each bowl (and, of course, 26 letters to choose from)?

	Repetition allowed	Repetition not allowed
Order matters	1. License plates n^k	2. People in line $(n)_k$
Order doesn't matter	4. Alphabet soup $\left(\!\binom{n}{k}\!\right)$	3. People in an elevator $\binom{n}{k}$

⟵ How to choose k objects from among n objects.

License plates

n choices	n choices	n choices	$\cdots$	n choices
↓	↓	↓		↓
___	___	___	$\cdots$	___

k positions

$$n \cdot n \cdot n \cdots n = n^k$$

People in line

n choices	$n-1$	$n-2$	$\cdots$	$n+1-k$
↓	↓	↓		↓
___	___	___	$\cdots$	___

k positions

$$n \cdot (n-1) \cdot (n-2) \cdots (n+1-k) \overset{(def)}{=\!=}: (n)_k$$

People in an elevator $\binom{n}{k} = ???$ "n choose k"

Alphabet soup $\left(\!\binom{n}{k}\!\right) = ???$ "n multi-choose k"

Exercise Find a formula for $\left(\!\binom{n}{k}\!\right)$.

The Binomial Coefficient $\binom{n}{k}$

- $\binom{n}{k}$ is read "n choose k." Here, n and k are integers. We are trying to answer the question: How many ways are there to choose k objects from among n objects if order doesn't matter and if repetition isn't allowed?

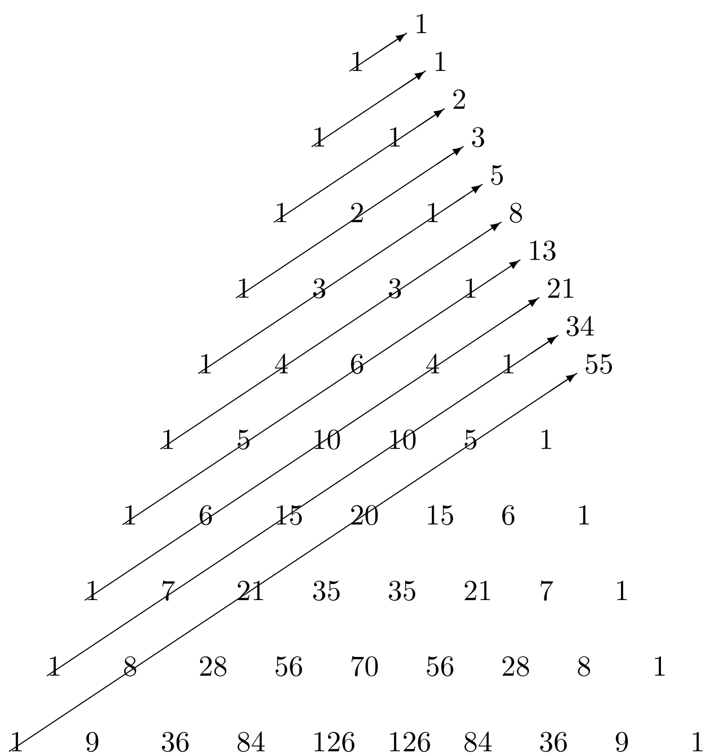
- **Recurrence relation**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

where $\binom{n}{k} \overset{(def)}{:=} 0$ if $k < 0$ or $k > n$ (why?), and $\binom{0}{0} \overset{(def)}{:=} 1$ (why?).

Exercise: Prove this.

Application: We use this recurrence to build "Pascal's Triangle." (For now, please ignore the diagonal sums.)



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The Binomial Coefficient $\binom{n}{k}$, continued

- **Generating function**

The Binomial Theorem $(x+1)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

(So, binomial coefficients are the coefficients that appear when we write $(x+1)^n$ as a power series.)

Exercise: Prove the Binomial Theorem. HINT: Use the recurrence relation for $\binom{n}{k}$.

Application 1: Plug in $x = 1$ to get

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \text{ sum of the entries in } n\text{th row of Pascal's Triangle}$$

Application 2: Plug in $x = -1$ to get

$$\begin{aligned} 0 = 0^n = (-1+1)^n &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} \\ &= \text{alternating sum of the entries in } n\text{th row of Pascal's Triangle} \end{aligned}$$

- **Explicit formula**

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

when $0 \le k \le n$. Here we take

$$n! \overset{(def)}{:=} (n)_n = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

when $n > 0$ and set

$$0! \overset{(def)}{:=} 1.$$

(Why define 0! this way?)

Exercise: Prove that this formula works. HINT: Use the recurrence relation for $\binom{n}{k}$.

Secret Santa

The Question Suppose n people attend a party. Each guest brings a gift. The gifts are placed in a bin. Each guest blindly picks one gift from the bin. What is the probability that no guest takes home the gift that he/she brought to the party?

Example Here's a way to depict one possible scenario if there are n guests at the party:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 & 3 \end{pmatrix} \begin{matrix} \longleftarrow \text{ Person} \\ \longleftarrow \text{ Gift he/she takes} \end{matrix}$$

Set-up for our solution

$$\mathcal{S}_n \overset{(def)}{:=} \text{the set of all possible scenarios}$$

$$\text{Then } |\mathcal{S}_n| = (n)_n = n!$$

$$\mathcal{D}_n \overset{(def)}{:=} \text{the set of all scenarios for which no guest takes home the gift he/she brought}$$

$$\text{Then } \frac{|\mathcal{D}_n|}{n!} = \text{probability that no guest takes home the gift he/she brought}$$

$$Fix(p_1, p_2, \ldots, p_r) \overset{(def)}{:=} \text{the set of all "fixed point" scenarios } \sigma \text{ for which each guest } p_1, p_2, \ldots, p_r \text{ takes home the gift he/she brought}$$

IMPORTANT NOTE: $Fix(2,4,5,7,8) \subset Fix(2,4,7) \subset Fix(2,7)$

A sloppy way to count $|\mathcal{D}_n|$

$$(*) \qquad |\mathcal{D}_n| \overset{???}{=} |\mathcal{S}_n| - \begin{pmatrix} |Fix(1)| \\ + \\ |Fix(2)| \\ + \\ \vdots \\ + \\ |Fix(n)| \end{pmatrix} + \begin{pmatrix} |Fix(1,2)| \\ + \\ |Fix(1,3)| \\ + \\ \vdots \\ + \\ |Fix(n-1,n)| \end{pmatrix} - \begin{pmatrix} |Fix(1,2,3)| \\ + \\ |Fix(1,2,4)| \\ + \\ \vdots \\ + \\ |Fix(n-2,n-1,n)| \end{pmatrix} + \cdots$$

So apparently many "fixed point" scenarios will be thrown out multiple times and added back multiple times to the count on the right hand side.

Claim *On the right hand side of* $(*)$, *the net result is that each "fixed point" scenario is thrown out exactly once.*

Secret Santa, continued

Proof of Claim. Consider a scenario σ which has r fixed points.

Example The scenario $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 & 3 \end{pmatrix}$ has $r = 6$ fixed points

Thrown out	$\binom{6}{1}$	times:	$-\binom{6}{1}$	-6
Added back	$\binom{6}{2}$	times:	$+\binom{6}{2}$	$+15$
Thrown out	$\binom{6}{3}$	times:	$-\binom{6}{3}$	-20
Added back	$\binom{6}{4}$	times:	$+\binom{6}{4}$	$+15$
Thrown out	$\binom{6}{5}$	times:	$-\binom{6}{5}$	-6
Added back	$\binom{6}{6}$	times:	$+\binom{6}{6}$	$+1$
				-1

In general, the net result for σ is:

$$-\binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r}$$

But, recall that

$$\binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} \;=\; 0,$$

so

$$-\binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} \;=\; -1$$

□

Conclusion: The probability is $\frac{1}{e}$ (!!!)

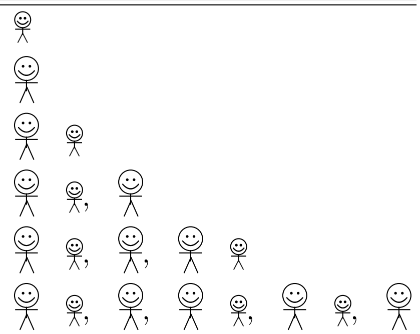
$$\begin{aligned}
|\mathcal{D}_n| &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n \binom{n}{n}(n-n)! \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \, (n-k)! \\
\frac{|\mathcal{D}_n|}{n!} &= \sum_{k=0}^{n} \frac{1}{n!} \, \frac{n!}{k!\,(n-k)!} (-1)^k \, (n-k)! \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \\
&= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \\
&\approx \frac{1}{e}
\end{aligned}$$

The Elf Reproduction Problem

A model for elf reproduction? It takes one year for a baby elf to mature into an adult elf. Each year an adult elf will "spontaneously reproduce" a baby elf. Elves have an extremely long lifespan.

The Question How many elves will there be after n years, if we start with one baby elf in year zero?

Year	# of elves
0	1
1	1
2	2
3	3
4	5
5	8
6	???



Let $f_n \overset{(def)}{:=}$ # of elves after n years

Recurrence relation

$$f_{n+2} = f_{n+1} + f_n$$

for $n \geq 0$, with $f_0 = 1$.

Exercise Prove this.

This is the famous *Fibonacci sequence*: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

Generating function Let $g(x) \overset{(def)}{:=} \sum_{k=0}^{\infty} f_k\, x^k$.

Claim $$g(x) = \frac{-1}{x^2 + x - 1} = \frac{-1}{(x-\alpha)(x-\beta)},$$
where α and β are the roots of $x^2 + x - 1 = 0$ with $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$.

Exercise Prove this.

Claim $$g(x) = \frac{-1}{(x-\alpha)(x-\beta)} = \frac{1}{\sqrt{5}}\frac{1}{(\alpha - x)} - \frac{1}{\sqrt{5}}\frac{1}{(\beta - x)} = \sum_{k=0}^{\infty} \frac{1}{\sqrt{5}}\left(\frac{1}{\alpha^{k+1}} - \frac{1}{\beta^{k+1}}\right) x^k$$

Exercise Prove this.

Explicit formula Equate coefficients to see that

$$f_k = \frac{1}{\sqrt{5}}\left(\frac{1}{\alpha^{k+1}} - \frac{1}{\beta^{k+1}}\right)$$

The Elf Reproduction Problem, continued

Proof of generating function claims

Claim Let $g(x) \overset{(def)}{:=} \sum_{k=0}^{\infty} f_k\, x^k$. Then $g(x) = \dfrac{-1}{x^2+x-1} = \dfrac{-1}{(x-\alpha)(x-\beta)}$,

where α and β are the roots of $x^2+x-1=0$ with $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$.

Proof. We have

$$\begin{aligned} y := g(x) = \sum_{k=0}^{\infty} f_k\, x^k &= 1 + x + \sum_{k=2}^{\infty} f_k\, x^k \\ &= 1 + x + \sum_{k=2}^{\infty} (f_{k-1} + f_{k-2})\, x^k \\ &= 1 + x + \sum_{k=2}^{\infty} f_{k-1}\, x^k + \sum_{k=2}^{\infty} f_{k-2}\, x^k \\ &= 1 + x + x\sum_{k=2}^{\infty} f_{k-1}\, x^{k-1} + x^2 \sum_{k=2}^{\infty} f_{k-2}\, x^{k-2} \\ &= 1 + x - x + x\sum_{k=1}^{\infty} f_{k-1}\, x^{k-1} + x^2 \sum_{k=2}^{\infty} f_{k-2}\, x^{k-2} \\ &= 1 + xy + x^2 y \end{aligned}$$

So $y = 1 + xy + x^2y$. Solve for y to get $y = \dfrac{-1}{x^2+x-1}$. □

Claim $g(x) = \dfrac{-1}{(x-\alpha)(x-\beta)} = \dfrac{1}{\sqrt{5}}\dfrac{1}{(\alpha - x)} - \dfrac{1}{\sqrt{5}}\dfrac{1}{(\beta - x)} = \sum_{k=0}^{\infty} \dfrac{1}{\sqrt{5}} \left(\dfrac{1}{\alpha^{k+1}} - \dfrac{1}{\beta^{k+1}} \right) x^k$

Proof. Using the Calculus II notion of "partial fractions" we write

$$\frac{-1}{(x-\alpha)(x-\beta)} = \frac{A}{\alpha - x} + \frac{B}{\beta - x}$$

Solving for A and B we get $A = 1/\sqrt{5}$ and $B = -1/\sqrt{5}$. Then

$$\frac{-1}{(x-\alpha)(x-\beta)} = \frac{1}{\sqrt{5}}\frac{1}{(\alpha - x)} - \frac{1}{\sqrt{5}}\frac{1}{(\beta - x)}$$

The function $\frac{1}{r-x}$ can be written as a power series as follows:

$$\frac{1}{r-x} = \frac{1}{r}\frac{1}{\left(1 - \frac{x}{r}\right)} = \frac{1}{r}\sum_{k=0}^{\infty} \frac{1}{r^k}\, x^k$$

Apply this to $\frac{1}{\alpha - x}$ and $\frac{1}{\beta - x}$. Then

$$\frac{1}{\sqrt{5}}\frac{1}{(\alpha - x)} - \frac{1}{\sqrt{5}}\frac{1}{(\beta - x)} = \frac{1}{\sqrt{5}\,\alpha}\sum_{k=0}^{\infty} \frac{1}{\alpha^k}\, x^k \; - \; \frac{1}{\sqrt{5}\,\beta}\sum_{k=0}^{\infty} \frac{1}{\beta^k}\, x^k = \sum_{k=0}^{\infty} \frac{1}{\sqrt{5}} \left(\frac{1}{\alpha^{k+1}} - \frac{1}{\beta^{k+1}} \right) x^k$$ □

The Elf Reproduction Problem, continued

A solution that uses matrix methods*: The set-up

Let x_n denote the number of baby elves at year n, and let y_n denote the number of adult elves at year n. We can view these as a vector:

$$\mathbf{x}_n \;:=\; \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

Let's suppose that at the nth year we have $x_n = x$ and $y_n = y$, so that $\mathbf{x}_n \;=\; \begin{bmatrix} x \\ y \end{bmatrix}$. Then in the following year, the population distribution will change as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} y \\ x+y \end{bmatrix},$$

since all x of the babies become adults while each of the y adults will have one baby. This is a linear transformation whose representing matrix in standard coordinates is $A \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Thus we see that:

$$\mathbf{x}_n \;=\; A\mathbf{x}_{n-1} \;=\; A^2\mathbf{x}_{n-2} \;=\; \cdots \;=\; A^n\mathbf{x}_0 \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In order to get an explicit formula for x_n and y_n, we need a way to compute A^n. If we could *diagonalize* A — that is, find a diagonal matrix D and an invertible matrix P so that $A = PDP^{-1}$ — then this would be easy, since we would have

$$A^n = (PDP^{-1})^n = (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})(PDP^{-1}) = PD^nP^{-1}$$

Eigenvalues and eigenvectors for A

We diagonalize by finding *eigenvalues* and *eigenvectors* for A. The eigenvalues are $\kappa = \frac{1+\sqrt{5}}{2}$ and $\lambda = \frac{1-\sqrt{5}}{2}$. Corresponding eigenvectors are $\mathbf{v} = \begin{bmatrix} \lambda \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} \kappa \\ -1 \end{bmatrix}$. Then we'll have $A = PDP^{-1}$ as desired if we set

$$D \overset{(def)}{:=} \begin{bmatrix} \kappa & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad P \overset{(def)}{:=} \begin{bmatrix} \lambda & \kappa \\ -1 & -1 \end{bmatrix} \quad \text{with} \quad P^{-1} \;=\; \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & -\kappa \\ 1 & \lambda \end{bmatrix}$$

Exercise: Confirm the previous statement.

Putting it all together

Now check that

$$\mathbf{x}_n \;=\; A^n\mathbf{x}_0 \;=\; PD^nP^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;=\; \frac{1}{\sqrt{5}} \begin{bmatrix} -\kappa^n\lambda + \lambda^n\kappa \\ \kappa^n - \lambda^n \end{bmatrix}$$

Then the total population at the nth year is $f_n = x_n + y_n = \frac{1}{\sqrt{5}}\left(\kappa^{n+1} - \lambda^{n+1}\right)$.

Exercise: Confirm the previous computations.

* This is a simplistic example of a phenomenon in population dynamics known as "discrete-time evolution."