Posets, Weyl Characters, and Representations of Semisimple Lie Algebras

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Abstract

This work-in-progress is intended as an exposition of the background material, results, and open problems of a particular poset theoretic study of Weyl characters and semisimple Lie algebra representations begun in the late 1970's and early 1980's in the work of Richard P. Stanley and Robert A. Proctor.

1. Introduction

Combinatorial representation theory is currently a flourishing area of research. Broadly speaking, the goal of this area is to advance understanding of algebraic structures and their representations using combinatorial methods, and vice-versa. For an excellent survey, see [BR].

Our focus here is on one particular corner of this area: a poset theoretic study of Weyl characters and semisimple Lie algebra representations. This subject began with certain work of Richard P. Stanley and Robert A. Proctor in the late 1970's and early 1980's. In papers such as [Sta1], it was clear that Stanley was aware of nice interactions between certain families of posets and Weyl characters. Proctor introduced the idea of semisimple Lie algebras acting on posets in papers such as [Pro1], [Pro2], [Pro3], and [Pro4]. Since that time, there has been interest in finding combinatorial models for Weyl characters and in constructing representations using combinatorial methods. These have been topics of interest for this author ([Don1], [Don2], [Don3], [Don4], [Don5]) as well as many other researchers ([Alv], [ADLP], [ADLMPPW], [DLP1], [DLP2], [DW], [HL], [KN], [LS], [LP], [Lit], [Mc], [Stem3], [Wil1], [Wil2], [Wil3], etc).

One goal of this work-in-progress is to provide a reasonably self-contained exposition of the main background ideas from combinatorics, Weyl group and Weyl character theory, and the representation theory of semisimple Lie algebras that are needed to understand the results of this subject. Another goal is to survey the main results and contributions of this subject since its inception in work of Stanley and Proctor, paying particular attention to developments over the past decade. Finally, we hope to inspire interest in the subject by showcasing some of the beautiful objects this study has produced and by pointing out many open problems. In many ways, this subject is still in its infancy.

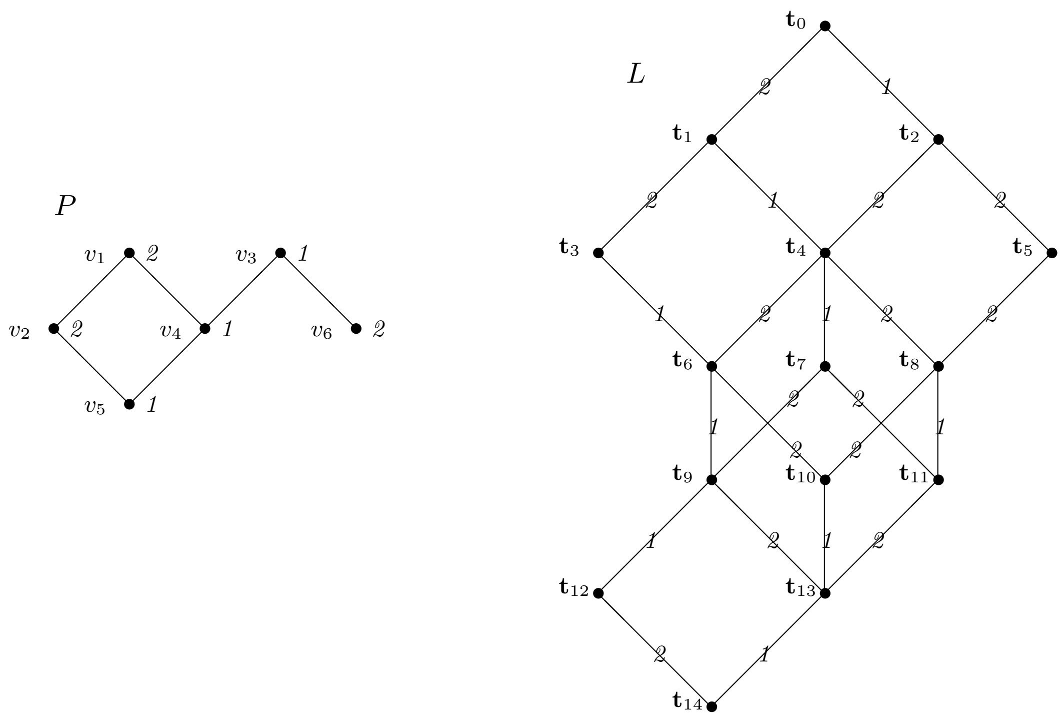
April 2008. In this edition, we mainly focus on providing a view of the combinatorial and Weyl character theoretic environment for our subject. In introducing our main combinatorial objects of interest, we recast some of the the conventional notions of finite partially ordered sets and finite distributive lattices by "coloring" vertices and/or edges. The development of these combinatorial ideas is largely self-contained. We also attempt to present basic Weyl character theory from a combinatorial starting point. This part of our exposition is not as self-contained, but key references are given for those parts of the development which require theory not included here.

2. Combinatorics background

Some of the definitions, notational conventions, and results of this chapter borrow from [Don4], [DLP1], [DLP2], [ADLP], [ADLMPPW], and [Sta2]. We use "R" (and when necessary, "Q") as a generic name for most of the combinatorial objects we define here ("edge-colored directed graph," "vertex-colored directed graph," "ranked poset"). The letter "P" is reserved for posets (and "vertex-colored" posets) that will be viewed as posets of irreducibles for distributive lattices; we reserve use of the letter "L" for reference to lattices and "edge-colored" lattices.

§2.1 Vertex- and edge-colored directed graphs. Let I be any set. An *edge-colored directed graph with edges colored by the set* I is a directed graph R with vertex set $\mathcal{V}(R)$ and directed-edge set $\mathcal{E}(R)$ together with a function $\mathbf{edgecolor}_R : \mathcal{E}(R) \longrightarrow I$ assigning to each edge of R an element ("color") from the set I. If an edge $\mathbf{s} \rightarrow \mathbf{t}$ in R is assigned color $i \in I$, we write $\mathbf{s} \xrightarrow{i} \mathbf{t}$. For $i \in I$, we let $\mathcal{E}_i(R)$ denote the set of edges in R of color i, so $\mathcal{E}_i(R) = \mathbf{edgecolor}_R^{-1}(i)$. If J is a subset of I, remove all edges from R whose colors are not in J; connected components of the resulting edge-colored directed graph are called *J-components* of R. For any $\mathbf{t}$ in R and any $J \subset I$, we let $\mathbf{comp}_J(\mathbf{t})$ denote the J-component of R containing $\mathbf{t}$. The *dual* R^* is the edge-colored directed graph whose vertex set $\mathcal{V}(R^*)$ is the set of symbols $\{\mathbf{t}^*\}_{\mathbf{t} \in R}$ together with colored edges $\mathcal{E}_i(R^*) := \{\mathbf{t}^* \xrightarrow{i} \mathbf{s}^* \,|\, \mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathcal{E}_i(R)\}$ for each $i \in I$. Let Q be another edge-colored directed graph with edge colors from I. If R and Q have disjoint vertex sets, then the *disjoint sum* $R \oplus Q$ is the edge-colored directed graph whose vertex set is the disjoint union $\mathcal{V}(R) \cup \mathcal{V}(Q)$ and whose colored edges are $\mathcal{E}_i(R) \cup \mathcal{E}_i(Q)$ for each $i \in I$. If $\mathcal{V}(Q) \subseteq \mathcal{V}(R)$ and $\mathcal{E}_i(Q) \subseteq \mathcal{E}_i(R)$ for each $i \in I$, then Q is an *edge-colored subgraph* of R. Let $R \times Q$ denote the edge-colored directed graph whose vertex set is the Cartesian product $\{(\mathbf{s}, \mathbf{t}) | \mathbf{s} \in R, \mathbf{t} \in Q\}$ and with colored edges $(\mathbf{s}_1, \mathbf{t}_1) \xrightarrow{i} (\mathbf{s}_2, \mathbf{t}_2)$ if and only if $\mathbf{s}_1 = \mathbf{s}_2$ in R with $\mathbf{t}_1 \xrightarrow{i} \mathbf{t}_2$ in Q or $\mathbf{s}_1 \xrightarrow{i} \mathbf{s}_2$ in R with $\mathbf{t}_1 = \mathbf{t}_2$ in Q. Two edge-colored directed graphs are *isomorphic* if there is a bijection between their vertex sets that preserves edges and edge colors. If R is an edge-colored directed graph with edges colored by the set I, and if $\sigma : I \longrightarrow I'$ is a mapping of sets, then we let R^σ be the edge-colored directed graph with edge color function $\mathbf{edgecolor}_{R^\sigma} := \sigma \circ \mathbf{edgecolor}_R$. We call R^σ a *recoloring* of R. Observe that $(R^*)^\sigma \cong (R^\sigma)^*$. We similarly define a *vertex-colored directed graph* with a function $\mathbf{vertexcolor}_R : \mathcal{V}(R) \longrightarrow I$ that assigns colors to the vertices of R. In this context, we speak of the *dual vertex-colored directed graph* R^*, the *disjoint sum* of two vertex-colored directed graphs with disjoint vertex sets, *isomorphism* of vertex-colored directed graphs, *recoloring*, etc. See Figures 2.1, 2.2, 2.3, and 2.4 for examples.

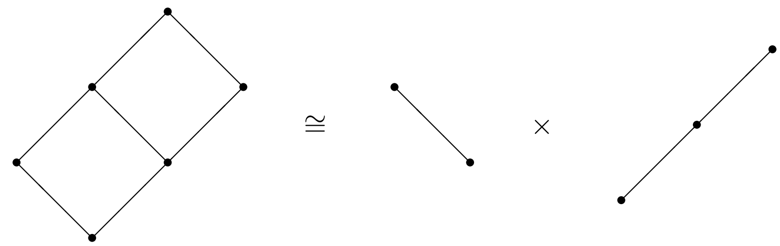
Figure 2.1: A vertex-colored poset P and an edge-colored distributive lattice L.
(The set of vertex colors for P and the set of edge colors for L are $\{1,2\}$.
Elements of P are denoted v_i and elements of L are denoted $\mathbf{t}_i$.
Edges in P and L are directed "up".)



§2.2 Finiteness hypothesis. In this paper, all directed graphs, including all partially ordered sets (discussed in the next subsection) will be assumed to be finite.

§2.3 Posets. A *partially ordered set* ('poset') is a set R together with a relation $\leq_R$ that is reflexive ($\mathbf{s} \leq_R \mathbf{s}$ for all $\mathbf{s} \in R$), transitive ($\mathbf{r} \leq_R \mathbf{s}$ and $\mathbf{s} \leq_R \mathbf{t} \Rightarrow \mathbf{r} \leq_R \mathbf{t}$ for all $\mathbf{r}, \mathbf{s}, \mathbf{t} \in R$), and antisymmetric ($\mathbf{s} \leq_R \mathbf{t}$ and $\mathbf{t} \leq_R \mathbf{s} \Rightarrow \mathbf{s} = \mathbf{t}$ for all $\mathbf{s}, \mathbf{t} \in R$). In this paper, we identify a poset $(R, \leq_R)$ with its Hasse diagram ([Sta2] p. 98): For elements $\mathbf{s}$ and $\mathbf{t}$ of a poset R, there is a directed edge $\mathbf{s} \rightarrow \mathbf{t}$ in the Hasse diagram if and only if $\mathbf{s} < \mathbf{t}$ and there is no $\mathbf{x}$ in R such that $\mathbf{s} < \mathbf{x} < \mathbf{t}$, i.e. $\mathbf{t}$ *covers* $\mathbf{s}$. Thus, terminology that applies to directed graphs (*connected*, *edge-colored*, *dual*, *vertex-colored*, etc) will also apply to posets. When we depict the Hasse diagram for a poset, its edges are directed 'up'. In an edge-colored poset R, we say the vertex $\mathbf{s}$ and the edge $\mathbf{s} \stackrel{i}{\rightarrow} \mathbf{t}$ are *below* $\mathbf{t}$, and the vertex $\mathbf{t}$ and the edge $\mathbf{s} \stackrel{i}{\rightarrow} \mathbf{t}$ are *above* $\mathbf{s}$. The vertex $\mathbf{s}$ is a *descendant* of $\mathbf{t}$, and $\mathbf{t}$ is an *ancestor* of $\mathbf{s}$. The edge-colored and vertex-colored directed graphs studied in this thesis will turn out to be posets. Given a subset Q of the elements of a poset R, let Q inherit the partial ordering of R; call Q a *subposet in the induced order*. Suppose $Q \subseteq R$ for another poset $(Q, \leq_Q)$, and suppose

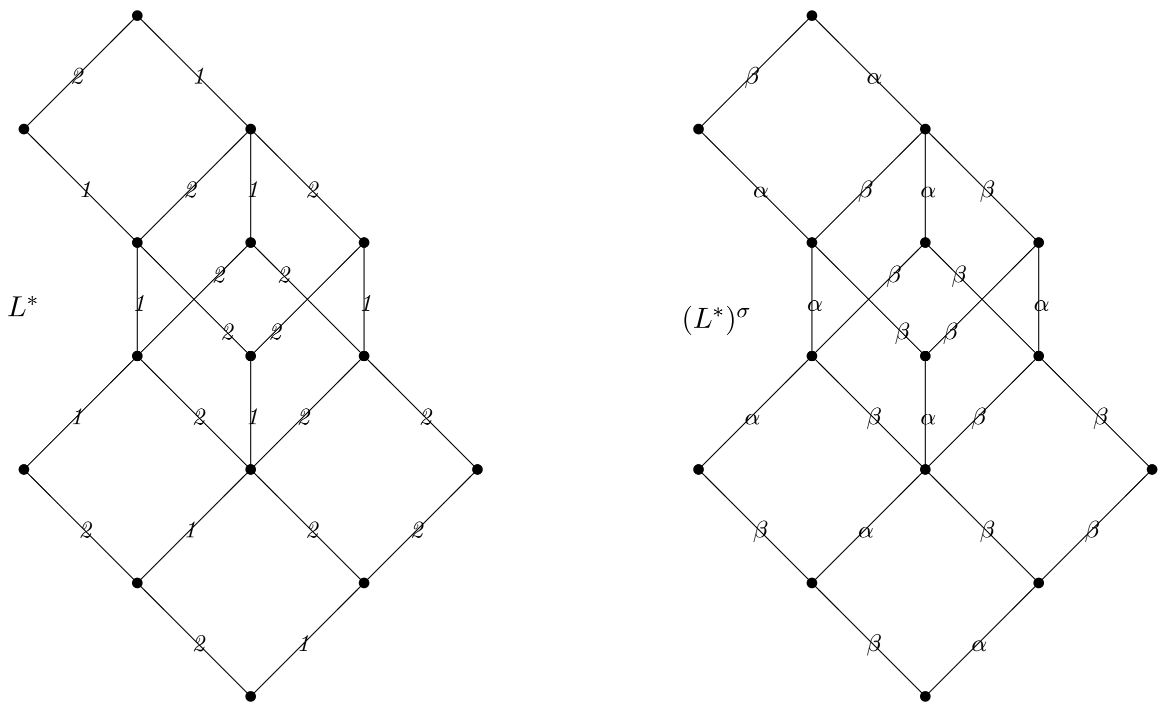
Figure 2.2: A product of chains.



that $\mathbf{s} \leq_Q \mathbf{t} \Rightarrow \mathbf{s} \leq_R \mathbf{t}$ for all $\mathbf{s}, \mathbf{t} \in Q$. Then Q is a *weak subposet* of R. The terminology "weak subposet" applies in the case that Q and R are vertex-colored (resp. edge-colored) if the colors of vertices (resp. edges) from Q are the same as their colors when viewed as vertices (resp. edges) of R. An *antichain* in R is a subset whose elements are pairwise incomparable with respect to the partial order. A *chain* in R is a subset whose elements are pairwise comparable.

For a directed graph R, a *rank function* is a surjective function $\rho : R \longrightarrow \{0, \ldots, l\}$ (where $l \geq 0$) with the property that if $\mathbf{s} \rightarrow \mathbf{t}$ in R, then $\rho(\mathbf{s}) + 1 = \rho(\mathbf{t})$; if such a rank function exists then R is the Hasse diagram for a poset — a *ranked* poset. We call l the *length* of R with respect to ρ, and the set $\rho^{-1}(i)$ is the *ith rank* of R. The *rank generating function* $RGF(R, q)$ for such a ranked poset R is the polynomial $\sum_{i=0}^{l} |\rho^{-1}(i)| q^i$ in the variable q. Given another ranked poset Q, a simple counting argument can be used to show that $RGF(Q \times R, q) = RGF(Q, q) \cdot RGF(R, q)$. A ranked poset that is connected has a unique rank function. A ranked poset R with rank function ρ and length l is *rank symmetric* if $|\rho^{-1}(i)| = |\rho^{-1}(l - i)|$ for $0 \leq i \leq l$. It is *rank unimodal* if there is an m such that $|\rho^{-1}(0)| \leq |\rho^{-1}(1)| \leq \cdots \leq |\rho^{-1}(m)| \geq |\rho^{-1}(m+1)| \geq \cdots \geq |\rho^{-1}(l)|$. It is *strongly Sperner* if for every $k \geq 1$, the largest union of k antichains is no larger than the largest union of k ranks. It has a *symmetric chain decomposition* if there exist chains $R_1, \ldots, R_k$ in R such that (1) as a set $R = R_1 \cup \cdots \cup R_k$ (disjoint union), and (2) for $1 \leq i \leq k$, $\rho(\mathbf{y}_i) + \rho(\mathbf{x}_i) = l$ and $\rho(\mathbf{y}_i) - \rho(\mathbf{x}_i) = l_i$, where $\mathbf{x}_i$ and $\mathbf{y}_i$ are respectively the minimal and maximal elements of the chain R_i, and l_i is the length of the chain R_i. See Figures 2.5 and 2.6. If R has a symmetric chain decomposition, then one can see that R is rank symmetric, rank unimodal, and strongly Sperner; however, the converse does not hold. In an edge-colored ranked poset R, $\mathbf{comp}_i(\mathbf{t})$ will be a ranked poset for each $\mathbf{t} \in R$ and $i \in I$. We let $l_i(\mathbf{t})$ denote the length of $\mathbf{comp}_i(\mathbf{t})$, and we let $\rho_i(\mathbf{t})$ denote the rank of $\mathbf{t}$ within this component. We define the *depth* of $\mathbf{t}$ in its i-component to be $\delta_i(\mathbf{t}) := l_i(\mathbf{t}) - \rho_i(\mathbf{t})$.

Figure 2.3: L^* and $(L^*)^\sigma$ for the lattice L from Figure 2.1.
(Here $\sigma(1) = \alpha$ and $\sigma(2) = \beta$.)



A *path from* $\mathbf{s}$ *to* $\mathbf{t}$ in a poset R is a sequence $(\mathbf{s}_0 = \mathbf{s}, \mathbf{s}_1, \ldots, \mathbf{s}_r = \mathbf{t})$ such that for $1 \leq p \leq r$ it is the case that either $\mathbf{s}_{p-1} \rightarrow \mathbf{s}_p$ or $\mathbf{s}_p \rightarrow \mathbf{s}_{p-1}$. We say this path has *length* r. In notating paths, we sometimes include the directed edges between sequence elements. The distance $\mathbf{dist}(\mathbf{s}, \mathbf{t})$ between $\mathbf{s}$ and $\mathbf{t}$ in a connected poset R is the minimum length achieved when all paths from $\mathbf{s}$ to $\mathbf{t}$ in R are considered. (For example, the distance from $\mathbf{t}_3$ to $\mathbf{t}_5$ in the lattice L from Figure 2.5 is $\mathbf{dist}(\mathbf{t}_3, \mathbf{t}_5) = 4$.) If R is a ranked poset and if $\mathbf{s} \leq \mathbf{t}$ in R, then $\mathbf{dist}(\mathbf{s}, \mathbf{t}) = \rho(\mathbf{t}) - \rho(\mathbf{s})$. We say a poset R *has no open vees* if (1) whenever $\mathbf{r} \rightarrow \mathbf{s}$ and $\mathbf{r} \rightarrow \mathbf{t}$ in R, then there exists a unique $\mathbf{u}$ in R such that $\mathbf{s} \rightarrow \mathbf{u}$ and $\mathbf{t} \rightarrow \mathbf{u}$, and (2) whenever $\mathbf{s} \rightarrow \mathbf{u}$ and $\mathbf{t} \rightarrow \mathbf{u}$ in R, then there exists a unique

Figure 2.4: The disjoint sum of the *2*-components
of the edge-colored lattice L from Figure 2.1.

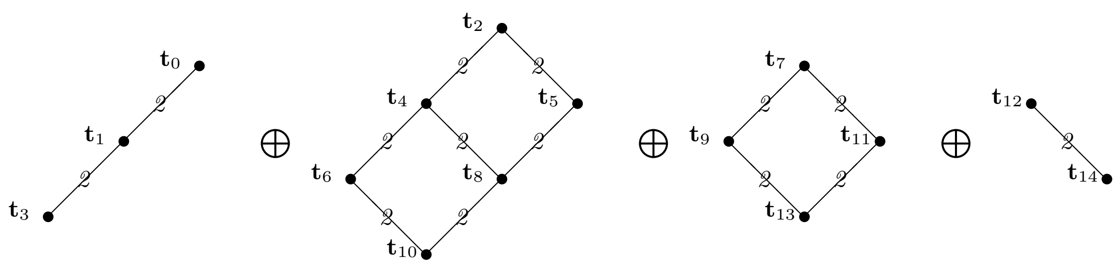
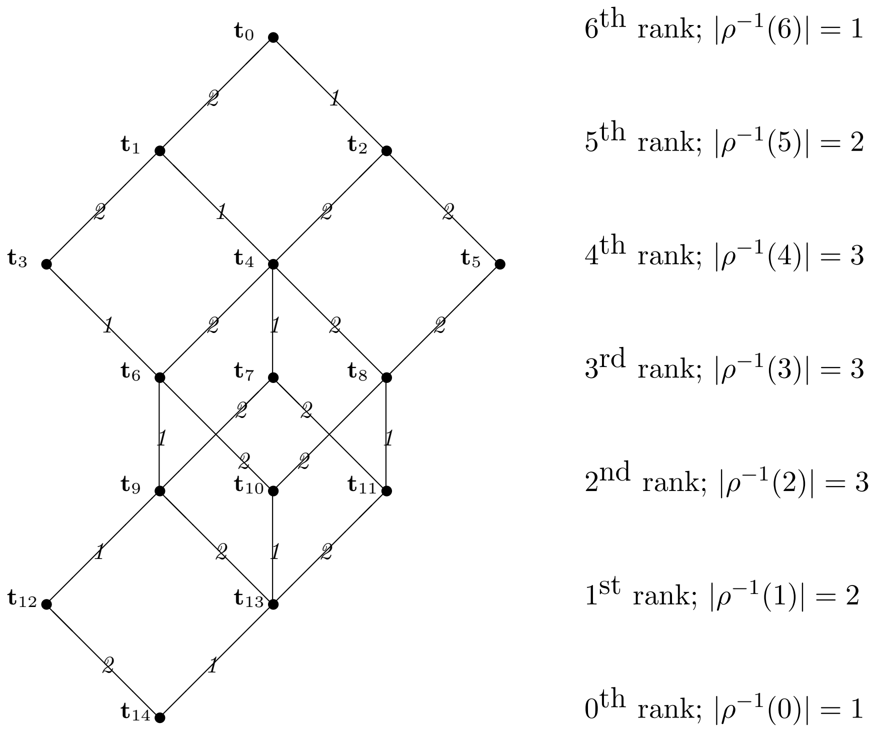
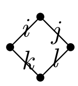


Figure 2.5: The lattice L from Figure 2.1 is rank symmetric and rank unimodal.
$RGF(L,q) = 1 + 2q + 3q^2 + 3q^3 + 3q^4 + 2q^5 + q^6$

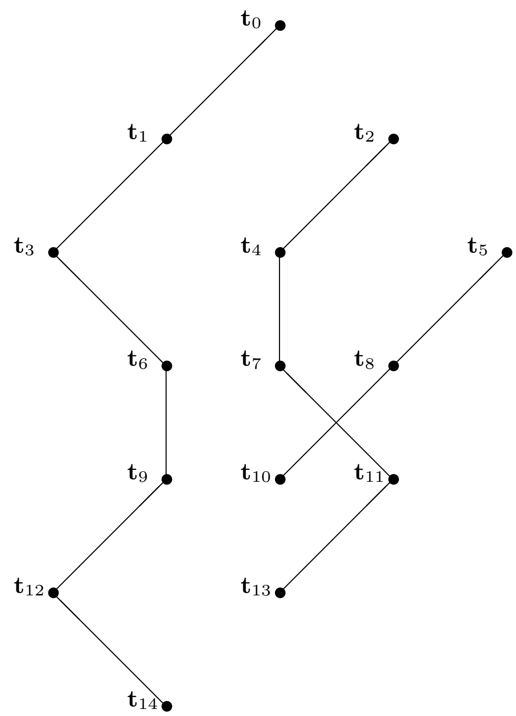


$\mathbf{r}$ in R such that $\mathbf{r} \to \mathbf{s}$ and $\mathbf{r} \to \mathbf{t}$. An edge-colored poset R has the *diamond coloring property* if whenever  is an edge-colored subgraph of the Hasse diagram for R, then $i = l$ and $j = k$.

Let R be an edge-colored ranked poset. For this paragraph, the elements of R will be denoted by $v_1, \ldots, v_n$, so $n = |R|$. For an integer $k \geq 0$, let $\bigwedge^k(R)$ denote the set of all k-element subsets of the vertex set of R. If $k > n$, then $\bigwedge^k(R) = \emptyset$. If $k = 0$ or $k = n$ then $\bigwedge^k(R)$ is a set with one element. For $\mathbf{s}, \mathbf{t} \in \bigwedge^k(R)$, write $\mathbf{s} \xrightarrow{i} \mathbf{t}$ if and only if $\mathbf{s}$ and $\mathbf{t}$ differ by exactly one element in the sense that $(\mathbf{s} - \mathbf{t}, \mathbf{t} - \mathbf{s}) = (\{v_p\}, \{v_q\})$ and $v_p \xrightarrow{i} v_q$ in R. Use the notation $\bigwedge^k(R)$ to refer to this edge-colored directed graph, which we call the kth *exterior power* of R. Similarly let $\mathbb{S}^k(R)$ denote the set of all k-element multisubsets of the vertex set of R and define colored, directed edges $\mathbf{s} \xrightarrow{i} \mathbf{t}$ between elements of $\mathbb{S}^k(R)$. Call $\mathbb{S}^k(R)$ the kth *symmetric power* of R. It can be shown that $\bigwedge^k(R)$ and $\mathbb{S}^k(R)$ are ranked posets whose covering relations are the colored, directed edges prescribed in this paragraph.

§2.4 Lattices, modular lattices, and distributive lattices. A *lattice* is a poset for which any two elements $\mathbf{s}$ and $\mathbf{t}$ have a unique least upper bound $\mathbf{s} \vee \mathbf{t}$ (the *join* of $\mathbf{s}$ and $\mathbf{t}$) and a unique greatest lower bound $\mathbf{s} \wedge \mathbf{t}$ (the *meet* of $\mathbf{s}$ and $\mathbf{t}$). That is, whenever $\mathbf{s} \leq \mathbf{x}$ and $\mathbf{t} \leq \mathbf{x}$ then $(\mathbf{s} \vee \mathbf{t}) \leq \mathbf{x}$,

Figure 2.6: The lattice L from Figure 2.1 has a symmetric chain decomposition.



and whenever $\mathbf{x} \leq \mathbf{s}$ and $\mathbf{x} \leq \mathbf{t}$ then $\mathbf{x} \leq (\mathbf{s} \wedge \mathbf{t})$. A lattice L is necessarily connected, and finiteness implies that there is a unique maximal element $\mathbf{max}(L)$ and a unique minimal element $\mathbf{min}(L)$. For any $\mathbf{r}, \mathbf{s}, \mathbf{t} \in L$, the facts that $\mathbf{r} \wedge (\mathbf{s} \wedge \mathbf{t}) = (\mathbf{r} \wedge \mathbf{s}) \wedge \mathbf{t}$ and $\mathbf{r} \vee (\mathbf{s} \vee \mathbf{t}) = (\mathbf{r} \vee \mathbf{s}) \vee \mathbf{t}$ follow easily from transitivity and antisymmetry of the partial order on L. That is, the meet and join operations are associative. Thus for a nonempty subset S of L, the meet $\wedge_{\mathbf{s} \in S}(\mathbf{s})$ and the join $\vee_{\mathbf{s} \in S}(\mathbf{s})$ are well-defined. We take $\wedge_{\mathbf{s} \in S}(\mathbf{s}) = \mathbf{min}(L)$ and $\vee_{\mathbf{s} \in S}(\mathbf{s}) = \mathbf{max}(L)$ if S is empty.

A lattice L is *modular* if it is ranked and $\rho(\mathbf{s}) + \rho(\mathbf{t}) = \rho(\mathbf{s} \vee \mathbf{t}) + \rho(\mathbf{s} \wedge \mathbf{t})$ for all $\mathbf{s}, \mathbf{t} \in L$. One can easily check that a modular lattice L is a ranked lattice with no open vees. If L is a lattice with no open vees, then one can see that L is ranked and for any $\mathbf{s}$ and $\mathbf{t}$, $\mathbf{dist}(\mathbf{s}, \mathbf{t}) = 2\rho(\mathbf{s} \vee \mathbf{t}) - \rho(\mathbf{s}) - \rho(\mathbf{t}) = \rho(\mathbf{s}) + \rho(\mathbf{t}) - 2\rho(\mathbf{s} \wedge \mathbf{t})$; hence L is a modular lattice (see [Sta2] Proposition 3.3.2). A lattice L is *distributive* if for any $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ in L it is the case that $\mathbf{r} \vee (\mathbf{s} \wedge \mathbf{t}) = (\mathbf{r} \vee \mathbf{s}) \wedge (\mathbf{r} \vee \mathbf{t})$ and $\mathbf{r} \wedge (\mathbf{s} \vee \mathbf{t}) = (\mathbf{r} \wedge \mathbf{s}) \vee (\mathbf{r} \wedge \mathbf{t})$. One can see that this distributive lattice L is a ranked lattice with no open vees. It follows that L is also a modular lattice. The following lemma shows how the modular lattice and diamond-coloring properties can interact. This lemma is used in the proofs of Theorem 2.6 and Proposition 2.8.

Lemma 2.1 *Let L be a diamond-colored modular lattice. Suppose $\mathbf{s} \leq \mathbf{t}$. Suppose $\mathbf{s} = \mathbf{r}_0 \stackrel{i_1}{\to} \mathbf{r}_1 \stackrel{i_2}{\to} \mathbf{r}_2 \stackrel{i_3}{\to} \cdots \stackrel{i_{p-1}}{\to} \mathbf{r}_{p-1} \stackrel{i_p}{\to} \mathbf{r}_p = \mathbf{t}$ and $\mathbf{s} = \mathbf{r}'_0 \stackrel{j_1}{\to} \mathbf{r}'_1 \stackrel{j_2}{\to} \mathbf{r}'_2 \stackrel{j_3}{\to} \cdots \stackrel{j_{p-1}}{\to} \mathbf{r}'_{p-1} \stackrel{j_p}{\to} \mathbf{r}'_p = \mathbf{t}$ are two paths from $\mathbf{s}$ up to $\mathbf{t}$. Then $\{i_1, i_2, \ldots, i_p\} = \{j_1, j_2, \ldots, j_p\}$. Moreover, if $\mathbf{r}_1$ and $\mathbf{r}'_{p-1}$ are incomparable, then $i_1 = j_p$.*

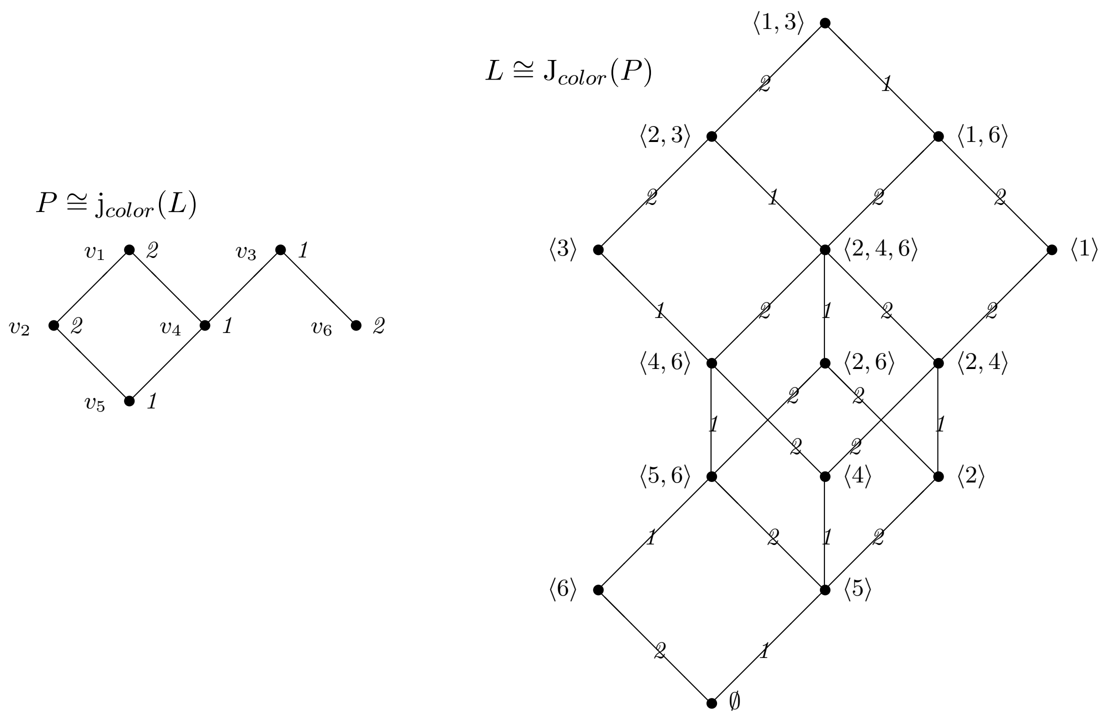
Proof. We use induction on the length p of the given paths to prove both claims of the lemma statement. If $p = 0$, then there is nothing to prove. For our induction hypothesis, we assume the theorem statement holds whenever $p \leq m$ for some nonnegative integer m. Suppose now that $p = m+1$. We consider two cases: (1) $\mathbf{r}_{p-1} = \mathbf{r}'_{p-1}$ and (2) $\mathbf{r}_{p-1} \neq \mathbf{r}'_{p-1}$. In case (1), if $\mathbf{r}_{p-1} = \mathbf{r}'_{p-1}$, then $i_p = j_p$. Moreover, the induction hypothesis applies to the paths $\mathbf{s} = \mathbf{r}_0 \stackrel{i_1}{\to} \mathbf{r}_1 \stackrel{i_2}{\to} \mathbf{r}_2 \stackrel{i_3}{\to} \cdots \stackrel{i_{p-1}}{\to} \mathbf{r}_{p-1} = \mathbf{r}'_{p-1}$ and $\mathbf{s} = \mathbf{r}'_0 \stackrel{j_1}{\to} \mathbf{r}'_1 \stackrel{j_2}{\to} \mathbf{r}'_2 \stackrel{j_3}{\to} \cdots \stackrel{j_{p-1}}{\to} \mathbf{r}'_{p-1} = \mathbf{r}_{p-1}$. It follows that $\{i_1, i_2, \ldots, i_{p-1}\} = \{j_1, j_2, \ldots, j_{p-1}\}$. Since $i_p = j_p$, we conclude that $\{i_1, i_2, \ldots, i_{p-1}, i_p\} = \{j_1, j_2, \ldots, j_{p-1}, j_p\}$. Note that in case (1), $\mathbf{r}_1 \leq \mathbf{r}_{p-1} = \mathbf{r}'_{p-1}$. So $\mathbf{r}_1$ and $\mathbf{r}'_{p-1}$ are comparable.

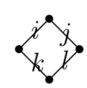
In case (2), if $\mathbf{r}_{p-1} \neq \mathbf{r}'_{p-1}$, then consider $\mathbf{x} := \mathbf{r}_{p-1} \wedge \mathbf{r}'_{p-1}$. Since $\mathbf{s} \leq \mathbf{r}_{p-1}$ and $\mathbf{s} \leq \mathbf{r}'_{p-1}$, it follows that $\mathbf{s} \leq \mathbf{x}$. Then consider a path $\mathbf{s} = \mathbf{r}''_0 \stackrel{k_1}{\to} \mathbf{r}''_1 \stackrel{k_2}{\to} \mathbf{r}''_2 \stackrel{k_3}{\to} \cdots \stackrel{k_{p-3}}{\to} \mathbf{r}''_{p-3} \stackrel{k_{p-2}}{\to} \mathbf{r}''_{p-2} = \mathbf{x}$. Note that since L is diamond-colored, we have $\mathbf{x} \stackrel{j_p}{\to} \mathbf{r}_{p-1}$ and $\mathbf{x} \stackrel{i_p}{\to} \mathbf{r}'_{p-1}$. Then by the induction hypothesis, we have $\{k_1, k_2, \ldots, k_{p-2}, j_p\} = \{i_1, i_2, \ldots, i_{p-2}, i_{p-1}\}$ and $\{k_1, k_2, \ldots, k_{p-2}, i_p\} = \{j_1, j_2, \ldots, j_{p-2}, j_{p-1}\}$. Then, $\{i_1, i_2, \ldots, i_{p-2}, i_{p-1}, i_p\} = \{k_1, k_2, \ldots, k_{p-2}, i_p, j_p\} = \{j_1, j_2, \ldots, j_{p-2}, j_{p-1}, j_p\}$, as desired. Now suppose that $\mathbf{r}_1$ and $\mathbf{r}'_{p-1}$ are incomparable. It follows that $\mathbf{r}_1$ and $\mathbf{x}$ are incomparable as well. Then we can apply the induction hypothesis to the paths $\mathbf{s} = \mathbf{r}''_0 \stackrel{k_1}{\to} \mathbf{r}''_1 \stackrel{k_2}{\to} \mathbf{r}''_2 \stackrel{k_3}{\to} \cdots \stackrel{k_{p-3}}{\to} \mathbf{r}''_{p-3} \stackrel{k_{p-2}}{\to} \mathbf{r}''_{p-2} = \mathbf{x} \stackrel{j_p}{\to} \mathbf{r}_{p-1}$ and $\mathbf{s} = \mathbf{r}_0 \stackrel{i_1}{\to} \mathbf{r}_1 \stackrel{i_2}{\to} \mathbf{r}_2 \stackrel{i_3}{\to} \cdots \stackrel{i_{p-1}}{\to} \mathbf{r}_{p-1}$. From this, we see that $i_1 = j_p$. This completes the induction step, and the proof. $\square$

The following discussion of edge-colored distributive lattices and certain related vertex-colored posets encompasses the classical uncolored situation (for example as in Ch. 3 of [Sta2]). These concepts have antecedents in work of Proctor and Stembridge (see e.g. [Pro3], [Pro4], [Stem2], [Stem1]), but there seems to be no standard treatment of these ideas. The main idea is that for a certain kind of edge-colored distributive lattice, all the information about the lattice can be compressed into a much smaller vertex-colored poset in such a way that the information can be fully recovered.

Edge-colored distributive lattices can be constructed as follows: Let P be a poset with vertices colored by a set I. An *order ideal* $\mathbf{x}$ from P is a vertex subset of P with the property that $u \in \mathbf{x}$ whenever $v \in \mathbf{x}$ and $u \leq v$ in P. For order ideals $\mathbf{x}$ and $\mathbf{y}$ from P, write $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x} \subseteq \mathbf{y}$ (subset

Figure 2.7: The lattice L from Figure 2.1 recognized as $\mathrm{J}_{color}(P)$.
(In this figure, each order ideal from P is identified by the indices of its maximal vertices.
For example, $\langle 2, 3\rangle$ in L denotes the order ideal $\{v_2, v_3, v_4, v_5, v_6\}$ in P.
A join irreducible in L is an order ideal $\langle k\rangle$ whose only maximal element is v_k.)



containment). This is a partial ordering on the set L of order ideals from P. With respect to this partial ordering, L is a distributive lattice: $\mathbf{x} \vee \mathbf{y} = \mathbf{x} \cup \mathbf{y}$ (set union) and $\mathbf{x} \wedge \mathbf{y} = \mathbf{x} \cap \mathbf{y}$ (set intersection) for all $\mathbf{x}, \mathbf{y} \in L$. One can easily see that $\mathbf{x} \rightarrow \mathbf{y}$ in L if and only if $\mathbf{x} \subset \mathbf{y}$ (proper containment) and $\mathbf{y} \setminus \mathbf{x} = \{v\}$ for some maximal element v of $\mathbf{y}$ (thought of as a subposet of P in the induced order). In this case, we declare that $\mathbf{edgecolor}_L(\mathbf{x} \rightarrow \mathbf{y}) := \mathbf{vertexcolor}_P(v)$, making L an edge-colored distributive lattice. One can easily check that whenever 

is an edge-colored subgraph of the Hasse diagram for L, then $i = l$ and $j = k$. Therefore L has the diamond-coloring property. The diamond-colored distributive lattice just constructed is given special notation: we write $L := \mathrm{J}_{color}(P)$. See Figure 2.7. Note that if $P \cong Q$ as vertex-colored posets, then $\mathrm{J}_{color}(P) \cong \mathrm{J}_{color}(Q)$ as edge-colored posets. Moreover, L is ranked with rank function given by $\rho(\mathbf{t}) = |\mathbf{t}|$, the number of elements in the subset $\mathbf{t}$ from P. In particular, the length of L is $|P|$.

The process described in the previous paragraph can be reversed. Given a diamond-colored distributive lattice L, an element $\mathbf{x}$ is *join irreducible* if $\mathbf{x} \neq \mathbf{min}(L)$ and whenever $\mathbf{x} = \mathbf{y} \vee \mathbf{z}$ then $\mathbf{x} = \mathbf{y}$ or $\mathbf{x} = \mathbf{z}$. One can see that $\mathbf{x}$ is join irreducible if and only if $\mathbf{x}$ has precisely one descendant $\mathbf{x}'$ in L, i.e. $|\{\mathbf{x}' \,|\, \mathbf{x}' \rightarrow \mathbf{x}\}| = 1$. Let P be the set of all join irreducible elements of L with the induced partial ordering. Color the vertices of P by the rule: $\mathbf{vertexcolor}_P(\mathbf{x}) := \mathbf{edgecolor}_L(\mathbf{x}' \rightarrow \mathbf{x})$. We call P the *vertex-colored poset of join irreducibles* and denote it by $P := \mathrm{j}_{color}(L)$. If $K \cong L$ is an isomorphism of diamond-colored lattices, then $\mathrm{j}_{color}(K) \cong \mathrm{j}_{color}(L)$ is an isomorphism of vertex-colored posets.

Example 2.2 Let P be an antichain whose elements all have the same color. Then the elements of $L := \mathrm{J}_{color}(P)$ are just the subsets of P. In particular, $|L| = 2^{|P|}$. Moreover, the rank $\rho_L(\mathbf{t})$ of a subset $\mathbf{t}$ from P is just $|\mathbf{t}|$. The join irreducible elements of L are just the singleton subsets of P. Covering relations in L are easy to describe: $\mathbf{s} \rightarrow \mathbf{t}$ if and only if $\mathbf{t}$ is formed from $\mathbf{s}$ by adding to $\mathbf{s}$ exactly one element from $P \setminus \mathbf{s}$. Any such lattice L is called a *Boolean lattice.* □

What follows is a dual to the above constructions of edge-colored distributive lattices. A *filter* from a vertex-colored poset P is a subset $\mathbf{x}$ with the property that if $u \in \mathbf{x}$ and $u \leq v$ in P then $v \in \mathbf{x}$. Note that for $\mathbf{x} \subseteq P$, $\mathbf{x}$ is a filter if and only if the set complement $P \setminus \mathbf{x}$ is an order ideal. Now partially order all filters from P by reverse containment: $\mathbf{x} \leq \mathbf{y}$ if and only if $\mathbf{x} \supseteq \mathbf{y}$ for filters $\mathbf{x}, \mathbf{y}$ from P. The resulting partially ordered set L is a distributive lattice. We color the edges of L as we did in the case of order ideals. The result is a diamond-colored distributive lattice which we denote by $L = \mathrm{M}_{color}(P)$. In the other direction, given a diamond-colored distributive lattice L, we say $\mathbf{x} \in L$ is *meet irreducible* if and only if $\mathbf{x} \neq \mathbf{max}(L)$ and whenever $\mathbf{x} = \mathbf{y} \wedge \mathbf{z}$ then $\mathbf{x} = \mathbf{y}$ or $\mathbf{x} = \mathbf{z}$. One can see that $\mathbf{x}$ is meet irreducible if and only if $\mathbf{x}$ has exactly one ancestor. Now consider the set P of meet irreducible elements in L with the order induced from L. Color the vertices of P in the same way we colored the vertices of the poset of join irreducibles. The vertex-colored poset P is the *poset of meet irreducibles* for L. Write $P = \mathrm{m}_{color}(L)$. We have $\mathrm{m}_{color}(P) \cong \mathrm{m}_{color}(Q)$ if $P \cong Q$ (an isomorphism of vertex-colored posets). We also have $\mathrm{M}_{color}(L) \cong \mathrm{M}_{color}(K)$ if $L \cong K$ (an isomorphism of diamond-colored distributive lattices).

The next result shows that the operations J_{color} (respectively, M_{color}) and j_{color} (respectively, m_{color}) are inverses in a certain sense. This is a straightforward generalization of the classical Fundamental Theorem of Finite Distributive Lattices (cf. Theorem 3.4.1 of [Sta2]). The latter result is formulated for uncolored posets and distributive lattices.

Theorem 2.3 **(The Fundamental Theorem of Finite Diamond-colored Distributive Lattices)** *(1) Let L be any diamond-colored distributive lattice. Then it is the case that $L \cong \mathrm{J}_{color}(\mathrm{j}_{color}(L)) \cong \mathrm{M}_{color}(\mathrm{m}_{color}(L))$. (2) Let P be any vertex-colored poset. Then $P \cong \mathrm{j}_{color}(\mathrm{J}_{color}(P)) \cong \mathrm{m}_{color}(\mathrm{M}_{color}(P))$.*

Proof. For (1), let $P := \mathrm{j}_{color}(L)$. Let $\mathbf{min} = \mathbf{min}(L)$ be the unique minimal element of L. For any $\mathbf{x} \in L$ set $\mathcal{I}_{\mathbf{x}} := \{\mathbf{y} \in P \,|\, \mathbf{y} \leq_L \mathbf{x}\}$. Observe that $\mathcal{I}_{\mathbf{x}}$ is an order ideal from P. Clearly $\vee_{\mathbf{y}\in\mathcal{I}_{\mathbf{x}}}(\mathbf{y}) \leq_L \mathbf{x}$. We claim that $\mathbf{x} = \vee_{\mathbf{y}\in\mathcal{I}_{\mathbf{x}}}(\mathbf{y})$. To see this we induct on the rank of $\mathbf{x}$. If $\mathbf{x} = \mathbf{min}$, then $\mathcal{I}_{\mathbf{x}} = \emptyset$, so the desired result follows. For our induction hypothesis, we suppose that $\mathbf{z} = \vee_{\mathbf{y}\in\mathcal{I}_{\mathbf{z}}}(\mathbf{y})$ for all $\mathbf{z}$ with $\rho_L(\mathbf{z}) \leq k$ for some integer $k \geq 0$. Suppose now that $\mathbf{x} \in L$ with $\rho_L(\mathbf{x}) = k+1$. First, consider the case that $\mathbf{x}$ is join irreducible. Then $\mathbf{x} \in \mathcal{I}_{\mathbf{x}}$, so the result $\mathbf{x} = \vee_{\mathbf{y}\in\mathcal{I}_{\mathbf{x}}}(\mathbf{y})$ follows immediately. Now suppose $\mathbf{x}$ is not join irreducible. Then we may write $\mathbf{x} = \mathbf{s} \vee \mathbf{t}$ for some $\mathbf{s} \neq \mathbf{x} \neq \mathbf{t}$. Since $\mathbf{s} \leq_L (\mathbf{s} \vee \mathbf{t})$ and $\mathbf{t} \leq_L (\mathbf{s} \vee \mathbf{t})$, then $\mathbf{s} <_L \mathbf{x}$ and $\mathbf{t} <_L \mathbf{x}$. In particular, $\rho_L(\mathbf{s}) \leq k$ and $\rho_L(\mathbf{t}) \leq k$. So the induction hypothesis applies to $\mathbf{s}$ and $\mathbf{t}$. That is, $\mathbf{s} = \vee_{\mathbf{y}\in\mathcal{I}_{\mathbf{s}}}(\mathbf{y})$ and $\mathbf{t} = \vee_{\mathbf{y}\in\mathcal{I}_{\mathbf{t}}}(\mathbf{y})$. Note also that $(\mathcal{I}_{\mathbf{s}} \cup \mathcal{I}_{\mathbf{t}}) \subseteq \mathcal{I}_{\mathbf{x}}$. Then,

$$\mathbf{x} = \mathbf{s} \vee \mathbf{t} = \vee_{\mathbf{y}\in(\mathcal{I}_{\mathbf{s}}\cup\mathcal{I}_{\mathbf{t}})}(\mathbf{y}) \leq_L \vee_{\mathbf{y}\in\mathcal{I}_{\mathbf{x}}}(\mathbf{y}) \leq_L \mathbf{x},$$

so we have equality all the way through. That is, $\vee_{\mathbf{y}\in\mathcal{I}_{\mathbf{x}}}(\mathbf{y}) = \mathbf{x}$.

We also claim that for any $\mathbf{x} \in L$, if $\mathbf{x} = \vee_{\mathbf{y}\in\mathcal{I}}(\mathbf{y})$ for some order ideal $\mathcal{I}$ from P, then $\mathcal{I} = \mathcal{I}_{\mathbf{x}}$ and $|\mathcal{I}_{\mathbf{x}}| = \rho_L(\mathbf{x})$. To see this, we use induction on the rank of $\mathbf{x}$. When $\rho_L(\mathbf{x}) = 0$, then $\mathbf{x} = \mathbf{min}$. In this case, if $\mathbf{x} = \vee_{\mathbf{y}\in\mathcal{I}}(\mathbf{y})$ for some order ideal $\mathcal{I}$ from P, then it must be the case that $\mathcal{I} = \emptyset$, hence $\mathcal{I} = \mathcal{I}_{\mathbf{x}}$ and $|\mathcal{I}_{\mathbf{x}}| = \rho_L(\mathbf{x})$. For our induction hypothesis, suppose the claim holds for all elements of L with rank no more than k for some positive integer k. Next suppose that for some $\mathbf{x} \in L$ we have $\rho_L(\mathbf{x}) = k+1$ and $\mathbf{x} = \vee_{\mathbf{y}\in\mathcal{I}}(\mathbf{y})$ for some order ideal $\mathcal{I}$ from P. Choose a maximal element $\mathbf{z}$ in $\mathcal{I}$. Then let $\mathcal{J} := \mathcal{I} \setminus \{\mathbf{z}\}$. Clearly $\mathcal{J}$ is an order ideal from P. Let $\mathbf{x}' := \vee_{\mathbf{y}\in\mathcal{J}}(\mathbf{y})$. Clearly $\mathbf{x}' \leq_L \mathbf{x}$. In order to apply the induction hypothesis to $\mathbf{x}'$, we need $\mathbf{x}' <_L \mathbf{x}$. Suppose otherwise, so $\mathbf{x}' = \mathbf{x}$. Then $\mathbf{x}' \neq \mathbf{min}$, and hence $J \neq \emptyset$. Further, $\mathbf{x}' = \mathbf{x} = \mathbf{z} \vee (\vee_{\mathbf{y}\in\mathcal{J}}(\mathbf{y})) = \mathbf{z} \vee \mathbf{x}'$ implies that $\mathbf{z} \leq_L \mathbf{x}'$. So $\mathbf{z} \wedge \mathbf{x}' = \mathbf{z}$. But then $\mathbf{z} = \mathbf{z} \wedge \mathbf{x}' = \mathbf{z} \wedge (\vee_{\mathbf{y}\in\mathcal{J}}(\mathbf{y})) = \vee_{\mathbf{y}\in\mathcal{J}}(\mathbf{z} \wedge \mathbf{y})$. Since $\mathbf{z}$ is join irreducible, then $\mathbf{z} \wedge \mathbf{y} = \mathbf{z}$ for all $\mathbf{y} \in \mathcal{J}$. Since $\mathcal{J} \neq \emptyset$, then for some $\mathbf{y} \in \mathcal{J}$ we have $\mathbf{z} \leq_L \mathbf{y}$. But $\mathbf{z}$ was chosen to be maximal in $\mathcal{I}$, and hence $\mathbf{z} \not\leq_L \mathbf{y}$ for all $\mathbf{y} \in \mathcal{J} = \mathcal{I} \setminus \{\mathbf{z}\}$. This is a contradiction, so we conclude that $\mathbf{x}' <_L \mathbf{x}$. Then $\rho_L(\mathbf{x}') < \rho_L(\mathbf{x})$, so the induction hypothesis applies to $\mathbf{x}'$. We get $\mathcal{J} = \mathcal{I}_{\mathbf{x}'}$. In particular, $|\mathcal{I}| = |\mathcal{I}_{\mathbf{x}'}| + 1$. Applying this reasoning to the particular order ideal $\mathcal{I}_{\mathbf{x}}$ we conclude that $|\mathcal{I}_{\mathbf{x}}| = |\mathcal{I}_{\mathbf{x}'}| + 1$. Of course if $\mathbf{t} \in \mathcal{I}$, then by definition $\mathbf{t} \leq_L \mathbf{x}$. Hence $\mathbf{t} \in \mathcal{I}_{\mathbf{x}}$. This shows that

$\mathcal{I} \subseteq \mathcal{I}_{\mathbf{x}}$. Since $|\mathcal{I}| = |\mathcal{I}_{\mathbf{x}}|$, we conclude that $\mathcal{I} = \mathcal{I}_{\mathbf{x}}$. Next suppose $\mathbf{x}' <_L \mathbf{x}'' <_L \mathbf{x}$ for some $\mathbf{x}'' \in L$. Then $\mathcal{I}_{\mathbf{x}'} \subset \mathcal{I}_{\mathbf{x}''} \subset \mathcal{I}_{\mathbf{x}}$, both proper containments. (Otherwise, $\mathbf{x}' = \vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{x}'}}(\mathbf{y}) = \vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{x}''}}(\mathbf{y}) = \mathbf{x}''$, etc.) So $|\mathcal{I}_{\mathbf{x}'}| < |\mathcal{I}_{\mathbf{x}''}| < |\mathcal{I}_{\mathbf{x}}|$. But $|\mathcal{I}_{\mathbf{x}'}| + 1 = |\mathcal{I}_{\mathbf{x}}|$, so both of the preceding inequalities cannot be strict. We conclude that there is no $\mathbf{x}'' \in L$ for which $\mathbf{x}' <_L \mathbf{x}'' <_L \mathbf{x}$. That is, $\mathbf{x}$ covers $\mathbf{x}'$. By the inductive hypothesis we have $\rho_L(\mathbf{x}') = |\mathcal{I}_{\mathbf{x}'}|$. So $\rho_L(\mathbf{x}) = \rho_L(\mathbf{x}') + 1 = |\mathcal{I}_{\mathbf{x}'}| + 1 = |\mathcal{I}_{\mathbf{x}}|$. This completes the proof of our claim.

Now consider the function $\phi : L \to \mathrm{J}_{color}(P)$ defined by $\phi(\mathbf{x}) := \mathcal{I}_{\mathbf{x}}$. We show that ϕ is a bijection. If $\mathcal{I}_{\mathbf{s}} = \mathcal{I}_{\mathbf{t}}$, then $\mathbf{s} = \vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{s}}}(\mathbf{y}) = \vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{t}}}(\mathbf{y}) = \mathbf{t}$. In particular, ϕ is injective. Now suppose $\mathcal{I}$ is an order ideal from P. Let $\mathbf{x} := \vee_{\mathbf{y} \in \mathcal{I}}(\mathbf{y})$. By the preceding paragraph, $\mathcal{I} = \mathcal{I}_{\mathbf{x}}$. So, ϕ is surjective.

We wish to show that $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in L if and only if $\mathcal{I}_{\mathbf{s}} \xrightarrow{i} \mathcal{I}_{\mathbf{t}}$ in $\mathrm{J}_{color}(P)$. First, suppose $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in L. It follows from the definitions that $\mathcal{I}_s \subseteq \mathcal{I}_{\mathbf{t}}$. Now $\mathbf{s} \neq \mathbf{t}$ since $\mathbf{t}$ covers $\mathbf{s}$ in L. Since $\mathcal{I}_{\mathbf{s}} = \phi(\mathbf{s})$ and $\mathcal{I}_{\mathbf{t}} = \phi(\mathbf{t})$ and ϕ is injective, then $\mathcal{I}_{\mathbf{s}} \neq \mathcal{I}_{\mathbf{t}}$. So $\mathcal{I}_s \subset \mathcal{I}_{\mathbf{t}}$ is a proper containment. Suppose $\mathcal{I}_{\mathbf{s}} \subseteq \mathcal{I} \subseteq \mathcal{I}_{\mathbf{t}}$. Since ϕ is surjective, then $\mathcal{I} = \mathcal{I}_{\mathbf{x}}$ for some $\mathbf{x} \in L$. But then $\vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{s}}}(\mathbf{y}) \leq_L \vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{x}}}(\mathbf{y}) \leq_L \vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{t}}}(\mathbf{y})$, and hence $\mathbf{s} \leq_L \mathbf{x} \leq_L \mathbf{t}$. Since $\mathbf{t}$ covers $\mathbf{s}$, then $\mathbf{s} = \mathbf{x}$ or $\mathbf{x} = \mathbf{t}$. Hence $\mathcal{I}_{\mathbf{s}} \to \mathcal{I}_{\mathbf{t}}$ in $\mathrm{J}_{color}(P)$. In particular, there is some $\mathbf{z} \in P$ such that $\mathcal{I}_{\mathbf{s}} = \mathcal{I}_{\mathbf{t}} \setminus \{\mathbf{z}\}$. Moreover, by the definition of J_{color}, $\mathcal{I}_{\mathbf{s}} \xrightarrow{j} \mathcal{I}_{\mathbf{t}}$ in $\mathrm{J}_{color}(P)$ where $j = \mathbf{vertexcolor}_P(\mathbf{z})$. Now j is just the color of the edge $\mathbf{z}' \xrightarrow{j} \mathbf{z}$ for the unique descendant $\mathbf{z}'$ of $\mathbf{z}$ in L. If $\mathbf{z} = \mathbf{t}$, then necessarily $\mathbf{z}' = \mathbf{s}$, and so $j = i$.

So now suppose that $\mathbf{z} \neq \mathbf{t}$. So we have $\mathbf{z} <_L \mathbf{t}$, and hence $\mathcal{I}_{\mathbf{z}} \subset \mathcal{I}_{\mathbf{t}}$. We claim that $\mathbf{z}' \leq_L \mathbf{s}$. To see this, apply the reasoning of the preceding paragraph to conclude that $\mathcal{I}_{\mathbf{z}'} \subset \mathcal{I}_{\mathbf{z}}$ with $\mathcal{I}_{\mathbf{z}} = \mathcal{I}_{\mathbf{z}'} \cup \{\mathbf{z}\}$. It follows that $\mathcal{I}_{\mathbf{z}'} \subset \mathcal{I}_{\mathbf{t}}$. Since $\mathbf{z} \notin \mathcal{I}_{\mathbf{z}'}$, $\mathcal{I}_{\mathbf{z}} \subset \mathcal{I}_{\mathbf{t}}$, and $\mathcal{I}_{\mathbf{s}} = \mathcal{I}_{\mathbf{t}} \setminus \{\mathbf{z}\}$, we get $\mathcal{I}_{\mathbf{z}'} \subseteq \mathcal{I}_{\mathbf{s}}$. Then $\mathbf{z}' = \vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{z}'}}(\mathbf{y}) \leq_L \vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{s}}}(\mathbf{y}) = \mathbf{s}$. Since $\mathbf{z}' \leq_L \mathbf{s}$, there is a path $(\mathbf{z}' = \mathbf{z}'_0, \mathbf{z}'_1, \ldots, \mathbf{z}'_p = \mathbf{s})$ such that for $1 \leq q \leq p$ it is the case that $\mathbf{z}'_q$ covers $\mathbf{z}'_{q-1}$. In particular, for each $1 \leq q \leq p$ there is a color i_q such that $\mathbf{z}'_{q-1} \xrightarrow{i_q} \mathbf{z}'_q$. Since $\mathbf{z}' \xrightarrow{j} \mathbf{z}$ and $\mathbf{z}' \xrightarrow{i_1} \mathbf{z}'_1$ and since L has no open vees, then there is a unique $\mathbf{z}_1$ such that $\mathbf{z} \to \mathbf{z}_1$ and $\mathbf{z}'_1 \to \mathbf{z}_1$. Since L is diamond-colored, then $\mathbf{z} \xrightarrow{i_1} \mathbf{z}_1$ and $\mathbf{z}'_1 \xrightarrow{j} \mathbf{z}_1$. Continue in this way, eventually obtaining a path $(\mathbf{z} = \mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_q)$ such that for $1 \leq q \leq p$ we have $\mathbf{z}_{q-1} \xrightarrow{i_q} \mathbf{z}_q$ and for $0 \leq q \leq p$ we have $\mathbf{z}'_q \xrightarrow{j} \mathbf{z}_q$. In particular, $\mathbf{s} \leq_L \mathbf{z}_p$ and $\mathbf{z} \leq_L \mathbf{z}_p$, so $\mathbf{s} \vee \mathbf{z} \leq_L \mathbf{z}_p$. We claim that $\mathbf{z}$ and $\mathbf{s}$ are not comparable. Otherwise, $\mathbf{s} \leq_L \mathbf{z}$ or $\mathbf{z} \leq_L \mathbf{s}$. In the latter case, we would have $\mathbf{z} \in \mathcal{I}_{\mathbf{s}}$, which is not true. In the former case, $\mathbf{s} \leq_L \mathbf{z} <_L \mathbf{t}$. Since $\mathbf{t}$ covers $\mathbf{s}$, then we must have $\mathbf{s} = \mathbf{z}$. But then $\mathbf{z} \in \mathcal{I}_{\mathbf{s}}$, which is not true. Since $\mathbf{s}$ and $\mathbf{z}$ are not comparable, then $\mathbf{s} < \mathbf{s} \vee \mathbf{z}$. Since $\mathbf{s} \vee \mathbf{z} \leq_L \mathbf{z}_p$ and $\mathbf{s} \xrightarrow{j} \mathbf{z}_p$, it follows that $\mathbf{z}_p = \mathbf{s} \vee \mathbf{z}$. But $\mathbf{s} \vee \mathbf{z} = (\vee_{\mathbf{y} \in \mathcal{I}_{\mathbf{s}}}(\mathbf{y})) \vee \mathbf{z} = \mathbf{t}$, and hence $\mathbf{t} \leq \mathbf{z}_p$. Since $\mathbf{s}$ is covered by both $\mathbf{z}_p$ and $\mathbf{t}$, this can only mean that $\mathbf{z}_p = \mathbf{t}$. Hence $j = i$. So $\mathcal{I}_{\mathbf{s}} \xrightarrow{i} \mathcal{I}_{\mathbf{t}}$ in $\mathrm{J}_{color}(P)$.

On the other hand, suppose $\mathcal{I}_{\mathbf{s}} \xrightarrow{i} \mathcal{I}_{\mathbf{t}}$ in $\mathrm{J}_{color}(P)$. Then $\mathcal{I}_{\mathbf{s}} = \mathcal{I}_{\mathbf{s}} \setminus \{\mathbf{z}\}$ for some $\mathbf{z} \in P$, where $i = \mathbf{vertexcolor}(\mathbf{z})$. That is, $\mathbf{z}' \xrightarrow{i} \mathbf{z}$ in L, where $\mathbf{z}'$ is the unique descendant of $\mathbf{z}$ in L. Then $\mathbf{s} <_L \mathbf{t}$. Since $\rho_L(\mathbf{s}) = |\mathcal{I}_{\mathbf{s}}|$ and $\rho_L(\mathbf{t}) = |\mathcal{I}_{\mathbf{t}}|$, the $\mathbf{s} \rightarrow \mathbf{t}$. Let j be the color of this edge, so $\mathbf{s} \xrightarrow{j} \mathbf{t}$. The preceding two paragraphs showed that we must have $\mathcal{I}_{\mathbf{s}} \xrightarrow{j} \mathcal{I}_{\mathbf{t}}$. Then $i = j$.

We conclude that ϕ is an edge and edge-color preserving bijection from L to $\mathrm{J}_{color}(\mathrm{j}_{color}(P))$. It follows that $L \cong \mathrm{J}_{color}(\mathrm{j}_{color}(P))$. The argument that $L \cong \mathrm{M}_{color}(\mathrm{m}_{color}(P))$ is entirely similar. This completes the proof of (1).

For (2), we only show $P \cong \mathrm{j}_{color}(\mathrm{J}_{color}(P))$ since the argument that $P \cong \mathrm{m}_{color}(\mathrm{M}_{color}(P))$ is entirely similar. Let $L := \mathrm{J}_{color}(P)$, and let $Q := \mathrm{j}_{color}(L)$. For any $v \in P$, let $\langle v \rangle := \{u \in P \,|\, u \leq_P v\}$. Observe that $\langle v \rangle$ is an order ideal with v as its unique maximal element. It follows that for an order ideal $\mathcal{I}$ from P we have $\mathcal{I} \rightarrow \langle v \rangle$ in L if and only if $\mathcal{I} = \langle v \rangle \setminus \{v\}$. Hence, $\langle v \rangle$ is join irreducible in L. So we define a mapping $\psi : P \rightarrow Q$ by $\psi(v) := \langle v \rangle$.

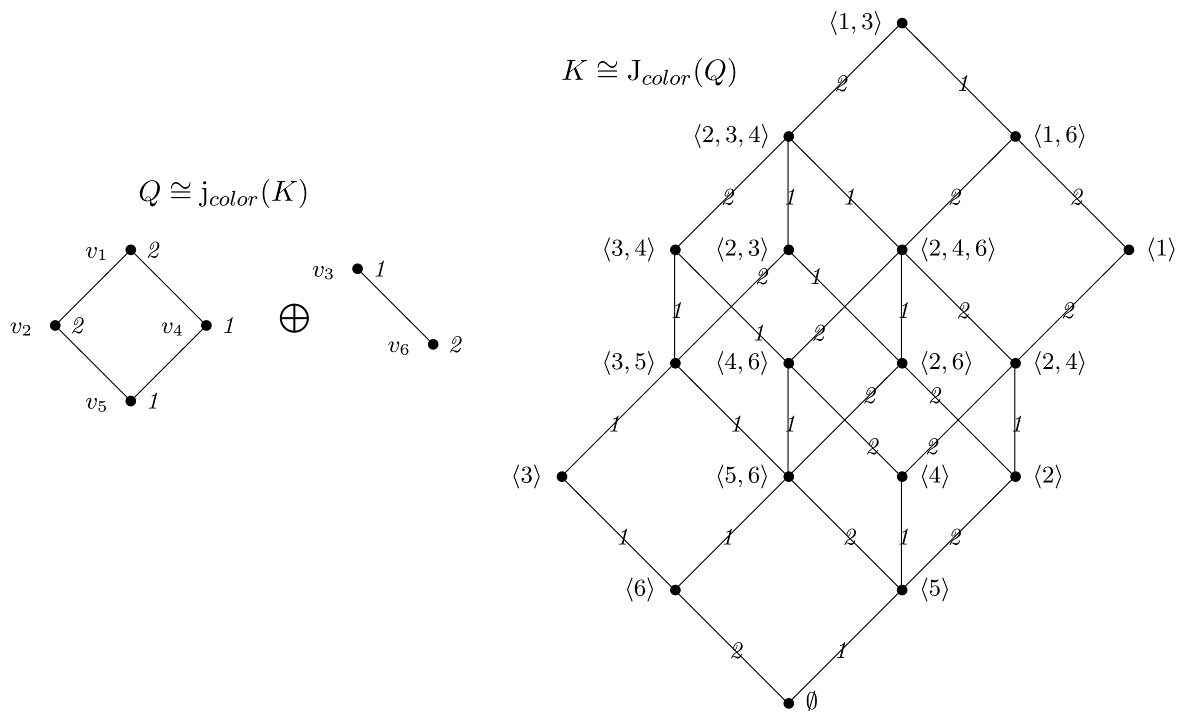
We claim that ψ is a bijection. Indeed, if $\psi(u) = \psi(v)$ for $u, v \in P$, then $\langle u \rangle = \langle v \rangle$. But then $u \leq_P v$ and $v \leq_P u$. Therefore $u = v$, and hence ψ is injective. On the other hand, if $\mathcal{I}$ is an order ideal from P that is join irreducible in L, then $\mathcal{I}$ must have a unique maximal element, say v. But then $\mathcal{I} = \langle v \rangle = \psi(v)$, so ψ is injective.

Finally we show ψ preserves edges and vertex colors. If $u \rightarrow v$ in P, then $\langle u \rangle <_Q \langle v \rangle$. Now if $\langle u \rangle \leq_Q \langle z \rangle \leq_Q \langle v \rangle$, it follows that $u \leq_P z \leq_P v$. Since v covers u in P, then $u = z$ or $z = v$, and hence $\langle u \rangle = \langle z \rangle$ or $\langle z \rangle = \langle v \rangle$. That is, $u \rightarrow v$ in P implies that $\psi(u) \rightarrow \psi(v)$ in Q. Conversely, if $\langle u \rangle \rightarrow \langle v \rangle$ in Q, then we must have $u <_P v$ in P. Suppose $u \leq_P z \leq_P v$. Then one easily sees that $\langle u \rangle \leq_Q \langle z \rangle \leq_P \langle v \rangle$, and hence $\langle u \rangle = \langle z \rangle$ or $\langle z \rangle = \langle v \rangle$. Then $u = z$ or $z = v$. That is, $\psi(u) \rightarrow \psi(v)$ in Q implies that $u \rightarrow v$ in P. As for vertex colors, observe that $\mathbf{vertexcolor}_P(v) = i$ if and only if $\langle v \rangle \setminus \{v\} \xrightarrow{i} \langle v \rangle$ in L if and only if $\mathbf{vertexcolor}_Q(\psi(v)) = i$. This completes the proof. $\square$

As a consequence, we note that a necessary and sufficient condition for an edge-colored distributive lattice L to be isomorphic (as an edge-colored poset) to $\mathrm{J}_{color}(P)$ or $\mathrm{M}_{color}(P)$ for some vertex-colored poset P is for L to have the diamond coloring property. We will often refer to P simply as the *poset of irreducibles*.

The details justifying the next result are routine.

Figure 2.8: An illustration of the principles that $\mathrm{J}_{color}(P_1 \oplus P_2) \cong \mathrm{J}_{color}(P_1) \times \mathrm{J}_{color}(P_2)$ and $\mathrm{j}_{color}(L_1 \times L_2) \cong \mathrm{j}_{color}(L_1) \oplus \mathrm{j}_{color}(L_2)$, cf. Proposition 2.4.
(As in Figure 2.7, here each order ideal from Q is identified by the indices of its maximal vertices. A join irreducible in K is an order ideal $\langle k \rangle$ whose only maximal element is v_k.)



Proposition 2.4 *Let P and Q be posets with vertices colored by a set I, and let K and L be diamond-colored distributive lattices with edges colored by I. In what follows, $*$, σ, $\oplus$, $\times$, and $\cong$ account for colors on vertices or edges as appropriate. (1) Then $\mathrm{J}_{color}(P^*) \cong (\mathrm{J}_{color}(P))^*$, $\mathrm{J}_{color}(P^\sigma) \cong (\mathrm{J}_{color}(P))^\sigma$ (recoloring), and $\mathrm{J}_{color}(P \oplus Q) \cong \mathrm{J}_{color}(P) \times \mathrm{J}_{color}(Q)$. (2) Also, $\mathrm{j}_{color}(L^*) \cong (\mathrm{j}_{color}(L))^*$, $\mathrm{j}_{color}(L^\sigma) \cong (\mathrm{j}_{color}(L))^\sigma$, and $\mathrm{j}_{color}(L \times K) \cong \mathrm{j}_{color}(L) \oplus \mathrm{j}_{color}(K)$. (3) Further, $\mathrm{M}_{color}(P^*) \cong (\mathrm{M}_{color}(P))^*$, $\mathrm{M}_{color}(P^\sigma) \cong (\mathrm{M}_{color}(P))^\sigma$, and $\mathrm{M}_{color}(P \oplus Q) \cong \mathrm{M}_{color}(P) \times \mathrm{M}_{color}(Q)$. (4) In addition it is the case that $\mathrm{m}_{color}(L^*) \cong (\mathrm{m}_{color}(L))^*$, $\mathrm{m}_{color}(L^\sigma) \cong (\mathrm{m}_{color}(L))^\sigma$, and $\mathrm{m}_{color}(L \times K) \cong \mathrm{m}_{color}(L) \oplus \mathrm{m}_{color}(K)$. (5) If $K \cong L$, then $\mathrm{j}_{color}(K) \cong \mathrm{m}_{color}(L)$. If $P \cong Q$, then $\mathrm{J}_{color}(P) \cong \mathrm{M}_{color}(Q)$.*

§2.5 Sublattices. Let L be a lattice with partial ordering $\leq_L$ and meet and join operations $\wedge_L$ and $\vee_L$ respectively. Let K be a vertex subset of L. Suppose that K has a lattice partial ordering $\leq_K$ of its own with meet and join operations $\wedge_K$ and $\vee_K$ respectively. We say K is a *sublattice* of L if for all $\mathbf{x}$ and $\mathbf{y}$ in K we have $\mathbf{x} \wedge_K \mathbf{y} = \mathbf{x} \wedge_L \mathbf{y}$ and $\mathbf{x} \vee_K \mathbf{y} = \mathbf{x} \vee_L \mathbf{y}$. It is easy to see that if K is a sublattice of L then for all $\mathbf{x}$ and $\mathbf{y}$ in K we have $\mathbf{x} \leq_K \mathbf{y}$ if and only if $\mathbf{x} \leq_L \mathbf{y}$. That is, K is a weak subposet of L and a subposet in the induced order.

Lemma 2.5 *Suppose K is a sublattice of L. Suppose K and L are ranked with rank functions $\rho^{(K)}$ and $\rho^{(L)}$ respectively. Suppose K and L have the same length. Then $\rho^{(K)}(\mathbf{x}) = \rho^{(L)}(\mathbf{x})$ for all $\mathbf{x}$ in K, and moreover for all $\mathbf{x}$ and $\mathbf{y}$ in K we have $\mathbf{x} \rightarrow \mathbf{y}$ in K if and only if $\mathbf{x} \rightarrow \mathbf{y}$ in L.*

Proof. Let l denote the common length of the ranked posets K and L. Take a chain in K $\mathbf{min}(K) = \mathbf{x}_0 \rightarrow \mathbf{x}_1 \rightarrow \cdots \rightarrow \mathbf{x}_l = \mathbf{max}(K)$ of longest length. Then, $\mathbf{x}_0 <_L \mathbf{x}_1 <_L \cdots <_L \mathbf{x}_l$, so $\rho^{(L)}(\mathbf{x}_l) \geq l + \rho^{(L)}(\mathbf{x}_0)$. Since L has length l, this must mean that $\rho^{(L)}(\mathbf{x}_l) = l$ and $\rho^{(L)}(\mathbf{x}_0) = 0$. So $\mathbf{x} = \mathbf{min}(L)$ and $\mathbf{x}_l = \mathbf{max}(L)$.

Now take any $\mathbf{x}$ in K. Then $\mathbf{x} = \mathbf{x}_r$ in some longest chain $\mathbf{x}_0 \rightarrow \mathbf{x}_1 \rightarrow \cdots \rightarrow \mathbf{x}_l$ in K. Now $(\rho^{(K)}(\mathbf{x}_0), \rho^{(K)}(\mathbf{x}_1), \ldots, \rho^{(K)}(\mathbf{x}_l) = (0, 1, \ldots, l)$. Since $(\rho^{(L)}(\mathbf{x}_0), \rho^{(L)}(\mathbf{x}_1), \ldots, \rho^{(L)}(\mathbf{x}_l)$ is an increasing sequence of integers with $\rho^{(L)}(\mathbf{x}_0)$ and $\rho^{(L)}(\mathbf{x}_l) = l$, then $(\rho^{(L)}(\mathbf{x}_0), \rho^{(L)}(\mathbf{x}_1), \ldots, \rho^{(L)}(\mathbf{x}_l) = (0, 1, \ldots, l)$ also. Hence $\rho^{(K)}(\mathbf{x}) = \rho^{(K)}(\mathbf{x}_r) = \rho^{(L)}(\mathbf{x}_r) = \rho^{(L)}(\mathbf{x})$.

Finally, let $\mathbf{x}$ and $\mathbf{y}$ be elements of K. Assume $\mathbf{x} \rightarrow \mathbf{y}$ in K. Then $x <_K \mathbf{y}$ and $\rho^{(K)}(\mathbf{x}) + 1 = \rho^{(K)}(\mathbf{y})$. So $x <_L \mathbf{y}$ in L and $\rho^{(L)}(\mathbf{x}) + 1 = \rho^{(L)}(\mathbf{y})$. Hence $\mathbf{x} \rightarrow \mathbf{y}$ is a covering relation in L as well. Clearly this argument reverses to show that if $\mathbf{x} \rightarrow \mathbf{y}$ in L then $\mathbf{x} \rightarrow \mathbf{y}$ in K. $\square$

When K satisfies the hypotheses of Lemma 2.5, we say K is a *full length* sublattice of L. Suppose L is an edge-colored lattice. Suppose K is a sublattice of L such that $\mathbf{x} \rightarrow \mathbf{y}$ in L whenever $\mathbf{x} \rightarrow \mathbf{y}$ in K. If K is also edge-colored and if the colors on edges of K match the colors when we view these as edges in L, then we say K is an *edge-colored sublattice* of L. The previous lemma gives us one way to know whether the edges of a sublattice are also edges of the 'parent' lattice. We now turn our attention to the special case of diamond-colored distributive lattices.

Theorem 2.6 *(1) Let P and Q be vertex-colored posets with vertices colored by a set I. Suppose that for each $i \in I$, the vertices of color i in P coincide with the vertices of color i in Q (so in particular $P = Q$ as vertex sets). Further suppose that P is a weak subposet of Q. Let $K := \mathrm{J}_{color}(Q)$ and $L := \mathrm{J}_{color}(P)$. Then K is a full-length edge-colored sublattice of L. (2) Conversely, suppose L is a diamond-colored distributive lattice with edges colored by a set I. Suppose K is a full-length edge-colored sublattice of L (so K is necessarily a diamond-colored distributive lattice). Let $P := \mathrm{j}_{color}(L)$ and $Q := \mathrm{j}_{color}(K)$. Then $P \cong P'$ (an isomorphism of vertex-colored posets) where P' is weak-subposet of Q, $P' = Q$ as vertex sets, and the color of a vertex in P' is the same as its color when viewed as a vertex in Q.*

Proof. The proof of (1) is easy. Let $\mathbf{x}$ be an order ideal from Q. It follows from the definitions that $\mathbf{x}$ is also an order ideal from P. So we get an inclusion $K = \mathrm{J}_{color}(Q) \subseteq \mathrm{J}_{color}(P) = L$. The length of K (resp. L) is the cardinality of Q (resp. P), and since $Q = P$ as vertex sets then K and L have the same length. Finally, note that for order ideals $\mathbf{x}$ and $\mathbf{y}$ from Q, $\mathbf{x} \vee_K \mathbf{y} = \mathbf{x} \cup \mathbf{y} = \mathbf{x} \vee_L \mathbf{y}$ and $\mathbf{x} \wedge_K \mathbf{y} = \mathbf{x} \cap \mathbf{y} = \mathbf{x} \wedge_L \mathbf{y}$.

For the proof of (2), we begin by choosing a join irreducible $\mathbf{x}$ in L. Let $\mathcal{F}_{\mathbf{x}} := \{\mathbf{y} \in K \,|\, \mathbf{x} \leq_L \mathbf{y}\}$. We claim that $\mathcal{F}_{\mathbf{x}}$ is a filter in K with a unique minimal element. First, if $\mathbf{y} \in \mathcal{F}_{\mathbf{x}}$ and $\mathbf{y} \leq_K \mathbf{y}'$ for some $\mathbf{y}' \in K$, then $\mathbf{y} \leq_L \mathbf{y}'$, and by transitivity of the partial order on L it follows that $\mathbf{x} \leq_L \mathbf{y}'$. Hence $\mathbf{y}' \in \mathcal{F}_{\mathbf{x}}$. This shows that $\mathcal{F}_{\mathbf{x}}$ is a filter in K. Second, if $\mathbf{y}$ and $\mathbf{y}'$ are both minimal elements of $\mathcal{F}_{\mathbf{x}}$, then whenever $\mathbf{x} \leq_L \mathbf{y}$ and $\mathbf{x} \leq_L \mathbf{y}'$ we will have $\mathbf{x} \leq_L (\mathbf{y} \wedge_L \mathbf{y}')$ and so $\mathbf{x} \leq_L (\mathbf{y} \wedge_K \mathbf{y}')$. Hence $(\mathbf{y} \wedge_K \mathbf{y}') \in \mathcal{F}_{\mathbf{x}}$. Now $(\mathbf{y} \wedge_K \mathbf{y}') \leq_K \mathbf{y}$ and $(\mathbf{y} \wedge_K \mathbf{y}') \leq_K \mathbf{y}'$. But $\mathbf{y}$ and $\mathbf{y}'$ are minimal elements of $\mathcal{F}_{\mathbf{x}}$. So $(\mathbf{y} \wedge_K \mathbf{y}') = \mathbf{y}$ and $(\mathbf{y} \wedge_K \mathbf{y}') = \mathbf{y}'$, i.e. $\mathbf{y} = \mathbf{y}'$. So $\mathcal{F}_{\mathbf{x}}$ has a unique minimal element.

Let $\mathbf{z}$ be the unique minimal element of $\mathcal{F}_{\mathbf{x}}$, let $\mathcal{D}_K(\mathbf{z}) \subset K$ be the set of descendants of $\mathbf{z}$ in K, and let $\mathbf{y}$ be the unique descendant of $\mathbf{x}$ in L. We claim that for any $\mathbf{z}' \in \mathcal{D}_K(\mathbf{z})$ we have $\mathbf{x} \vee_L \mathbf{z}' = \mathbf{z}$ and $\mathbf{x} \wedge_L \mathbf{z}' = \mathbf{y}$. To see this, note that when $\mathbf{z}' \to \mathbf{z}$ in K, we cannot have $\mathbf{x} \leq_L \mathbf{z}'$ or else $\mathbf{z}$ will not be minimal in $\mathcal{F}_{\mathbf{x}}$. So we cannot have $\mathbf{z}' = \mathbf{x} \vee_L \mathbf{z}'$. Therefore, $\mathbf{z}' <_L \mathbf{x} \vee_L \mathbf{z}'$. Then $\rho^{(L)}(\mathbf{x} \vee_L \mathbf{z}') \geq \rho^{(L)}(\mathbf{z})$. But since $\mathbf{x} \leq_L \mathbf{z}$ and $\mathbf{z}' <_L \mathbf{z}$, we have $\rho^{(L)}(\mathbf{x} \vee_L \mathbf{z}') \leq \rho^{(L)}(\mathbf{z})$. Hence $\rho^{(L)}(\mathbf{x} \vee_L \mathbf{z}') = \rho^{(L)}(\mathbf{z})$. It now follows that $\mathbf{z} = \mathbf{x} \vee_L \mathbf{z}'$. Next, since $\mathbf{x} \not\leq_L \mathbf{z}'$, then $\mathbf{x} \wedge_L \mathbf{z}' <_L \mathbf{x}$. But $\rho^{(L)}(\mathbf{x} \wedge_L \mathbf{z}') = \rho^{(L)}(\mathbf{x}) + \rho^{(L)}(\mathbf{z}') - \rho^{(L)}(\mathbf{x} \wedge_L \mathbf{z}') = \rho^{(L)}(\mathbf{x}) + \rho^{(L)}(\mathbf{z}) - 1 - \rho^{(L)}(\mathbf{x} \wedge_L \mathbf{z}') = \rho^{(L)}(\mathbf{x}) - 1$. Thus $\mathbf{x} \wedge_L \mathbf{z}' \to \mathbf{x}$. But since $\mathbf{y}$ is the only element of L covered by $\mathbf{x}$, then $\mathbf{x} \wedge_L \mathbf{z}' = \mathbf{y}$.

Next we claim that $\mathbf{z}$ has exactly one descendant in K, i.e. $|\mathcal{D}_K(\mathbf{z})| = 1$. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{D}_K(\mathbf{z})$. Let $\mathbf{z}' := \mathbf{z}_1 \wedge_K \mathbf{z}_2$. We will show that $\mathbf{z}' \vee_L \mathbf{x} = \mathbf{z}$ and $\mathbf{z}' \wedge_L \mathbf{x} = \mathbf{y}$. Since $\mathbf{y} \leq_L \mathbf{z}_i$ $(i = 1, 2)$ by the previous paragraph, then $\mathbf{y} \leq_L \mathbf{z}_1 \wedge_L \mathbf{z}_2 = \mathbf{z}_1 \wedge_K \mathbf{z}_2 = \mathbf{z}'$. Since we also have $\mathbf{y} \leq_L \mathbf{x}$, then $\mathbf{y} \leq_L \mathbf{z}' \wedge_L \mathbf{x}$. Since $\mathbf{z}' \wedge_L \mathbf{x} \leq_L \mathbf{x}$ and $\mathbf{y} \to \mathbf{x}$, the only way to have $\mathbf{y} <_L \mathbf{z}' \wedge_L \mathbf{x}$ is if $\mathbf{x} = \mathbf{z}' \wedge_L \mathbf{x}$. But then we would have $\mathbf{x} \leq_L \mathbf{z}'$, which would mean $\mathbf{z}' \in \mathcal{F}_{\mathbf{x}}$. Then $\mathbf{z} \leq_K \mathbf{z}'$ by the minimality of $\mathbf{z}$. This contradicts the fact that $\mathbf{z}' \leq_K \mathbf{z}_1 <_K \mathbf{z}$. So $\mathbf{y} = \mathbf{z}' \wedge_L \mathbf{x}$. Next, using a result from the previous paragraph we see that $\mathbf{z}' \vee_L \mathbf{x} = (\mathbf{z}_1 \wedge_K \mathbf{z}_2) \vee_L \mathbf{x} = (\mathbf{z}_1 \wedge_L \mathbf{z}_2) \vee_L \mathbf{x} = (\mathbf{z}_1 \vee_L \mathbf{x}) \wedge_L (\mathbf{z}_2 \vee_L \mathbf{x}) = \mathbf{z} \wedge_L \mathbf{z} = \mathbf{z}$. Now, $\rho^{(L)}(\mathbf{z}') + \rho^{(L)}(\mathbf{x}) = \rho^{(L)}(\mathbf{z}) + \rho^{(L)}(\mathbf{y})$. Since $\rho^{(L)}(\mathbf{y}) = \rho^{(L)}(\mathbf{x}) - 1$, we have $\rho(\mathbf{z}') = \rho^{(L)}(\mathbf{z}) - 1$. Hence $\rho^{(L)}(\mathbf{z}') = \rho^{(L)}(\mathbf{z}_i)$ for $i = 1, 2$. This can only happen if $\mathbf{z}_1 = \mathbf{z}' = \mathbf{z}_1 \wedge_K \mathbf{z}_2 = \mathbf{z}_2$. Hence $\mathbf{z}_1 = \mathbf{z}_2$. Next we argue that $\mathcal{D}_K(\mathbf{z})$ is nonempty. We have that $\mathbf{x} \leq_L \mathbf{z}$ and (since $\mathbf{x}$ is join irreducible) $\rho^{(L)}(\mathbf{x}) > 0$. Therefore $\rho^{(K)}(\mathbf{z}) > 0$, so $\mathbf{z}$ is not the unique minimal element of K. In particular, $\mathcal{D}_K(\mathbf{z})$ is nonempty. So $\mathbf{z}$ is join irreducible in K.

With P and Q as in the theorem statement, we define a function $\phi : P \to Q$ by $\phi(\mathbf{x}) = \mathbf{z}$, where $\mathbf{x}$ and $\mathbf{z}$ are as in the preceding paragraphs. Next we show that ϕ is surjective. Let $\mathbf{z} \in L$ be any join irreducible in K. Suppose $\mathbf{z}$ is also join irreducible in L. It follows that $\mathbf{z}$ is the unique minimal element of $\mathcal{F}_{\mathbf{z}}$. That is, $\mathbf{z} = \phi(\mathbf{z})$.

So now suppose $\mathbf{z}$ is not join irreducible in L. Let $\mathbf{z}'$ be the unique element of K such that $\mathbf{z}' \to \mathbf{z}$. Define a set $\mathcal{S}_{\mathbf{z}} := \{\mathbf{y} \in L \setminus K \,|\, \mathbf{y} \vee_L \mathbf{z}' = \mathbf{z}\}$. Since $\mathbf{z}$ is not join irreducible in L, it follows that $\mathcal{S}_{\mathbf{z}}$ is nonempty. We claim $\mathcal{S}_{\mathbf{z}}$ has a unique minimal element. Indeed, suppose $\mathbf{y}$ and $\mathbf{y}'$ are minimal elements in $\mathcal{S}_{\mathbf{z}}$. Then $(\mathbf{y} \wedge_L \mathbf{y}') \vee_L \mathbf{z}' = (\mathbf{y} \vee_L \mathbf{z}') \wedge_L (\mathbf{y}' \vee_L \mathbf{z}') = \mathbf{z} \wedge_L \mathbf{z} = \mathbf{z}$. Since $\mathbf{y} <_L \mathbf{z}$ and $\mathbf{y}' <_L \mathbf{z}$, then $\mathbf{y} \wedge_L \mathbf{y}' <_L \mathbf{z}$. If $(\mathbf{y} \wedge_L \mathbf{y}') \in K$, then $(\mathbf{y} \wedge_L \mathbf{y}') <_K \mathbf{z}$. Then it must be the case that $(\mathbf{y} \wedge_L \mathbf{y}') \leq_K \mathbf{z}'$ since any path from $\mathbf{y} \wedge_L \mathbf{y}'$ up to $\mathbf{z}$ and that stays in K must pass through $\mathbf{z}'$. But then we would have $(\mathbf{y} \wedge_L \mathbf{y}') \vee_L \mathbf{z}' = \mathbf{z}'$ instead of $(\mathbf{y} \wedge_L \mathbf{y}') \vee_L \mathbf{z}' = \mathbf{z}$. Then $(\mathbf{y} \wedge_L \mathbf{y}')$ is in $L \setminus K$ and hence in $\mathcal{S}_{\mathbf{z}}$. Minimality of $\mathbf{y}$ and $\mathbf{y}'$ in $\mathcal{S}_{\mathbf{z}}$ then forces us to have $\mathbf{y} = (\mathbf{y} \wedge_L \mathbf{y}') = \mathbf{y}'$. Let $\mathbf{x}$ denote the unique minimal element of $\mathcal{S}_{\mathbf{z}}$.

We have two claims: $\mathbf{x}$ is join irreducible in L, and $\mathbf{z}$ is the unique minimal element of $\mathcal{F}_{\mathbf{x}}$. Let $\mathbf{x}' := \mathbf{x} \wedge_L \mathbf{z}'$. Since $\rho(\mathbf{x}') = \rho(\mathbf{x}) + \rho(\mathbf{z}') - \rho(\mathbf{z}) = \rho(\mathbf{x}) - 1$, then $\mathbf{x}' \to \mathbf{x}$. Suppose $\mathbf{x}'' \to \mathbf{x}$ for some $\mathbf{x}'' \neq \mathbf{x}'$. It cannot be the case that $\mathbf{x}'' \leq_L \mathbf{z}'$, because otherwise $\mathbf{x}' \leq_L \mathbf{z}'$ and $\mathbf{x}'' \leq_L \mathbf{z}'$ means that $\mathbf{x} = (\mathbf{x}' \vee_L \mathbf{x}'') \leq_L \mathbf{z}'$, a contradiction. Further, we have that $\mathbf{x}'' \in K$. Otherwise we would have $\mathbf{x}'' \in L \setminus K$, and since $\mathbf{x}'' \not\leq_L \mathbf{z}'$ then $(\mathbf{x}'' \vee_L \mathbf{z}') = \mathbf{z}$. But then $\mathbf{x}''$ would be in $\mathcal{S}_{\mathbf{z}}$, violating minimality of $\mathbf{x}$. So $\mathbf{x}'' \in K$ and $\mathbf{x}'' \not\leq_L \mathbf{z}'$. Then there is a path from $\mathbf{x}''$ up to $\mathbf{z}$ that stays in K. But since $\mathbf{z}$ is join irreducible in K, then such a path must pass through $\mathbf{z}'$, implying that $\mathbf{x}'' \leq_K \mathbf{z}'$. But then $\mathbf{x}'' \leq_L \mathbf{z}'$, a contradiction. Therefore $\mathbf{x}'$ can be the only descendant of $\mathbf{x}$, hence $\mathbf{x}$ is join irreducible in L. Now if $\mathbf{w} \in \mathcal{F}_{\mathbf{x}}$, then from the facts that $\mathbf{x} <_L \mathbf{w}$ and $\mathbf{x} <_L \mathbf{z}$ we get $\mathbf{x} \leq_L (\mathbf{w} \wedge_L \mathbf{z})$. Since $(\mathbf{w} \wedge_L \mathbf{z}) = (\mathbf{w} \wedge_K \mathbf{z})$, then $(\mathbf{w} \wedge_L \mathbf{z}) \in K$, so we cannot have $\mathbf{x} = (\mathbf{w} \wedge_L \mathbf{z})$. Then $\mathbf{x} <_L (\mathbf{w} \wedge_L \mathbf{z})$. If $(\mathbf{w} \wedge_L \mathbf{z}) <_L \mathbf{z}$, then we would have $\mathbf{x} \leq_L \mathbf{z}'$, which is not the case. So $(\mathbf{w} \wedge_L \mathbf{z}) = \mathbf{z}$, and hence $\mathbf{z} \leq_L \mathbf{w}$. So $\mathbf{z}$ is the unique minimal element of $\mathcal{F}_{\mathbf{x}}$. That is, $\mathbf{z} = \phi(\mathbf{x})$.

Our work in the preceding paragraphs shows that any join irreducible in K is the image under ϕ of a join irreducible in L. That is, ϕ is surjective. Since $|P| = |Q|$ (K and L have the same length), then ϕ is therefore a bijection. Suppose that $\mathbf{z} = \phi(\mathbf{x}) \neq \mathbf{x}$ for some $\mathbf{x} \in P$ and $\mathbf{z} \in Q$. Let $\mathbf{x}'$ be the unique descendant of $\mathbf{x}$ in L, with $\mathbf{x}' \xrightarrow{i} \mathbf{x}$ for some color i. Let $\mathbf{z}'$ be the unique descendant of $\mathbf{z}$ in

K, with $\mathbf{z}' \xrightarrow{j} \mathbf{z}$ for some color j. Choose paths $\mathbf{x}' = \mathbf{r}_0 \xrightarrow{i=i_1} \mathbf{x} = \mathbf{r}_1 \xrightarrow{i_2} \mathbf{r}_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{p-1}} \mathbf{r}_{p-1} \xrightarrow{i_p} \mathbf{r}_p = \mathbf{z}$ and $\mathbf{x} = \mathbf{r}'_0 \xrightarrow{j_1} \mathbf{r}'_1 \xrightarrow{j_2} \mathbf{r}'_2 \xrightarrow{j_3} \cdots \xrightarrow{j_{p-1}} \mathbf{z}' = \mathbf{r}'_{p-1} \xrightarrow{j=j_p} \mathbf{r}'_p = \mathbf{z}$ from $\mathbf{x}'$ up to $\mathbf{z}$. One path goes through $\mathbf{x}$ and the other through $\mathbf{z}'$. In particular, Lemma 2.1 applies, so $i = i_1 = j_p = j$. Since $\mathbf{vertexcolor}_P(\mathbf{x}) = i = j$ and $\mathbf{vertexcolor}_Q(\mathbf{z}) = j = i$, it follows that ϕ preserves vertex colors.

To complete the proof of (2), we show that for $\mathbf{u}$ and $\mathbf{v}$ in P, $\mathbf{u} \leq_P \mathbf{v}$ implies that $\phi(\mathbf{u}) \leq_Q \phi(\mathbf{v})$. To see this, first note that $\mathbf{u}$ and $\mathbf{v}$ are join irreducible elements of L with $\mathbf{u} \leq_L \mathbf{v}$. Consider $\mathcal{F}_{\mathbf{u}}$ and $\mathcal{F}_{\mathbf{v}}$. If $\mathbf{w} \in \mathcal{F}_{\mathbf{v}}$, then $\mathbf{w} \in K$ and $\mathbf{v} \leq_L \mathbf{w}$. Then $\mathbf{u} \leq_L \mathbf{w}$ as well, so $\mathbf{w} \in \mathcal{F}_{\mathbf{u}}$. So $\mathcal{F}_{\mathbf{u}} \supseteq \mathcal{F}_{\mathbf{v}}$. Therefore $\phi(\mathbf{u}) \leq_L \phi(\mathbf{v})$. Since $\phi(\mathbf{u})$ and $\phi(\mathbf{v})$ are both in K, then we have $\phi(\mathbf{u}) \leq_K \phi(\mathbf{v})$. Viewing $\phi(\mathbf{u})$ and $\phi(\mathbf{v})$ as elements of Q, we then have $\phi(\mathbf{u}) \leq_Q \phi(\mathbf{v})$. $\square$

To set up our next result we require some further notation. For elements $\mathbf{s}, \mathbf{t}$ in any poset R, the *interval* $[\mathbf{s}, \mathbf{t}]$ is the set $\{\mathbf{x} \in R \,|\, \mathbf{s} \leq_R \mathbf{x} \leq_R \mathbf{t}\}$ with partial order induced by R. One can check that the Hasse diagram for $[\mathbf{s}, \mathbf{t}]$ is just the induced subgraph of R on the vertices of $[\mathbf{s}, \mathbf{t}]$. Then we can regard $[\mathbf{s}, \mathbf{t}]$ as an edge-colored subposet of R in the induced order, if R is edge-colored. In a diamond-colored modular lattice L, it is not hard to see that any interval $[\mathbf{s}, \mathbf{t}]$ is naturally an edge-colored sublattice of L. Our next result concerns the distributive lattice structure of certain intervals in diamond-colored distributive lattices.

Proposition 2.7 *Let L be a diamond-colored distributive lattice. Let $\mathbf{t} \in L$. Let D be a subset of the descendants of $\mathbf{t}$. For any $\mathbf{s} \in D$, let $\mathbf{vertexcolor}_D(\mathbf{s}) := \mathbf{edgecolor}_L(\mathbf{s} \to \mathbf{t})$. Let $\mathbf{r} := \wedge_{\mathbf{s}\,\in\,D}(\mathbf{s})$. Then $[\mathbf{x}, \mathbf{t}] \cong \mathrm{M}_{color}(D)$ and $D \subseteq [\mathbf{x}, \mathbf{t}]$ if and only if $\mathbf{x} = \mathbf{r}$. Similarly let A be a subset of the ancestors of $\mathbf{t}$. For any $\mathbf{s} \in A$, let $\mathbf{vertexcolor}_A(\mathbf{s}) := \mathbf{edgecolor}_L(\mathbf{t} \to \mathbf{s})$. Let $\mathbf{u} := \vee_{\mathbf{s}\,\in\,A}(\mathbf{s})$. Then $[\mathbf{t}, \mathbf{x}] \cong \mathrm{J}_{color}(A)$ and $A \subseteq [\mathbf{t}, \mathbf{x}]$ if and only if $\mathbf{x} = \mathbf{u}$.*

Proof. In this proof we only address the claim concerning the set D since the proof for the claim concerning A is entirely similar. In the notation of the proposition statement, suppose $[\mathbf{x}, \mathbf{t}] \cong \mathrm{M}_{color}(D)$ and $D \subseteq [\mathbf{x}, \mathbf{t}]$. Let $\phi : \mathrm{M}_{color}(D) \to [\mathbf{x}, \mathbf{t}]$ be the edge and edge-color preserving bijection. Since the unique maximal (resp. minimal) elements must correspond under the bijection ϕ, then $\phi(\emptyset) = \mathbf{t}$ (resp. $\phi(D) = \mathbf{x}$). For any $\mathbf{s} \in D$ we have $\{\mathbf{s}\} \to \emptyset$ in $\mathrm{M}_{color}(D)$. Then $\phi(\{\mathbf{s}\})$ must be covered by $\mathbf{t}$. So $\phi(D) \subseteq D$, and since ϕ is a bijection we have that $\phi(D) = D$. Then $\phi(D) = \phi(\cup_{\mathbf{s} \in D}(\mathbf{s})) = \wedge_{\mathbf{s} \in D}(\phi(\{\mathbf{s}\}))$, where the meet is computed in L. But since $\phi(D) = D$, then $\wedge_{\mathbf{s} \in D}(\phi(\{\mathbf{s}\})) = \wedge_{\mathbf{s} \in D}(\mathbf{s})$, which is just $\mathbf{r}$. That is, $\phi(D) = \mathbf{r}$. Then $\mathbf{x} = \mathbf{r}$.

For the converse, suppose that $\mathbf{x} = \mathbf{r}$. Now each $\mathbf{s} \in D$ is a descendant of $\mathbf{t}$, so $\mathbf{s} \leq_L \mathbf{t}$. By the definition of $\mathbf{r}$ we have $\mathbf{r} \leq_L \mathbf{s}$. So $\mathbf{s} \in [\mathbf{r}, \mathbf{t}]$. That is, $D \subseteq [\mathbf{r}, \mathbf{x}]$. Note that since all elements of D are pairwise incomparable, then the filters from D are just the subsets of D. For a subset S of D, it follows from the definitions that $\mathbf{r} \leq_L \wedge_{\mathbf{s}\in S}(\mathbf{s}) \leq_L \mathbf{t}$, so $\wedge_{\mathbf{s}\in S}(\mathbf{s}) \in [\mathbf{r},\mathbf{t}]$. Now define $\psi : \mathrm{M}_{color}(D) \to [\mathbf{r},\mathbf{t}]$ by the rule $\psi(S) := \wedge_{\mathbf{s}\in S}(\mathbf{s})$ for each subset $S \subseteq D$. Note that $\psi(\emptyset) = \mathbf{t}$ and $\psi(D) = \mathbf{r}$. We claim that if $S \xrightarrow{i} T$ in $\mathrm{M}_{color}(D)$ then $\psi(S) \xrightarrow{i} \psi(T)$ in $[\mathbf{r},\mathbf{t}]$. Now $S \xrightarrow{i} T$ in $\mathrm{M}_{color}(D)$ if and only if $|S| = |T| + 1$, $S = T \cup \{\mathbf{s}\}$ for some $\mathbf{s} \in D$, and $\mathbf{vertexcolor}_D(\mathbf{s}) = i$. To establish our claim we induct on the size of $|S|$. If $|S| = 1$ then $S = \{\mathbf{s}\}$ for some $\mathbf{s} \in D$ and $T = \emptyset$. Then $\psi(S) = \mathbf{s}$ and $\psi(T) = \mathbf{t}$. Clearly $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in this case. For our induction hypothesis we assume that $X \xrightarrow{i} Y$ in $\mathrm{M}_{color}(D)$ implies $\psi(X) \xrightarrow{i} \psi(Y)$ whenever X has no more than k elements, for some positive integer k. Now suppose $S \xrightarrow{i} T$ with $|S| = k + 1$. So $|S| = |T| + 1$, $S = T \cup \{\mathbf{s}\}$ for some $\mathbf{s} \in D$, and $\mathbf{vertexcolor}_D(\mathbf{s}) = i$. Let $Y := T \setminus \{\mathbf{u}\}$ for some $\mathbf{u} \in T$ with $\mathbf{vertexcolor}_D(\mathbf{u}) = j$, and let $X = S \setminus \{\mathbf{u}\}$. Then $Y = X \setminus \{\mathbf{s}\}$. Then $T \xrightarrow{j} Y$, $X \xrightarrow{i} Y$, and $S \xrightarrow{j} X$. Now by the induction hypothesis, $\psi(T) \xrightarrow{j} \psi(Y)$ and $\psi(X) \xrightarrow{i} \psi(Y)$. Then $\psi(Y) = (\psi(X) \vee \psi(T))$. We claim that $\psi(S) = (\psi(X) \wedge \psi(T))$. Let $\mathbf{z} = \psi(Y)$. Then $\psi(X) = \mathbf{z} \wedge \mathbf{s}$, $\psi(T) = \mathbf{z} \wedge \mathbf{u}$, and $\psi(S) = \mathbf{z} \wedge (\mathbf{s} \wedge \mathbf{u})$. So $\psi(S) = \mathbf{z} \wedge (\mathbf{s} \wedge \mathbf{u}) = (\mathbf{z} \wedge \mathbf{z}) \wedge (\mathbf{s} \wedge \mathbf{u}) = (\mathbf{z} \wedge \mathbf{s}) \wedge (\mathbf{z} \wedge \mathbf{u}) = (\psi(X) \wedge \psi(T))$. Our diamond-colored distributive lattice L can have no open vees, and since $\psi(T) \xrightarrow{j} \psi(Y)$ and $\psi(X) \xrightarrow{i} \psi(Y)$ we must therefore have $\psi(S) \xrightarrow{j} \psi(X)$ and $\psi(S) \xrightarrow{i} \psi(T)$. This completes the induction step, and the proof of our claim.

Let $d = |D|$. Let $D = S^{(0)} \to S^{(1)} \to S^{(2)} \to \cdots \to S^{(d-1)} \to S^{(d)} = \emptyset$ be a chain of maximal length in $\mathrm{M}_{color}(D)$. Then $\mathbf{r} = \psi(D) \to \psi(S^{(1)}) \to \cdots \to \psi(S^{(d-1)}) \to \psi(S^{(d)}) = \mathbf{t}$, a chain of maximal length in $[\mathbf{r},\mathbf{t}]$. In particular, the length of $[\mathbf{r},\mathbf{t}]$ is d. In the paragraph preceding the proposition it was noted that intervals in diamond-colored modular lattices are edge-colored sublattices. We now invoke the distributivity hypothesis for L: in this setting we have that $[\mathbf{r},\mathbf{t}]$ is a diamond-colored distributive lattice. Since $[\mathbf{r},\mathbf{t}]$ has length d as a ranked poset, it follows that $[\mathbf{r},\mathbf{t}]$ must have precisely d meet irreducibles. But each $\mathbf{s} \in D$ is meet irreducible in $[\mathbf{r},\mathbf{t}]$, so the set D must account for all meet irreducibles in $[\mathbf{r},\mathbf{t}]$. Therefore, $[\mathbf{r},\mathbf{t}] \cong \mathrm{M}_{color}(D)$ by Theorem 2.3. □

Note that any two descendants (respectively ancestors) of a given element of a poset are incomparable. It follows then that the intervals $[\mathbf{r},\mathbf{t}]$ and $[\mathbf{t},\mathbf{u}]$ of Proposition 2.7 are Boolean lattices, cf. Example 2.2.

The next result concerns the structure of J-components of a diamond-colored modular lattice.

Proposition 2.8 *Let L be a diamond-colored modular lattice with edge colors from a set I. If $\mathbf{t} \in L$ and $J \subseteq I$, then $\mathbf{comp}_J(\mathbf{t})$ is the Hasse diagram for a diamond-colored modular lattice.*

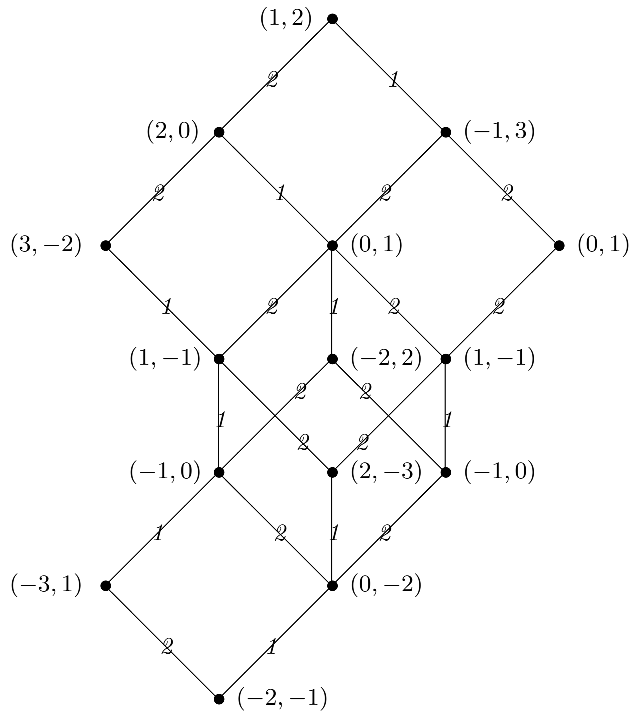
Moreover, $\mathbf{comp}_J(\mathbf{t})$ *is a sublattice of* L*, and a covering relation in* $\mathbf{comp}_J(\mathbf{t})$ *is also a covering relation in* L*. If* L *is a distributive lattice, then so is* $\mathbf{comp}_J(\mathbf{t})$*.*

Proof. Let $K := \mathbf{comp}_J(\mathbf{t})$. Then K is a poset with partial order $\leq_K$ given as follows: For $\mathbf{x}$ and $\mathbf{y}$ in K, $\mathbf{x} \leq_K \mathbf{y}$ if and only if there is a set $\{\mathbf{s}_q \in K \,|\, 0 \leq q \leq p\}$ for which $\mathbf{x} = \mathbf{s}_0 \xrightarrow{i_1} \mathbf{s}_1 \xrightarrow{i_2} \mathbf{s}_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{p-1}} \mathbf{s}_{p-1} \xrightarrow{i_p} \mathbf{s}_p = \mathbf{y}$ is a sequence of edges in L with each $i_q \in J$. To see that this is a partial order, observe that $\mathbf{x} \leq_K \mathbf{y}$ implies that $\mathbf{x} \leq_L \mathbf{y}$. It is also easy to see that the edges $\mathbf{p} \xrightarrow{j} \mathbf{q}$ in K are precisely the covering relations for this partial order.

To complete the proof, it suffices to show the following: $\mathbf{x} \vee_L \mathbf{y}$ and $\mathbf{x} \wedge_L \mathbf{y}$ are in K whenever $\mathbf{x}, \mathbf{y} \in K$. We actually make the stronger claim that any shortest path from $\mathbf{x}$ to $\mathbf{y}$ in K is also a shortest path in L, and moreover $\mathbf{x} \vee_L \mathbf{y} \in K$ and $\mathbf{x} \wedge_L \mathbf{y} \in K$. Suppose $\mathbf{dist}_K(\mathbf{x}, \mathbf{y}) = p$. By exchanging a 'valley' for a 'peak', then any shortest path in K from $\mathbf{x}$ to $\mathbf{y}$ can be modified to be 'single-peaked' and to use the same multiset of edge colors as used in the original shortest path. In particular, the resulting single-peaked path will be in K. So we may assume we have a shortest path $\mathbf{x} = \mathbf{r}_0 \xrightarrow{i_1} \mathbf{r}_1 \xrightarrow{i_2} \cdots \xrightarrow{i_q} \mathbf{r}_q \xleftarrow{i_{q+1}} \mathbf{r}_{q+1} \xleftarrow{i_{q+2}} \cdots \xleftarrow{i_p} \mathbf{r}_p = \mathbf{y}$ from $\mathbf{x}$ to $\mathbf{y}$ in K. Clearly, then, we have $\mathbf{x} \vee_L \mathbf{y} \leq_L \mathbf{r}_q$. So we can find a path from $\mathbf{x}$ up to $\mathbf{r}_q$ that goes through $\mathbf{x} \vee_L \mathbf{y}$. By Lemma 2.1, it follows that this path will only use edges with colors from the set J. In other words, we get a path from $\mathbf{x}$ up to $\mathbf{x} \vee_L \mathbf{y}$ that stays in K. Similarly argue that there is a path from $\mathbf{y}$ up to $\mathbf{x} \vee_L \mathbf{y}$ that stays in K. Putting these two together we have a path from $\mathbf{x}$ to $\mathbf{y}$ that has length no more than p. If $\mathbf{x} \vee_L \mathbf{y} <_L \mathbf{r}_q$, then we will have a path in K shorter than our given shortest path, a contradiction. Therefore $\mathbf{x} \vee_L \mathbf{y} = \mathbf{r}_q \in K$. It follows that the shortest path in K from $\mathbf{x}$ to $\mathbf{y}$ given originally is also shortest in L, since we have $\mathbf{dist}_L(\mathbf{x}, \mathbf{y}) = [\rho(\mathbf{x} \vee_L \mathbf{y}) - \rho(\mathbf{x})] + [\rho(\mathbf{x} \vee_L \mathbf{y}) - \rho(\mathbf{y})] = \mathbf{dist}_K(\mathbf{x}, \mathbf{y})$. A similar argument shows that $\mathbf{x} \wedge_L \mathbf{y}$ is also in K, thus completing the proof. □

§2.6 A first look at the M-structure property. Let R be a ranked poset whose Hasse diagram edges are colored with colors taken from a totally ordered set I_n of cardinality n. For $i \in I_n$ and $\mathbf{s}$ in R, set $m_i(\mathbf{s}) := \rho_i(\mathbf{s}) - \delta_i(\mathbf{s}) = 2\rho_i(\mathbf{s}) - l_i(\mathbf{s})$, where ρ_i, δ_i, and l_i are defined as in §2.3. Fix an n-dimensional real vector space V with basis $\{\omega_i\}_{i \in I_n}$. Define a mapping $wt_R : R \to V$ by the rule $wt_R(\mathbf{s}) := \sum_{i \in I_n} m_i(\mathbf{s})\omega_i$, and call this vector the *weight* of $\mathbf{s}$. Given a matrix $M = (M_{pq})_{p,q \in I_n}$, then for fixed $i \in I_n$ let $M^{(i)}$ be the "ith row" vector $\sum_{j \in I_n} M_{ij}\omega_j$. We say R has the *M-structure property* if $wt_R(\mathbf{s}) + M^{(i)} = wt_R(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{i} \mathbf{t}$ for some $i \in I_n$, that is, for all $j \in I_n$ we have $m_j(\mathbf{s}) + M_{ij} = m_j(\mathbf{t})$ if $\mathbf{s} \xrightarrow{i} \mathbf{t}$. We also say R is an *M-structured* poset. It can be easily shown that if the edge color function $\mathbf{edgecolor}_R : \mathcal{E}(R) \longrightarrow I_n$ is surjective, then the all of the M_{ij}'s

Figure 2.9: For each element $\mathbf{t}$ of the lattice L from Figure 2.1,
we compute $wt_L(\mathbf{t}) = (m_1(\mathbf{t}), m_2(\mathbf{t}))$.



are uniquely determined integers and that $M_{ii} = 2$ for all $i \in I_n$. One can check by hand that the edge-colored distributive lattice of Figure 2.9 has the M-structure property for the matrix $M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The following result shows how the M-structure property interacts with some of our usual poset operations.

Proposition 2.9 *Let Q and R be ranked posets with edges colored by a set I_n. Let $M = (M_{ij})_{i,j \in I_n}$ be a real matrix. Suppose Q and R have the M-structure property. (1) Then so do $Q \oplus R$, $Q \times R$, and R^*. Let $J \subseteq I_n$, and let M' be the submatrix $(M_{ij})_{i,j \in J}$ of M. Then for each $\mathbf{t} \in R$, the J-component $\mathbf{comp}_J(\mathbf{t})$ is a ranked poset with edges colored by J and with the M'-structure property. (2) Suppose now that M is nonsingular. Then for any nonnegative integer k, $\bigwedge^k(R)$ and $\mathbb{S}^k(R)$ have the M-structure property. Moreover, if R is connected and $wt_R(\mathbf{s}) = wt_R(\mathbf{t})$, then $\rho(\mathbf{s}) = \rho(\mathbf{t})$.*

Proof. For (1), that $Q \oplus R$ has the M-structure property is an easy consequence of the definitions. Now consider R^*. One can easily check that $m_j(\mathbf{x}^*) = -m_j(\mathbf{x})$ for all $j \in I_n$ and $\mathbf{x} \in R$. Suppose $\mathbf{t}^* \xrightarrow{i} \mathbf{s}^*$ is an edge in R^*. Then $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in R. So for any $j \in I_n$ we have $m_j(\mathbf{s}) + M_{ij} = m_j(\mathbf{t})$. Then $-m_j(\mathbf{s}^*) + M_{ij} = -m_j(\mathbf{t}^*)$, whence $m_j(\mathbf{s}^*) = m_j(\mathbf{t}^*) + M_{ij}$. So R^* has the M-structure property. Next consider $Q \times R$. Let $\rho_i^\times$, $\delta_i^\times$, and $m_i^\times$ be the relevant color i functions for the edge-colored ranked poset $Q \times R$. For $\mathbf{p} \in Q$ and $\mathbf{s} \in R$, one can easily check that the color i component $\mathbf{comp}_i(\mathbf{p}, \mathbf{s})$ of $(\mathbf{p}, \mathbf{s})$ in $Q \times R$ is edge-color isomorphic to $\mathbf{comp}_i(\mathbf{p}) \times \mathbf{comp}_i(\mathbf{s})$. Then $\rho_i^\times(\mathbf{p}, \mathbf{s}) = \rho_i^Q(\mathbf{p}) + \rho_i^R(\mathbf{s})$ and $\delta_i^\times(\mathbf{p}, \mathbf{s}) = \delta_i^Q(\mathbf{p}) + \delta_i^R(\mathbf{s})$. It follows that $m_i^\times(\mathbf{p}, \mathbf{s}) = m_i^Q(\mathbf{p}) + m_i^R(\mathbf{s})$. Now suppose $(\mathbf{p}, \mathbf{s}) \xrightarrow{i} (\mathbf{q}, \mathbf{t})$ in $Q \times R$. Then either $\mathbf{p} \xrightarrow{i} \mathbf{q}$ with $\mathbf{s} = \mathbf{t}$ or $\mathbf{s} \xrightarrow{i} \mathbf{t}$ with $\mathbf{p} = \mathbf{q}$. Without loss of generality, we assume that $\mathbf{p} \xrightarrow{i} \mathbf{q}$ with $\mathbf{s} = \mathbf{t}$. Then for any $j \in I_n$, $m_i^\times(\mathbf{p}, \mathbf{s}) + M_{ij} = m_i^Q(\mathbf{p}) + M_{ij} + m_i^R(\mathbf{s}) = m_i^Q(\mathbf{q}) + m_i^R(\mathbf{t}) = m_i^\times(\mathbf{q}, \mathbf{t})$. So $Q \times R$ has the M-structure property.

Let $\mathcal{C} := \mathbf{comp}_J(\mathbf{t})$ for some fixed $\mathbf{t} \in R$ and $J \subseteq I_n$. Let $\mathbf{x} \in \mathcal{C}$ be such that $\rho(\mathbf{x}) = \min\{\rho(\mathbf{z}) \,|\, \mathbf{z} \in \mathcal{C}\}$. Similarly let $\mathbf{y} \in \mathcal{C}$ be such that $\rho(\mathbf{y}) = \max\{\rho(\mathbf{z}) \,|\, \mathbf{z} \in \mathcal{C}\}$. Define $\rho' : \mathcal{C} \to \{0, \ldots, \rho(\mathbf{y}) - \rho(\mathbf{x})\}$ by the rule $\rho'(\mathbf{z}) := \rho(\mathbf{z}) - \rho(\mathbf{x})$. One can easily check now that ρ' is a rank function for $\mathcal{C}$. For $i \in J$ and $\mathbf{p} \in \mathcal{C}$, let $\rho_i'(\mathbf{p})$, $\delta_i'(\mathbf{p})$, and $m_i'(\mathbf{p})$ denote the respective color i functions for the edge-colored ranked poset $\mathcal{C}$. For any $\mathbf{p} \in \mathcal{C}$ and $i \in J$, we have all vertices and edges of $\mathbf{comp}_i(\mathbf{p})$ contained in $\mathcal{C}$. So it follows that $\rho_i'(\mathbf{p}) = \rho_i(\mathbf{p})$ and $\delta_i'(\mathbf{p}) = \delta_i(\mathbf{p})$. Then $m_i'(\mathbf{p}) = m_i(\mathbf{p})$, where $m_i(\mathbf{p})$ is calculated in R. Then for $i \in J$ and $\mathbf{p} \xrightarrow{i} \mathbf{q}$ in $\mathcal{C}$, it now follows that $m_j'(\mathbf{p}) + M_{ij} = m_j'(\mathbf{q})$ for all $j \in J$. Then $\mathcal{C}$ has the M'-structure property.

For (2), we suppose M is nonsingular. Suppose R is connected and $wt_R(\mathbf{s}) = wt_R(\mathbf{t})$. Since R is connected it is possible to find a path $\mathcal{P}$ from $\mathbf{s}$ to $\mathbf{t}$. For each $i \in I_n$, let

$$a_i := |\{\mathbf{p} \xrightarrow{i} \mathbf{q} \,|\, \mathbf{p} \text{ and } \mathbf{q} \text{ are successive elements in the path } \mathcal{P} \text{ with } \mathbf{p} \text{ before } \mathbf{q}\}|$$

and let

$$d_i := |\{\mathbf{p} \xrightarrow{i} \mathbf{q} \,|\, \mathbf{q} \text{ and } \mathbf{p} \text{ are successive elements in the path } \mathcal{P} \text{ with } \mathbf{q} \text{ before } \mathbf{p}\}|.$$

Think of a_i (resp. d_i) as counting 'ascents' (resp. 'descents') of color i in the path $\mathcal{P}$. Then

$$wt_R(\mathbf{s}) + \sum_{i \in I_n} (a_i - d_i) M^{(i)} = wt_R(\mathbf{t}),$$

which implies that $\sum_{i \in I_n} (a_i - d_i) M^{(i)} = 0$. Since M is nonsingular, then the $M^{(i)}$'s are linearly independent. Then $a_i = d_i$ for each $i \in I_n$. Since we add one to the rank of $\mathbf{s}$ for each ascent in $\mathcal{P}$ and subtract one for each descent as we move along $\mathcal{P}$ from $\mathbf{s}$ to $\mathbf{t}$, it follows that $\rho(\mathbf{s}) = \rho(\mathbf{t})$.

Keeping the hypothesis that M is nonsingular, we will show that $\bigwedge^k(R)$ has the M-structure property. (Since the argument that $\mathbb{S}^k(R)$ is M-structured is similar to the the argument for the kth exterior power, we omit the details of that proof.) We follow the notation of §2.3. For any $\mathbf{s} \in \bigwedge^k(R)$, define $\mu_i(\mathbf{s}) := \sum_{v_j \in \mathbf{s}} m_i(v_j)$, where $m_i(v_j) = \rho_i(v_j) - \delta_i(v_j)$ is calculated in R for each v_j. Then define $\mu(\mathbf{s}) := (\mu_i(\mathbf{s}))_{i\in I_n}$. First, note that if $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in $\bigwedge^k(R)$, then $(\mathbf{s}-\mathbf{t}, \mathbf{t}-\mathbf{s}) = (\{v_p\},\{v_q\})$ with $v_p \xrightarrow{i} v_q$ in R. Then $m_j(v_p) + M_{ij} = m_j(v_q)$ in R. It now follows that $\mu_j(\mathbf{s}) + M_{ij} = \left(\sum_{v_r\in\mathbf{s}} m_j(v_r)\right) + M_{ij} = m_j(v_p) + M_{ij} + \sum_{v_r\neq v_p} m_j(v_r) = m_j(v_q) + \sum_{v_r\neq v_p} m_j(v_r) = \sum_{v_r\in\mathbf{t}} m_j(v_r) = \mu_j(\mathbf{t})$. From this it follows that $\mu(\mathbf{s}) + M^{(i)} = \mu(\mathbf{t})$.

Now suppose $\mathbf{s} = \mathbf{r}_0 \xrightarrow{i_1} \mathbf{r}_1 \xrightarrow{i_2} \mathbf{r}_2 \xrightarrow{i_3} \cdots \xrightarrow{i_p} \mathbf{r}_p = \mathbf{s}$ in $\bigwedge^k(R)$. Then $\mu(\mathbf{s}) = \mu(\mathbf{s}) + \sum_{i\in I_n} a_i M^{(i)}$, where a_i counts the number of times there is an edge of color i in our given path from $\mathbf{s}$ to itself. So, $\sum_{i\in I_n} a_i M^{(i)} = 0$. Since M is nonsingular, then the $M^{(i)}$'s are linearly independent. So each $a_i = 0$. Hence $\bigwedge^k(R)$ is acyclic, so we may define a partial order on $\bigwedge^k(R)$ in the following way: $\mathbf{s} \leq \mathbf{t}$ if and only if there is an 'ascending' path $\mathbf{s} = \mathbf{r}_0 \xrightarrow{i_1} \mathbf{r}_1 \xrightarrow{i_2} \mathbf{r}_2 \xrightarrow{i_3} \cdots \xrightarrow{i_p} \mathbf{r}_p = \mathbf{t}$ from $\mathbf{s}$ to $\mathbf{t}$ in $\bigwedge^k(R)$. Suppose $\mathbf{s} \xrightarrow{i} \mathbf{t}$ and that $\mathbf{s} \leq \mathbf{x} \leq \mathbf{t}$. So then we have an ascending path $\mathbf{s} = \mathbf{r}_0 \xrightarrow{i_1} \mathbf{r}_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{q-1}} \mathbf{r}_{q-1} \xrightarrow{i_q} \mathbf{r}_q = \mathbf{x} \xrightarrow{i_{q+1}} \mathbf{r}_{q+1} \xrightarrow{i_{q+2}} \cdots \xrightarrow{i_p} \mathbf{r}_p = \mathbf{t}$. In this case we get $M^{(i)} = \sum_{i\in I_n} a_i M^{(i)}$, where a_i is as before. Then $a_j = \delta_{ij}$, from which we see that $\mathbf{x} = \mathbf{s}$ or $\mathbf{x} = \mathbf{t}$. Hence each $\mathbf{s} \xrightarrow{i} \mathbf{t}$ is a covering relation for the partial order on $\bigwedge^k(R)$.

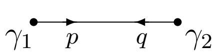
Finally, we show that $\bigwedge^k(R)$ is ranked. It suffices to show this on each connected component of $\bigwedge^k(R)$. So let $\mathcal{C}$ be such a connected component. An ordered pair of elements $(\mathbf{x}, \mathbf{y})$ from $\bigwedge^k(R)$ is *ascending of color* i if $\mathbf{x} \xrightarrow{i} \mathbf{y}$ and *descending of color* i if $\mathbf{y} \xrightarrow{i} \mathbf{x}$. For a path $\mathcal{P} = (\mathbf{s} = \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_p = \mathbf{t})$, let $a_i(\mathcal{P})$ count the number of ascending pairs of color i in the path $\mathcal{P}$ and $d_i(\mathcal{P})$ count the number of descending pairs of color i. Let $\sigma(\mathcal{P}) := \sum_{i\in I_n}(a_i - d_i)$. Call this quantity the *signed length* of the path $\mathcal{P}$. Define a new relation $<_{\mathcal{C}}$ on $\mathcal{C}$ by declaring $\mathbf{s} <_{\mathcal{C}} \mathbf{t}$ if and only if there is a path $\mathcal{P}$ from $\mathbf{s}$ to $\mathbf{t}$ such that $\sigma(\mathcal{P}) > 0$. Then define $\leq_{\mathcal{C}}$ by the rule that $\mathbf{s} \leq_{\mathcal{C}} \mathbf{t}$ if and only if $\mathbf{s} <_{\mathcal{C}} \mathbf{t}$ or $\mathbf{s} = \mathbf{t}$. We claim that $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{C}$. Clearly $\leq_{\mathcal{C}}$ is reflexive. Use concatenation of paths to see that $\leq_{\mathcal{C}}$ is transitive. Finally, we check that $\leq_{\mathcal{C}}$ is asymmetric. Suppose $\mathbf{s} \leq_{\mathcal{C}} \mathbf{t}$ and $\mathbf{t} \leq_{\mathcal{C}} \mathbf{s}$. If $\mathbf{s} \neq \mathbf{t}$, then $\mathbf{s} <_{\mathcal{C}} \mathbf{t}$ and $\mathbf{t} <_{\mathcal{C}} \mathbf{s}$. So there is a path $\mathcal{P}$ from $\mathbf{s}$ to $\mathbf{t}$ for which $\sigma(\mathcal{P}) > 0$ and a path $\mathcal{P}'$ from $\mathbf{t}$ to $\mathbf{s}$ for which $\sigma(\mathcal{P}') > 0$. But now we can see that $\mu(\mathbf{s}) + \sum_{i\in I_n}(a_i(\mathcal{P}) - d_i(\mathcal{P}))M^{(i)} = \mu(\mathbf{t}) = \mu(\mathbf{s}) - \sum_{i\in I_n}(a_i(\mathcal{P}') - d_i(\mathcal{P}'))M^{(i)}$. By linear independence of the $M^{(i)}$'s we conclude that $(a_i(\mathcal{P}) - d_i(\mathcal{P})) + (a_i(\mathcal{P}') - d_i(\mathcal{P}')) = 0$ for all $i \in I_n$. But then $\sigma(\mathcal{P}) = \sum_{i\in I_n}(a_i(\mathcal{P}) - d_i(\mathcal{P})) = -\sum_{i\in I_n}(a_i(\mathcal{P}') - d_i(\mathcal{P}')) = -\sigma(\mathcal{P}')$. So $\sigma(\mathcal{P})$ and $\sigma(\mathcal{P}')$ cannot both be positive. From this contradiction we conclude that $\mathbf{s} = \mathbf{t}$. Then $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{C}$.

Now choose $\mathbf{x}$ to be a minimal element of $\mathcal{C}$ with respect to this partial order. For any $\mathbf{s} \in \mathcal{C}$, we declare $\rho_{\mathcal{C}}(\mathbf{s}) := \sigma(\mathcal{P})$, where $\mathcal{P}$ is any path from $\mathbf{x}$ to $\mathbf{s}$. We claim that $\rho_{\mathcal{C}}(\mathbf{s})$ does not depend on the choice of path from $\mathbf{x}$ to $\mathbf{s}$. To see this, suppose $\mathcal{Q}$ is another path from $\mathbf{x}$ to $\mathbf{s}$. Then from the facts that $\mu(\mathbf{x}) + \sum_{i \in I_n}(a_i(\mathcal{P}) - d_i(\mathcal{P}))M^{(i)}$ and $\mu(\mathbf{x}) + \sum_{i \in I_n}(a_i(\mathcal{Q}) - d_i(\mathcal{Q}))M^{(i)}$, we deduce that $a_i(\mathcal{P}) - d_i(\mathcal{P}) = a_i(\mathcal{Q}) - d_i(\mathcal{Q})$ for all $i \in I_n$. Hence $\sigma(\mathcal{P}) = \sigma(\mathcal{Q})$. Since $\mathbf{x}$ is minimal with respect to the partial order $\leq_{\mathcal{C}}$ on $\mathcal{C}$, it must be the case that $\rho_{\mathcal{C}}(\mathbf{s}) = 0$ for all $\mathbf{s} \in \mathcal{C}$. Finally, suppose $\mathbf{s} \xrightarrow{i} \mathbf{t}$ is a covering relation in $\bigwedge^k(R)$ for elements $\mathbf{s}$ and $\mathbf{t}$ in $\mathcal{C}$. Then any path $\mathcal{P}$ from $\mathbf{x}$ to $\mathbf{s}$ may be extended via $\mathbf{s} \xrightarrow{i} \mathbf{t}$ to a path $\mathcal{Q}$ from $\mathbf{x}$ to $\mathbf{t}$. Then $\sigma(\mathcal{Q}) = \sigma(\mathcal{P}) + 1$, and hence $\rho_{\mathcal{C}}(\mathbf{t}) = \rho_{\mathcal{C}}(\mathbf{s}) + 1$. Then $\rho_{\mathcal{C}}$ is a rank function for $\mathcal{C}$. It follows that $\bigwedge^k(R)$ is ranked. $\square$

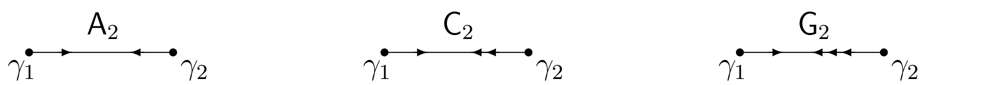
3. Weyl groups and Weyl characters

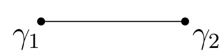
Much of the discussion of Weyl groups and Weyl characters in the following subsections is borrowed from [Don6], [Don7], and [ADLMPPW] as well as standard treatments like [Hum1], [Hum2], [Bour], and [BB].

§3.1 GCM graphs and Dynkin diagrams. Following [Don7] we take as our starting point some given simple graph Γ on n nodes. In particular, Γ has no loops and no multiple edges. Nodes $\{\gamma_i\}_{i\in I_n}$ for Γ are indexed by elements of some fixed totally ordered set I_n of size n (usually $I_n = \{1 < 2 < \cdots < n\}$). For each pair of adjacent nodes γ_i and γ_j in Γ, choose two negative integers M_{ij} and M_{ji}. Extend this to an $n \times n$ matrix $M = (M_{ij})_{i,j\in I_n}$ where, in addition to the negative integers M_{ij} and M_{ji} on edges of Γ, we have $M_{ii} := 2$ for all $i \in I_n$ and $M_{ij} := 0$ if there is no edge in Γ between nodes γ_i and γ_j. We call the pair (Γ, M) a *GCM graph*, since M is a 'generalized Cartan matrix' as in [Kac] and [Kum]. Such matrices are the starting point for the study of Kac–Moody algebras. More importantly for us, these matrices also encode information about certain geometric representations of Weyl groups. Such representations provide a suitable environment for studying Weyl characters, which can be thought of as special multivariate Laurent polynomials which exhibit symmetry under the actions of the Weyl groups.

We say a GCM graph (Γ, M) is *connected* if Γ is. We depict a generic connected two-node GCM graph as 

, where $p = -M_{12}$ and $q = -M_{21}$. We use special names and notation to refer to two-node GCM graphs which have $p = 1$ and $q = 1$, 2, or 3 respectively:

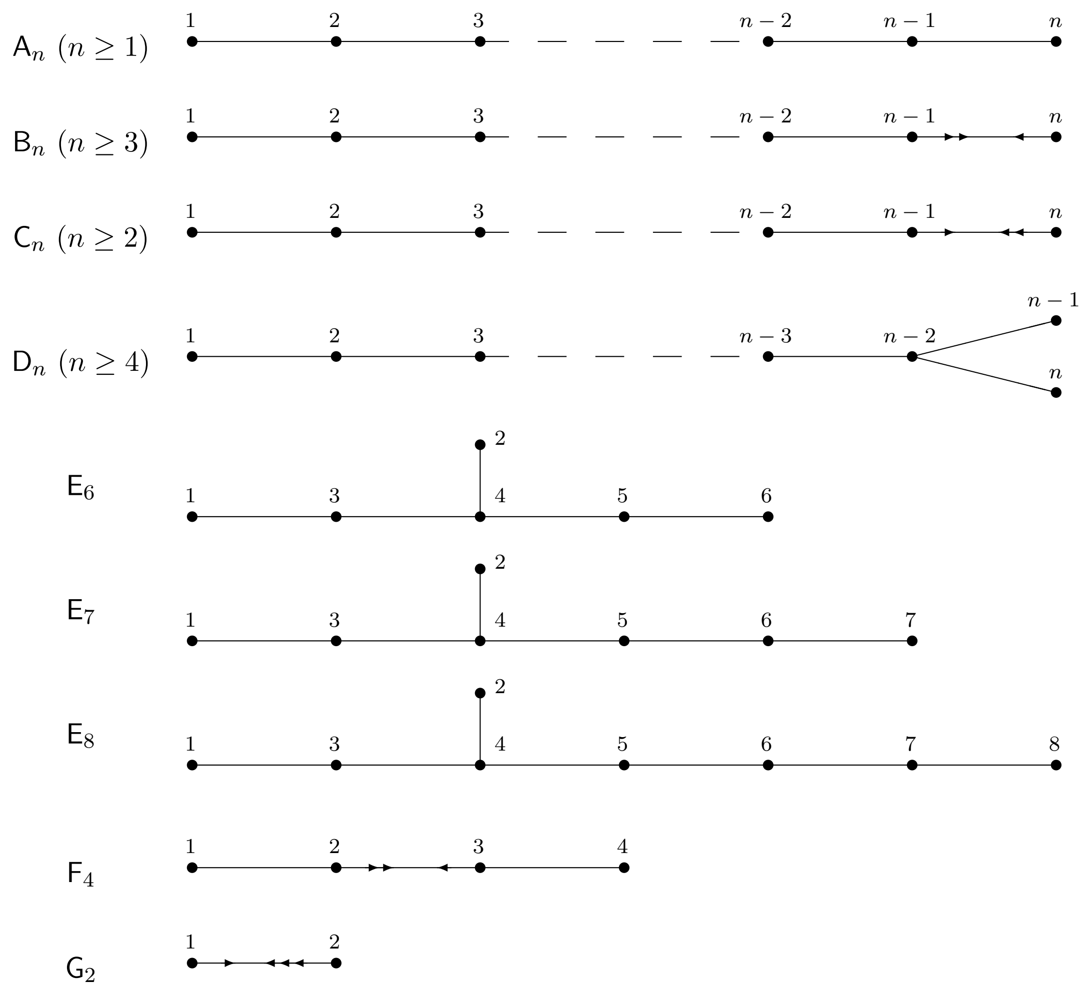


When $p = 1$ and $q = 1$ it is convenient to use the graph 

to represent the GCM graph A_2. A GCM graph (Γ, M) is a *Dynkin diagram of finite type* (or *Dynkin diagram*, for short) if each 'connected component' of (Γ, M) (in the obvious sense, defined below) is one of the graphs of Figure 3.1; in this case the matrix M is called a *Cartan matrix*. We number the nodes of connected Dynkin diagrams of finite type as in §11.4 of [Hum1]. The special two-node GCM graphs A_2, C_2, and G_2 above are Dynkin diagrams with Cartan matrices $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, and $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

The following language concerning GCM graphs is sometimes useful. Given two GCM graphs $\mathfrak{g}_1 = (\Gamma_1, (A_{ij})_{i,j\in I_n})$ and $\mathfrak{g}_2 = (\Gamma_2, (B_{ij})_{i,j\in J_m})$, the *disjoint sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the GCM graph (Γ, M)

Figure 3.1: Connected Dynkin diagrams of finite type.



with graph $\Gamma = \Gamma_1 \oplus \Gamma_2$ (a disjoint sum of undirected graphs in the obvious way, analogous to §2.1, and with nodes indexed by the disjoint union $I_n \dot\cup J_m$) and generalized Cartan matrix $M = \begin{pmatrix} P & O \\ O' & Q \end{pmatrix}$ (a block diagonal matrix in the obvious sense, where O and O' are a zero matrices of appropriate size). These GCM graphs are *isomorphic* if there is a bijection $\sigma : I_n \to J_m$ with respect to which $A_{ij} = B_{\sigma(i),\sigma(j)}$ for all $i, j \in I_n$. If I'_m is a subset of the index set I_n of a GCM graph (Γ, M), then let Γ' be the subgraph of Γ with nodes indexed I'_m and the induced set of edges, and let M' be the corresponding submatrix of the generalized Cartan matrix M; we call (Γ', M') a *GCM subgraph* of (Γ, M). (For example, in Figure 3.1 one can see that C_3 is a GCM subgraph of F_4.) The GCM subgraph (Γ', M') is a *connected component* if Γ' is a connected component of Γ. Given a one-to-one function $\sigma : I_n \to J_n$, obtain a graph Γ^σ by recoloring the nodes of the undirected graph Γ as in §2.1. Then the GCM graph $\mathfrak{g}^\sigma = (\Gamma^\sigma, M^\sigma)$ is the *re-coloring* of the GCM graph $\mathfrak{g}$, where $(M^\sigma)_{\sigma(i),\sigma(j)} := M_{i,j}$ for all $i, j \in I_n$. We let $\mathfrak{g}^\mathsf{T} := (\Gamma, M^\mathsf{T})$, so that $(\mathfrak{g}^\mathsf{T})^\mathsf{T} = \mathfrak{g}$.

§3.2 Weyl groups and geometric representations. For the remainder of this chapter, let $\mathfrak{g} := (\Gamma, M)$ be a fixed GCM graph with index set I_n. The development in this subsection basically follows [BB] and [Don7]. For $i \neq j$ in I_n, declare

$$m_{ij} = \begin{cases} k_{ij} & \text{if } M_{ij}M_{ji} = 4\cos^2(\pi/k_{ij}) \text{ for some integer } k_{ij} \geq 2 \\ \infty & \text{if } M_{ij}M_{ji} \geq 4 \end{cases}$$

We have $m_{ij} = 2$ (respectively $3, 4, 6$) if $M_{ij}M_{ji} = 0$ (resp. $1, 2, 3$). Let $\mathcal{W} := \mathcal{W}_{\mathfrak{g}}$ be the group generated by $\{s_i\}_{i \in I_n}$ subject to relations $s_i^2 = \varepsilon$ for all $i \in I_n$ and $(s_i s_j)^{m_{ij}} = \varepsilon$ for all $i \neq j$ in I_n. (Conventionally, $m_{ij} = \infty$ means there is no relation between generators s_i and s_j.) Then $\mathcal{W}$ is called a *Weyl group*, and it is a special kind of Coxeter group.

Let V be a real vector space freely generated by vectors $\{\alpha_i\}_{i \in I_n}$. The α_i's are called *simple roots*. For each $i \in I_n$, define a linear transformation $S_i : V \to V$ by setting $S_i(\alpha_j) = \alpha_j - M_{ji}\alpha_i$ for each $j \in I_n$ and extending linearly.* The next result follows from Proposition 3.13 of [Kac] or Proposition 1.3.21 of [Kum] (see also §2 of [Don7]). Here $GL(V)$ is the group of invertible linear transformations on V and Id denotes the identity transformation on V.

Lemma 3.1 *For each $i \in I_n$, $S_i^2 = \mathrm{Id}$. In particular, $S_i \in GL(V)$. Now take $i \neq j$ in I_n. If m_{ij} is finite, then $(S_iS_j)^{m_{ij}} = \mathrm{Id}$. If $m_{ij} = \infty$, then the subgroup of $GL(V)$ generated by $\{S_i, S_j\}$ is infinite.*

The above lemma guarantees that the mapping $s_i \mapsto S_i$ extends uniquely to a group homomorphism $\phi : \mathcal{W} \to GL(V)$. Our next result, which is Theorem 4.2.7 of [BB], says that this mapping is injective. In the language of group representations we state this as:

Theorem 3.2 *The representation ϕ of $\mathcal{W}$ in the previous paragraph is faithful.*

§3.3 Finiteness hypothesis. Of interest to us are GCM graphs whose corresponding Weyl groups are finite. These have the following well-known classification (see e.g. [Hum1] or [Hum2]):

Theorem 3.3 *The Weyl group $\mathcal{W}$ is finite if and only if the connected components of $\mathfrak{g}$ are Dynkin diagrams of finite-type from Figure 3.1.*

Two of the most famous Dynkin diagram classification results come from Lie theory: the Dynkin diagrams of Figure 3.1 are in one-to-one correspondence with the finite-dimensional complex simple Lie algebras and the finite-dimensional irreducible Kac-Moody algebras. For examples of other Dynkin diagram classifications, see [HHSV], [Pro5], and [Pro6]. From here on, we restrict our

*This 'transpose' of the usual definition ($S_i(\alpha_j) = \alpha_j - M_{ij}\alpha_i$) facilitates connections with certain results such as the root system and weights results of Chapter III of [Hum1].

attention to the finite cases unless stated otherwise. For connected Dynkin diagrams of finite type, we have the following important observation: one can verify case-by-case that the associated Cartan matrices are invertible.

§3.4 A Euclidean representation of the Weyl group. We would like to realize each transformation S_i as a reflection 'with respect to' α_i. Such a geometric realization of the Weyl group $\mathcal{W}$ will require an inner product $\langle\cdot,\cdot\rangle$ on V. The derivation of the inner product in this subsection is an interpretation of standard material. Assuming for the moment that such an inner product exists, we investigate in this paragraph its interactions with the Cartan matrix M. Relative to this inner product, the reflection $S : V \to V$ in the hyperplane orthogonal to some fixed nonzero α will act on vectors v in V by the rule $S(v) = v - 2\frac{\langle v,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$. Applied to the transformations S_i acting on vectors α_j, we determine that $M_{ji} = 2\frac{\langle\alpha_j,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}$. Symmetry of the inner product now gives

$$M_{ji}\langle\alpha_i,\alpha_i\rangle = M_{ij}\langle\alpha_j,\alpha_j\rangle. \tag{1}$$

If $\mathfrak{g}$ is connected, fix the length of one of the end node simple roots. Then using the preceding relation, the remaining simple root lengths can be computed in terms of the fixed simple root length and entries from the Cartan matrix M. For A–D–E graphs, only one simple root length is possible. Inspection of the other connected Dynkin diagrams of finite type ($\mathsf{B}_n, \mathsf{C}_n, \mathsf{F}_4, \mathsf{G}_2$) shows that each has two root lengths. In the B–C–F cases, 'long' simple roots have squared length twice that of 'short' roots. For G_2, the long simple root α_2 has squared length three times that of the short simple root α_1. If $\mathfrak{g}$ is not connected, then we must choose a squared length for short simple roots in each connected component of $\mathfrak{g}$. With such a fixed choice of short simple root lengths for $\mathfrak{g}$, one can now determine that

$$\langle\alpha_j,\alpha_i\rangle = \frac{1}{2}\langle\alpha_i,\alpha_i\rangle M_{ji} \tag{2}$$

for all $i, j \in I_n$. So our hypothetical inner product is determined by the preceding relations (1) and (2) together with the choices for short simple root lengths for connected components of $\mathfrak{g}$. With this discussion in mind, now define a bilinear form B on V so that for each $i \in I_n$, $B(\alpha_i,\alpha_i)$ coincides with the choices for squared lengths of simple roots indicated above, and where $B(\alpha_i,\alpha_j) := \frac{1}{2}B(\alpha_j,\alpha_j)M_{ij}$ for all $i, j \in I_n$.

Theorem 3.4 *The bilinear form B defined above is symmetric and nondegenerate. Moreover, the Weyl group $\mathcal{W}$ preserves B in the sense that $B(w.v_1, w.v_2) = B(v_1, v_2)$ for all $w \in \mathcal{W}$ and $v_1, v_2 \in V$. Finally, relative to the form B each S_i is a reflection with respect to α_i: $S_i(v) = v - 2\frac{B(v,\alpha_i)}{B(\alpha_i,\alpha_i)}\alpha_i$ for all $v \in V$.*

It suffices to prove Theorem 3.4 for connected Dynkin diagrams. This can be done case by case. From here on, we use $\langle \cdot, \cdot \rangle$ to denote the inner product B of the preceding paragraph and theorem. Given $\langle \cdot, \cdot \rangle$, we call $\phi : \mathcal{W} \to GL(V, \langle \cdot, \cdot \rangle)$ a *Euclidean representation* of $\mathcal{W}$. Let $\mathcal{O}(V, \langle \cdot, \cdot \rangle)$ be the orthogonal group for the Euclidean space $(V, \langle \cdot, \cdot \rangle)$. A consequence of the preceding theorem is that $\phi(\mathcal{W}) \cong \mathcal{W}$ is actually a subgroup of $\mathcal{O}(V, \langle \cdot, \cdot \rangle)$. From here on, we consider ϕ to be a Euclidean representation for $\mathcal{W}_{\mathfrak{g}}$ with respect to some fixed choice of inner product.

Suppose $\mathfrak{g} = (\Gamma_1, A = (A_{ij})_{i,j \in I_n})$ and $\mathfrak{h} = (\Gamma_2, B = (B_{ij})_{i,j \in J_m})$ are connected Dynkin diagrams with corresponding Weyl groups $\mathcal{W}_{\mathfrak{g}} = \langle s_i \rangle_{i \in I_n}$ and $\mathcal{W}_{\mathfrak{h}} = \langle t_j \rangle_{j \in J_m}$. Let $\phi : \mathcal{W}_{\mathfrak{g}} \to GL(V_1, \langle \cdot, \cdot \rangle_1)$ and $\psi : \mathcal{W}_{\mathfrak{h}} \to GL(V_2, \langle \cdot, \cdot \rangle_2)$ be Euclidean representations of $\mathcal{W}_{\mathfrak{g}}$ and $\mathcal{W}_{\mathfrak{h}}$ respectively, with $V_1 := \mathrm{span}_{\mathbb{R}}(\{\alpha_i\}_{i \in I_n})$ and $V_2 := \mathrm{span}_{\mathbb{R}}(\{\beta_j\}_{j \in J_m})$ for simple roots $\{\alpha_i\}_{i \in I_n}$ and $\{\beta_j\}_{j \in J_m}$ respectively. We say ϕ and ψ are *isomorphic* if there is a bijection $\sigma : I_n \to J_m$ such that the mapping $s_i \mapsto t_{\sigma(i)}$ extends to a group isomorphism from $\mathcal{W}_{\mathfrak{g}}$ to $\mathcal{W}_{\mathfrak{h}}$ and such that the linear transformation $T : V_1 \to V_2$ induced by the set mapping $\alpha_i \mapsto \beta_{\sigma(i)}$ is 'angle-preserving', i.e. for some fixed (necessarily positive) real scalar κ we have $\langle T(u), T(v) \rangle_2 = \kappa \langle u, v \rangle_1$ for all $u, v \in V_1$. To emphasize the role of the bijection σ we say that ϕ and ψ are *isomorphic via* σ. In particular, it follows that for any two choices of inner products on V_1 from Theorem 3.4, the corresponding Euclidean representations of $\mathcal{W}_{\mathfrak{g}}$ are isomorphic. Some other results concerning isomorphic Euclidean representations are explored in Lemma 3.6. The Euclidean representations corresponding to the connected Dynkin diagrams of finite type are pairwise nonisomorphic (even though the corresponding Weyl groups are not all distinct — in particular, $\mathcal{W}_{\mathsf{B}_n} \cong \mathcal{W}_{\mathsf{C}_n}$).

Now relax the connectedness hypothesis for $\mathfrak{g}$ and $\mathfrak{h}$. Suppose a connected component $\mathfrak{g}'$ of $\mathfrak{g}$ has nodes indexed by a subset $J \subseteq I_n$. Let $V_1' = \mathrm{span}_{\mathbb{R}}(\{\alpha_i\}_{i \in J})$, so V_1' is a subspace of V with the induced inner product $\langle \cdot, \cdot \rangle_1'$. It is easy to see that the mapping $\phi' : \mathcal{W}_{\mathfrak{g}'} \to GL(V_1', \langle \cdot, \cdot \rangle_1')$ is a Euclidean representation of $\mathcal{W}_{\mathfrak{g}'}$. We say Euclidean representations ϕ and ψ of $\mathcal{W}_{\mathfrak{g}}$ and $\mathcal{W}_{\mathfrak{h}}$ are *isomorphic* if there is some one-to-one correspondence $\mathfrak{g}' \mapsto \mathfrak{h}'$ of connected components of $\mathfrak{g}$ and $\mathfrak{h}$ such that ϕ' and ψ' are isomorphic.

§3.5 Roots and root systems. Write $w.v$ for $\phi(w)(v)$ whenever $w \in \mathcal{W}$ and $v \in V$. As in [Hum2] and [BB], we define the *root system* $\mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i \in I_n})$ to be the set $\phi(\mathcal{W}_{\mathfrak{g}})(\{\alpha_i\}_{i \in I_n}) = \{w.\alpha_i\}_{i \in I_n, w \in \mathcal{W}}$. Set $\Phi := \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i \in I_n})$. Elements of Φ are *roots*. A root $\alpha = \sum k_i \alpha_i$ is *positive* if each $k_i \geq 0$ and is *negative* if each $k_i \leq 0$. The sets Φ^+ and Φ^- of positive and negative roots can be seen to partition Φ (see §3 of [Don7]). For any $i, j \in I_n$, by definition $s_j.\alpha_i = \alpha_i - M_{ij} \alpha_j$.

Since any $w \in \mathcal{W}$ is a product of s_j's, then by iterating the previous computation we see that $w.\alpha_i$ is an integral linear combination of simple roots. That is, when $\alpha = \sum k_i\alpha_i$, then each $k_i \in \mathbb{Z}$. Now, each $w \in \mathcal{W}$ permutes Φ. To see this, note that for any $w \in \mathcal{W}$ and $\alpha, \beta \in \Phi$, (1) $w.\alpha \in \Phi$ by definition so $w(\Phi) \subseteq \Phi$, (2) $\alpha = w.(w^{-1}.\alpha)$ so $\Phi \subseteq w(\Phi)$, and (3) if $w.\alpha = w.\beta$ then $w^{-1}.(w.\alpha) = w^{-1}.(w.\beta)$ so $\alpha = \beta$. So we have an induced action of $\mathcal{W}$ on Φ. Two root systems $\Phi := \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i\in I_n})$ and $\Psi := \mathcal{R}(\mathfrak{h}, \psi, \{\beta_j\}_{j\in J_m})$ are *isomorphic* (respectively, *isomorphic via* σ) if the Euclidean representations ϕ and ψ are isomorphic (respectively, isomorphic via σ).

For any $\alpha \in \Phi$, define $\alpha^\vee := \frac{2}{\langle\alpha,\alpha\rangle}\alpha$. Observe that $\langle\alpha_i, \alpha_j^\vee\rangle = M_{ij}$ for all $i, j \in I_n$. Let $\Phi^\vee := \{\alpha^\vee\}_{\alpha\in\Phi}$. Based on the following lemma, we call $\Phi^\vee$ the *dual root system* for Φ.

Lemma 3.5 *We have $\Phi^\vee = \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i^\vee\}_{i\in I_n})$ (an equality of sets), and moreover $\alpha^\vee = w.\alpha_i^\vee$ for $w \in \mathcal{W}$ if and only if $\alpha = w.\alpha_i$.*

Proof. To prove the lemma, it suffices to show that $\alpha^\vee = w.\alpha_i^\vee$ for $w \in \mathcal{W}$ if and only if $\alpha = w.\alpha_i$. Suppose $\alpha = w.\alpha_i$. Since $\langle\alpha,\alpha\rangle = \langle w.\alpha_i, w\alpha_i\rangle = \langle\alpha_i,\alpha_i\rangle$, then $\alpha^\vee = \frac{2}{\langle\alpha,\alpha\rangle}\alpha = \frac{2}{\langle\alpha_i,\alpha_i\rangle}w.\alpha_i = w.\alpha_i^\vee$. Conversely, suppose $\alpha^\vee = w.\alpha_i^\vee$. Then $\langle\alpha^\vee,\alpha^\vee\rangle = \langle w.\alpha_i^\vee, w.\alpha_i^\vee\rangle = \langle\alpha_i^\vee,\alpha_i^\vee\rangle$. Now for any $\beta \in \Phi$, $\langle\beta^\vee,\beta^\vee\rangle = \frac{4}{\langle\beta,\beta\rangle}$. So from our previous calculation, it follows that $\frac{4}{\langle\alpha,\alpha\rangle} = \frac{4}{\langle\alpha_i,\alpha_i\rangle}$, and hence $\langle\alpha,\alpha\rangle = \langle\alpha_i,\alpha_i\rangle$. Then from $\frac{2}{\langle\alpha,\alpha\rangle}\alpha = \alpha^\vee = w.\alpha_i^\vee = \frac{2}{\langle\alpha_i,\alpha_i\rangle}w.\alpha_i$, we deduce that $\alpha = w.\alpha_i$. $\square$

For this paragraph, assume that $\mathfrak{g}$ is connected. According to the discussion of the previous section, simple roots have two possible lengths, which we call *long* or *short*. (If only one simple root length is possible i.e. in the A-D-E cases, the adjectives "short" and "long" are interchangeable.) Note that if $\alpha \in \Phi$ with $\alpha = w.\alpha_i$ for some $w \in \mathcal{W}$ and simple root α_i, then $\langle\alpha,\alpha\rangle = \langle w.\alpha_i, w.\alpha_i\rangle = \langle\alpha_i,\alpha_i\rangle$. So α has the same length as α_i. With this in mind, we let $\Phi_{long} = \{\alpha \in \Phi \,|\, \alpha = w.\alpha_i \text{ for } w \in \mathcal{W} \text{ and } \alpha_i \text{ long}\}$ be the set of *long* roots, and analogously define the set Φ_{short} of *short* roots. We also have Φ^+_{long} (the set of positive roots that are long) and Φ^+_{short} (the set of positive roots that are short).

Lemma 3.6 *Suppose $\mathfrak{g} = (\Gamma_1, A = (A_{ij})_{i,j\in I_n})$ and $\mathfrak{h} = (\Gamma_2, B = (B_{ij})_{i,j\in J_m})$ are connected Dynkin diagrams with corresponding Weyl groups $\mathcal{W}_\mathfrak{g} = \langle s_i\rangle_{i\in I_n}$ and $\mathcal{W}_\mathfrak{h} = \langle t_j\rangle_{j\in J_m}$. Let $\phi : \mathcal{W}_\mathfrak{g} \to GL(V_1, \langle\cdot,\cdot\rangle_1)$ and $\psi : \mathcal{W}_\mathfrak{h} \to GL(V_2, \langle\cdot,\cdot\rangle_2)$ be isomorphic Euclidean representations of $\mathcal{W}_\mathfrak{g}$ and $\mathcal{W}_\mathfrak{h}$ respectively, with $V_1 := \mathrm{span}_\mathbb{R}(\{\alpha_i\}_{i\in I_n})$ and $V_2 := \mathrm{span}_\mathbb{R}(\{\beta_j\}_{j\in J_m})$ for simple roots $\{\alpha_i\}_{i\in I_n}$ and $\{\beta_j\}_{j\in J_m}$ respectively. As in §3.4, let $\sigma : I_n \to J_m$ be the associated bijection and $T : V_1 \to V_2$ the associated angle-preserving linear transformation. Let $\Phi := \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i\in I_n})$ and $\Psi := \mathcal{R}(\mathfrak{h}, \psi, \{\beta_j\}_{j\in J_m})$. Let $(i_1, i_2, \ldots, i_p)$ be a sequence of elements from I_n. (1) For all $i, j \in I_n$,*

$A_{ij} = B_{\sigma(i),\sigma(j)}$. *(2) For all* $v \in V_1$, $T(s_{i_1}s_{i_2}\cdots s_{i_p}.v) = t_{\sigma(i_1)}t_{\sigma(i_2)}\cdots t_{\sigma(i_p)}.T(v)$. *(3) For* $j \in I_n$, *let* $\alpha := s_{i_1}s_{i_2}\cdots s_{i_p}.\alpha_j$ *and* $\beta := t_{\sigma(i_1)}t_{\sigma(i_2)}\cdots t_{\sigma(i_p)}.\beta_{\sigma(j)}$. *If* α *is positive in* Φ *(resp. long, short), then* β *is positive in* Ψ *(resp. long, short).*

Proof. For (1), $A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle = \frac{2\langle \alpha_i,\alpha_j\rangle_1}{\langle \alpha_j,\alpha_j\rangle_1} = \frac{2\kappa\langle \alpha_i,\alpha_j\rangle_1}{\kappa\langle \alpha_j,\alpha_j\rangle_1} = \frac{2\langle T(\alpha_i),T(\alpha_j)\rangle_2}{\langle T(\alpha_j),T(\alpha_j)\rangle_2} = \frac{2\langle \beta_{\sigma(i)},\beta_{\sigma(j)}\rangle_2}{\langle \beta_{\sigma(j)},\beta_{\sigma(j)}\rangle_2} = \langle \beta_{\sigma(i)}, \beta_{\sigma(j)}^\vee \rangle_2 = B_{\sigma(i),\sigma(j)}$. For (2), it suffices to show that $T(s_i.\alpha_j) = t_{\sigma(i)}.\beta_{\sigma(j)}$ for all $i, j \in I_n$. To see this, observe that for any $k \in I_n$, $\langle T(s_i.\alpha_j), \beta_{\sigma(j)}^\vee \rangle_2 = \frac{2}{\kappa\langle \alpha_k,\alpha_k\rangle_1}\kappa\langle s_i.\alpha_j, \alpha_k\rangle_1 = \langle s_i.\alpha_j, \alpha_k^\vee\rangle_1 = \langle \alpha_j - A_{ji}\alpha_i, \alpha_k^\vee\rangle_1 = A_{jk} - A_{ji}A_{ik} = B_{\sigma(j),\sigma(k)} - B_{\sigma(j),\sigma(i)}B_{\sigma(i),\sigma(k)} = \langle \beta_{\sigma(j)} - B_{\sigma(j),\sigma(i)}\beta_{\sigma(i)}, \beta_{\sigma(k)}^\vee\rangle_2 = \langle t_{\sigma(i)}.\beta_{\sigma(j)}, \beta_{\sigma(k)}^\vee\rangle_2$. Since this is true for all $k \in I_n$, then it must be the case that $T(s_i.\alpha_j) = t_{\sigma(i)}.\beta_{\sigma(j)}$, as desired. For (3), let α and β be as in the lemma statement. Suppose $\alpha = \sum k_i\alpha_i \in \Phi^+$. Then $T(\alpha) = \sum k_i\beta_{\sigma(i)}$. But by (2), $T(\alpha) = \beta$. Hence $\beta \in \Psi^+$. From (1), it follows that if $\alpha_j \in \Phi_{long}$ (resp. Φ_{short}), then $\beta_{\sigma(j)} \in \Psi_{long}$ (resp. Ψ_{short}). Since α has the same length as α_j and β has the same length as β_j, then $\alpha \in \Phi_{long}$ (resp. Φ_{short}) implies that $\beta \in \Psi_{long}$ (resp. Ψ_{short}). □

For connected $\mathfrak{g}$, give Φ the following partial ordering: write $\alpha \leq \beta$ for roots α and β if and only if $\beta - \alpha = \sum k_i\alpha_i$ with each k_i nonnegative. View Φ^+, Φ^+_{long} and Φ^+_{short} as subposets of Φ in the induced order. If $\alpha \in \Phi^+$, write $\alpha = \sum k_i\alpha_i$ for nonnegative integers k_i. The *height* of α, denoted $ht(\alpha)$, is defined to be the quantity $\sum k_i$. The following facts can be understood by studying the so-called 'adjoint' and 'short adjoint' representations of the finite-dimensional complex simple Lie algebras.

Facts 3.7 *Keep the notation of the previous paragraph as well as the assumption that* $\mathfrak{g}$ *is connected. The posets of roots* Φ^+ *and* Φ^+_{short} *are ranked, connected posets with (in each case) rank function given by* $\rho(\alpha) = ht(\alpha) - 1$. *The minimal roots for* Φ^+ *(respectively,* Φ^+_{short}*) are the simple roots (resp. short simple roots). Each has a unique maximal root.*

In the setting of these results, the maximal root $\overline{\omega}$ for Φ^+ is called the *highest long root*. For Φ^+_{short} the maximal root $\overline{\omega}_{short}$ is the *highest short root*.

The *transpose* representation and root system defined next are helpful in explicitly identifying long and short roots. For this definition, however, $\mathfrak{g}$ need not be connected. Let $V^\top$ be the real vector space freely generated by $\{\alpha_i^\top\}_{i\in I_n}$, and define $\phi^\top : W_{\mathfrak{g}} \to V^\top$ by the rule $\phi^\top(s_i)(\alpha_j^\top) = \alpha_j^\top - M_{ji}^\top\alpha_i^\top$. Give $V^\top$ an inner product $\langle \cdot, \cdot \rangle_\top$ as in Theorem 3.4 above using the matrix $M^\top$. Then set $\Phi^\top := \mathcal{R}(\mathfrak{g}, \phi^\top, \{\alpha_i^\top\}_{i\in I_n})$. (Evidently, the root systems $\mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i\in I_n})$ and $\mathcal{R}(\mathfrak{g}^\top, \phi^\top, \{\alpha_i^\top\}_{i\in I_n})$ are isomorphic via the identity bijection on I_n.)

Proposition 3.8 *Let $\mathfrak{g}$ be connected. For all $w \in \mathcal{W}$ and $j \in I_n$, it is the case that $w.\alpha_j$ is positive (resp. long, short) in Φ if and only if $w.\alpha_j^\vee$ is positive (resp. short, long) in $\Phi^\vee$ if and only if $w.\alpha_j^\mathsf{T}$ is positive (resp. short, long) in Φ^T. Moreover, $\Phi^\vee = \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i^\vee\}_{i\in I_n})$ and $\Phi^\mathsf{T} = \mathcal{R}(\mathfrak{g}, \phi^\mathsf{T}, \{\alpha_i^\mathsf{T}\}_{i\in I_n})$ are isomorphic via the identity bijection on I_n.*

Proof. First we show that $\Phi^\vee = \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i^\vee\}_{i\in I_n})$ and $\Phi^\mathsf{T} = \mathcal{R}(\mathfrak{g}, \phi^\mathsf{T}, \{\alpha_i^\mathsf{T}\}_{i\in I_n})$ are isomorphic via the identity bijection. We have that $\langle \alpha_i^\mathsf{T}, \alpha_j^\mathsf{T}\rangle_\mathsf{T} = \frac{1}{2}\langle \alpha_j^\mathsf{T}, \alpha_j^\mathsf{T}\rangle_\mathsf{T} M_{ij}^\mathsf{T} = \frac{1}{2}\langle \alpha_j^\mathsf{T}, \alpha_j^\mathsf{T}\rangle_\mathsf{T} M_{ji}$. On the other hand, we have $\langle \alpha_i^\vee, \alpha_j^\vee\rangle = \frac{4}{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle}\langle\alpha_i,\alpha_j\rangle = \frac{2}{\langle\alpha_j,\alpha_j\rangle}M_{ji} = \frac{1}{2}\langle\alpha_j^\vee, \alpha_j^\vee\rangle M_{ji}$. So $\frac{\langle\alpha_i^\mathsf{T},\alpha_j^\mathsf{T}\rangle_\mathsf{T}}{\langle\alpha_i^\vee,\alpha_j^\vee\rangle} = \frac{\langle\alpha_j^\mathsf{T},\alpha_j^\mathsf{T}\rangle_\mathsf{T}}{\langle\alpha_j^\vee,\alpha_j^\vee\rangle}$. That the latter ratio is constant for all $j \in I_n$ can be proved by checking cases. For example, in the A–D–E cases, all α_j^T's have the same length and all $\alpha_j^\vee$'s have the same length.

From this result and Lemma 3.6, we conclude that $w.\alpha_j^\vee$ is positive (resp. short, long) if and only if $w.\alpha_j^\mathsf{T}$ is positive (resp. short, long). Now $w.\alpha_j^\vee = \sum k_i\alpha_i^\vee$ if and only if $w.\alpha_j = \sum \frac{\langle\alpha_j,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}k_i\alpha_i$. Then $w.\alpha_j^\vee$ is positive if and only if $w.\alpha_j$ is positive. It is easy to see that there are two distinct root lengths in Φ if and only if there are two distinct root lengths in $\Phi^\vee$. Therefore, to show that $w.\alpha_j$ is long (resp. short) if and only if $w.\alpha_j^\vee$ is short (resp. long), it suffices to consider those cases with two distinct root lengths. In such cases, if $w.\alpha_j$ is long in Φ, then α_j is long. Then there is a simple root α_k such that $\langle\alpha_j,\alpha_j\rangle > \langle\alpha_k,\alpha_k\rangle$. Then $\frac{4}{\langle\alpha_j,\alpha_j\rangle} < \frac{4}{\langle\alpha_k,\alpha_k\rangle}$, hence $\langle\alpha_j^\vee,\alpha_j^\vee\rangle < \langle\alpha_k^\vee,\alpha_k^\vee\rangle$. So, $\alpha_j^\vee$ and $w.\alpha_j^\vee$ are short. This argument is easily modified to show that if $w.\alpha_j$ is short, then $w.\alpha_j^\vee$ is long. Similarly see that if $w.\alpha_j^\vee$ is short (resp. long), then $w.\alpha_j$ is long (resp. short). $\square$

§3.6 Weights. Some of the following recasts parts of §13 of [Hum1]. Using our inner product $\langle\cdot,\cdot\rangle$ we obtain another special basis for V, the basis of 'fundamental weights'. The following proposition shows how this basis is obtained and uniquely characterized.

Proposition 3.9 *Let $A = (A_{jk})_{j,k\in I_n}$ be a real $n \times n$ matrix. Define $\omega_j := \sum_{k\in I_n} A_{jk}\alpha_k$. Then $S_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$ for all $i, j \in I_n$ if and only if $A = M^{-1}$ if and only if $\langle\omega_j, \alpha_i^\vee\rangle = \delta_{ij}$ for all $i, j \in I_n$.*

Proof. Suppose $S_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$ for all $i, j \in I_n$. Then for fixed $i, j \in I_n$ we have

$$\begin{aligned} S_i(\omega_j) &= \omega_j - \delta_{ij}\alpha_i \\ &= \left(\sum_{k\in I_n} A_{jk}\alpha_k\right) - \delta_{ij}\alpha_i \\ &= \left(\sum_{k\neq i} A_{jk}\alpha_k\right) + (A_{ji} - \delta_{ij})\alpha_i. \end{aligned}$$

But we also have

$$\begin{aligned} S_i(\omega_j) &= \sum_{k\in I_n} A_{jk} S_i(\alpha_k) \\ &= \sum_{k\in I_n} A_{jk}(\alpha_k - M_{ki}\alpha_i) \\ &= \sum_{k\in I_n} (A_{jk}\alpha_k - A_{jk}M_{ki}\alpha_i) \\ &= \sum_{k\neq i} A_{jk}\alpha_k + \left(A_{ji} - \sum_{k\in I_n} A_{jk}M_{ki}\right)\alpha_i. \end{aligned}$$

Then $\sum_{k\in I_n} A_{jk}M_{ki} = \delta_{ij}$. Since this is true for all $i,j \in I_n$, we conclude that $A = M^{-1}$.

Now suppose $A = M^{-1}$. Fix $i,j\in I_n$. Then

$$\langle \omega_j, \alpha_i^\vee\rangle = \sum_{k\in I_n} A_{jk}\langle \alpha_k, \alpha_i^\vee\rangle = \sum_{k\in I_n} A_{jk}\frac{2\langle \alpha_k,\alpha_i\rangle}{\langle \alpha_i,\alpha_i\rangle} = \sum_{k\in I_n} A_{jk}M_{ki} = \delta_{ij}.$$

The crucial step in this calculation is our application of the identity (2) from §3.4.

Finally suppose $\langle \omega_j, \alpha_i^\vee\rangle = \delta_{ij}$ for all $i,j\in I_n$. Then for fixed $i,j\in I_n$ we have

$$\begin{aligned} S_i(\omega_j) &= S_i\left(\sum_{k\in I_n} A_{jk}\alpha_k\right) \\ &= \sum_{k\in I_n} A_{jk}(\alpha_k - M_{ki}\alpha_i) \\ &= \omega_j - \left(\sum_{k\in I_n} A_{jk}M_{ki}\right)\alpha_i \\ &= \omega_j - \left(\sum_{k\in I_n} A_{jk}\frac{2\langle \alpha_k,\alpha_i\rangle}{\langle \alpha_i,\alpha_i\rangle}\right)\alpha_i \\ &= \omega_j - \left(\sum_{k\in I_n} A_{jk}\langle \alpha_k,\alpha_i^\vee\rangle\right)\alpha_i \\ &= \omega_j - \langle \omega_j,\alpha_i^\vee\rangle\alpha_i \\ &= \omega_j - \delta_{ij}\alpha_i. \end{aligned}$$

This completes the proof. $\square$

In view of this result, we define the basis of *fundamental weights* $\{\omega_i\}_{i\in I_n}$ to be the unique basis for V satisfying the equivalent conditions of Proposition 3.9. As a consequence we see that for each $i \in I_n$, $\alpha_i = \sum_{j\in I_n} M_{ij}\omega_j$, i.e. the ith simple root is identified with the ith row of the

Cartan matrix relative to the basis of fundamental weights. Let $\Lambda \subset V$ be the set of all vectors in the integer linear span of $\{\omega_i\}_{i \in I_n}$. Vectors in Λ are *weights*, and we call Λ the *lattice of weights*. (Here 'lattice' is used in the sense of the $\mathbb{Z}$-span of a basis.) A weight $\lambda \in \Lambda$ is *dominant* (*strongly dominant*) if $\lambda = \sum m_i \omega_i$ with each m_i nonnegative (positive). Denote by Λ^+ the set of dominant weights.

Lemma 3.10 *We have $\Phi \subset \Lambda$. Moreover each $w \in \mathcal{W}$ permutes Λ, and we have an induced action of $\mathcal{W}$ on Λ.*

Proof. Let $i \in I_n$. Since $\alpha_i = \sum_{j \in I_n} M_{ij} \omega_j$, it follows that $\alpha_i \in \Lambda$. Since each $\alpha \in \Phi$ is an integral linear combination of α_i's, it follows that $\Phi \subset \Lambda$. To complete the proof of the lemma, it suffices to show that w permutes Λ for each $w \in \mathcal{W}$. Let $\lambda = \sum m_i \omega_i$. Then $w.\lambda = \sum m_i w.\omega_i$. Now each $s_j.\omega_i = \omega_i - \delta_{ij}\alpha_j \in \Lambda$. Since w is a product of s_j's, then by iterating the previous computation we see that $w.\omega_i \in \Lambda$ for each $i \in I_n$. It follows that $w.\lambda \in \Lambda$. Now for any $\nu \in \Lambda$, we have $\nu = w.(w^{-1}.\nu)$. Since $\phi(w) \in GL(V)$, it follows that $\phi(w)$ is one-to-one. So we have shown that $\phi(w)|_\Lambda : \Lambda \to \Lambda$ is a bijection. $\square$.

Given a subset $J \subseteq I_n$, let $\mathcal{W}_J$ be the subgroup of $\mathcal{W}$ generated by $\{s_j\}_{j \in J}$. A dominant weight λ is J^c-*dominant* if when we write $\lambda = \sum_{i \in I_n} m_i \omega_i$, then $m_j > 0$ if and only if $j \notin J$. It can be shown that the results of [Hum2] §5.13 extend to the setting of our geometric representation of the Weyl group $\mathcal{W}$. It follows that $\mathcal{W}_J$ is the stablizer of λ under the action of $\mathcal{W}$ on Λ. So, by the 'orbit-stablizer' theorem, we have $|\mathcal{W}| = |\mathcal{W}\lambda||\mathcal{W}_J|$. When $\mathfrak{g}$ is connected, we apply this to the special cases of the sets Φ_{long} and Φ_{short} of long and short roots respectively. In §3.11 below we show how one can use a game played on the Dynkin diagram $\mathfrak{g}$ to determine the highest root and highest short root. Using this technique one can determine that for A_n, $\overline{\omega} = \omega_1 + \omega_n$. For B_n, $\overline{\omega} = \omega_2$ and $\overline{\omega}_{short} = \omega_1$. For C_n, $\overline{\omega} = 2\omega_1$ and $\overline{\omega}_{short} = \omega_2$. For D_n, $\overline{\omega} = \omega_2$. For E_6, $\overline{\omega} = \omega_2$. For E_7, $\overline{\omega} = \omega_2$. For E_8, $\overline{\omega} = \omega_2$. For F_4, $\overline{\omega} = \omega_1$ and $\overline{\omega}_{short} = \omega_4$. For G_2, $\overline{\omega} = \omega_2$ and $\overline{\omega}_{short} = \omega_1$. Therefore, the highest long and short roots are dominant weights. In fact, it can be seen that all roots of Φ_{long} (resp. Φ_{short}) are conjugate under the action of $\mathcal{W}$.* We therefore obtain the following result, which gives us a nice way to compute the order of the Weyl group.

*Briefly, any two long (respectively, short) simple roots are connected by an 'ON-path', in the language of [Don7]. It follows from Theorem 3.2 of [Don7] that these simple roots are conjugate under the $\mathcal{W}$-action. Applying Corollary 3.27 and Proposition 3.28, it follows that any long (resp. short) root is conjugate to some long (resp. short) simple root. It then follows that any two long (resp. short) roots are conjugate under the $\mathcal{W}$-action.

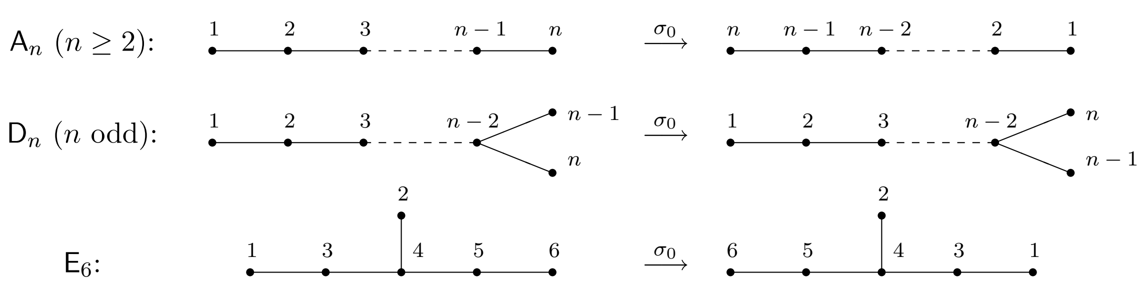
Theorem 3.11 *With $\mathfrak{g}$ connected, we have $\overline{\omega}$ (resp. $\overline{\omega}_{short}$) as the highest long (resp. short) root. Then $\overline{\omega}$ (resp. $\overline{\omega}_{short}$) is nonzero and dominant. Moreover, $\mathcal{W}\overline{\omega} = \Phi_{long}$ (resp. $\mathcal{W}\overline{\omega}_{short} = \Phi_{short}$). Suppose $\overline{\omega}$ (resp. $\overline{\omega}_{short}$) is J^c-dominant. Then $|\mathcal{W}| = |\Phi_{long}||\mathcal{W}_J|$ (resp. $|\mathcal{W}| = |\Phi_{short}||\mathcal{W}_J|$).*

§3.7 The longest element of the Weyl group. The material in this section is standard, see e.g. [Hum2] or [BB]. A finite Weyl group has a unique 'longest' element, where length is measured as follows: In any Weyl group, an element w may be written as a product $s_{i_1} \cdots s_{i_p}$. Any shortest such expression is a *reduced expression* for w, and the *length* of w is $\ell(w) := p$. Thus if $\mathcal{W}$ is finite, there is an upper bound on the lengths of group elements. The following result can be derived from standard facts (see e.g. Exercise 5.6.2 of [Hum2]).

Proposition 3.12 *In a finite Weyl group, there is exactly one longest element, denoted w_0. We have $w_0^2 = \varepsilon$. Moreover, there is a permutation $\sigma_0 : I_n \longrightarrow I_n$ such that for each $i \in I_n$, $w_0.\alpha_i = -\alpha_{\sigma_0(i)}$.*

Observe that since $\langle \alpha_i, \alpha_j^\vee \rangle = M_{ij}$ for all $i, j \in I_n$ then $\langle \alpha_{\sigma_0(i)}, \alpha_{\sigma_0(j)}^\vee \rangle = M_{\sigma_0(i),\sigma_0(j)}$. In particular, σ_0 is a symmetry of the Dynkin diagram $\mathfrak{g}$ in the sense that $\mathfrak{g} \cong \mathfrak{g}^{\sigma_0}$. Since $w_0^2 = \varepsilon$ in $\mathcal{W}$ then σ_0^2 is the identity permutation. It also follows from the proposition that when w_0 acts on Λ, then $\omega_i \mapsto -\omega_{\sigma_0(i)}$ for each $i \in I_n$: $\langle -w_0.\omega_i, \alpha_j^\vee \rangle = \langle \omega_i, -w_0.\alpha_j^\vee \rangle = \langle \omega_i, \alpha_{\sigma_0(j)}^\vee \rangle = \delta_{i,\sigma_0(j)}$, hence $w_0.\omega_i = -\omega_{\sigma_0(i)}$. Thus, $w_0.(\sum m_i\omega_i) = -\sum m_i\omega_{\sigma_0(i)}$. So once the action of w_0 on V is known (see §3.11 below) then one can compute σ_0. One finds that for connected Dynkin diagrams, σ_0 is trivial except in the cases A_n ($n \geq 2$), D_{2k+1} ($k \geq 2$), and E_6; see Figure 3.2.

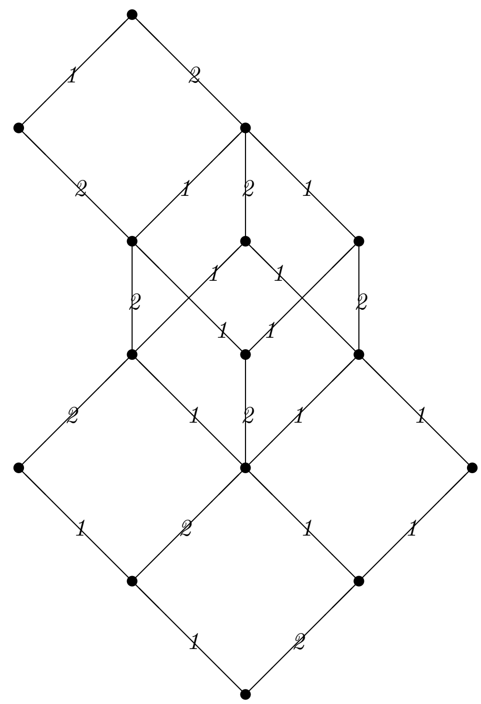
Figure 3.2: Action of the permutation σ_0 when σ_0 is not the identity.



If R is a ranked poset with edges colored by the set I_n, then the σ_0-*recolored dual* $R^\triangle$ is the edge-colored poset $(R^{\sigma_0})^* \cong (R^*)^{\sigma_0}$. See Figure 3.3 for an example.

§3.8 The M-structure property (again). Let R be a ranked poset with edges colored by the set I_n. We say R has the $\mathfrak{g}$-*structure property* if R has the M-structure property for the Cartan matrix M associated to $\mathfrak{g}$ with weight function $wt_R : R \longrightarrow \Lambda$ such that $wt_R(\mathbf{s}) = \sum_{j \in I_n} m_j(\mathbf{s})\omega_j$. Thus R has the $\mathfrak{g}$-structure property if and only if for each simple root α_i we have

Figure 3.3: $L^{\triangle}$ for the edge-colored lattice L from Figure 2.1.
Here regard L to be edge-colored by the nodes of A_2.



$wt_R(\mathbf{s}) + \alpha_i = wt_R(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in R. This condition depends not only on $\mathfrak{g}$ (information from the corresponding Dynkin diagram) but also on the combinatorics of R.

Let us temporarily assume only that (Γ, M) is a GCM graph with nodes indexed by I_n. If R is a ranked poset with edges colored by the set I_n, then the edge-coloring function $\mathbf{edgecolor}_R : \mathcal{E}(R) \to I_n$ is *sufficiently surjective* if for each connected component of (Γ, M) there is a node γ_i and an edge $\mathbf{s} \to \mathbf{t}$ with $\mathbf{edgecolor}_R(\mathbf{s} \to \mathbf{t}) = i$. The following results are from [Don8].

Theorem 3.13 *Let $M = (M_{i,j})_{i,j \in I_n}$ be a real matrix. (1) If there is a diamond-colored distributive lattice L with surjective edge-coloring function $\mathbf{edgecolor}_L : \mathcal{E}(L) \to I_n$ and having the M-structure property, then M must be a generalized Cartan matrix. (2) Suppose (Γ, M) is a GCM graph with nodes indexed by I_n. Suppose R is a ranked poset with sufficiently surjective edge-coloring function $\mathbf{edgecolor}_R : \mathcal{E}(R) \to I_n$. If R has the M-structure property, then $\mathbf{edgecolor}_R$ is surjective and (Γ, M) is a Dynkin diagram of finite type.*

Since our proof of part (2) of this theorem applies results from [Don6] concerning the so-called 'numbers game', we defer the proof of part (2) until §3.11.

Proof of Theorem 3.13.1. For (1), let $i \in I_n$ and choose an edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in L. Then for any $j \in I_n$ we have $m_j(\mathbf{s}) + M_{ij} = m_j(\mathbf{t})$. Since $m_j(\mathbf{s})$ and $m_j(\mathbf{t})$ are integers, it follows that M_{ij} is an integer. Since $\rho_i(\mathbf{s}) + 1 = \rho_i(\mathbf{t})$ and $\delta_i(\mathbf{s}) - 1 = \delta_i(\mathbf{t})$, then from $m_i(\mathbf{s}) + M_{ii} = m_i(\mathbf{t})$ it follows that $M_{ii} = 2$.

Pick $i, j \in I_n$ with $i \neq j$. First, suppose there is an $\{i, j\}$-component K in L which has at least one edge of color i and at least one of color j. By Theorem 2.3 and Proposition 2.8 we may write $K = \mathrm{J}_{color}(Q)$ for $Q = \mathrm{j}_{color}(K)$. Let d_i count the number of color i vertices in Q. Similarly define d_j. Since K has both color i and color j edges, d_i and d_j are both positive. Let $\mathbf{x}$ be the unique maximal element of K and let $\mathbf{y}$ be the unique minimal element. Then

$$wt_L(\mathbf{y}) + d_i\alpha_i + d_j\alpha_j = wt_L(\mathbf{x}).$$

In particular

$$m_i(\mathbf{y}) + d_iM_{ii} + d_jM_{ji} = m_i(\mathbf{x}) \qquad \text{and} \qquad m_j(\mathbf{y}) + d_iM_{ij} + d_jM_{jj} = m_j(\mathbf{x}).$$

Then $d_jM_{ji} = (-m_i(\mathbf{y}) - d_i) + (m_i(\mathbf{x}) - d_i)$ and $d_iM_{ij} = (m_j(\mathbf{x}) - d_j) + (-m_j(\mathbf{y}) - d_j)$. Since $m_i(\mathbf{x}) = l_i(\mathbf{x}) \leq d_i$, $m_j(\mathbf{x}) = l_j(\mathbf{x}) \leq d_j$, $-m_i(\mathbf{y}) = l_i(\mathbf{y}) \leq d_i$, and $-m_j(\mathbf{y}) = l_j(\mathbf{y}) \leq d_j$, then we see that $(m_i(\mathbf{x}) - d_i) + (-m_i(\mathbf{y}) - d_i) \leq 0$ and $(m_j(\mathbf{x}) - d_j) + (-m_j(\mathbf{y}) - d_j) \leq 0$. Hence $d_jM_{ji} \leq 0$ and $d_iM_{ij} \leq 0$. Since d_i and d_j are both positive, then $M_{ij} \leq 0$ and $M_{ji} \leq 0$.

Suppose that in this situation, $M_{ij} = 0$. Then $(m_j(\mathbf{x}) - d_j) + (-m_j(\mathbf{y}) - d_j) = 0$, and hence $m_j(\mathbf{x}) = d_j = l_j(\mathbf{x})$ and $-m_i(\mathbf{y}) = d_j = l_j(\mathbf{y})$. In particular, starting at the order ideal $\mathbf{y}$ in L, it is possible to add to $\mathbf{y}$ d_j color j vertices to get an order ideal $\mathbf{z}$ that is a vertex in K. At this point, $\mathbf{z}$ must be the minimum vertex of the color i component containing $\mathbf{x}$, and further we must be able to add d_i color i vertices to $\mathbf{z}$ to get $\mathbf{x}$. In particular, $m_i(\mathbf{x}) = d_i = l_i(\mathbf{x})$. Similarly from $\mathbf{x}$ we can remove d_j color j vertices to get an order ideal $\mathbf{w}$ in K that is the maximal vertex in the i-component of $\mathbf{y}$. In the same way as before we get $-m_i(\mathbf{y}) = d_i = l_i(\mathbf{y})$. So $(-m_i(\mathbf{y}) - d_i) + (m_i(\mathbf{x}) - d_i) = 0 = d_jM_{ji}$, so $M_{ji} = 0$.

At this point we know that if there is an $\{i, j\}$-component in L that has edges of both colors i and j, then $M_{ij} \leq 0$, $M_{ji} \leq 0$, and $M_{ij} = 0$ if and only if $M_{ji} = 0$. So now suppose that every $\{i, j\}$-component in L uses at most one of the colors i or j. Pick an edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in L. Since neither $\mathbf{s}$ nor $\mathbf{t}$ has an incident edge of color j, then $m_j(\mathbf{t}) = m_j(\mathbf{s}) = 0$. But $m_j(\mathbf{t}) = m_j(\mathbf{s}) + M_{ij}$, so therefore $M_{ij} = 0$. By looking at an edge of color j in L one can similarly conclude that $M_{ji} = 0$.

We thus conclude that M is a matrix of integers with $M_{ii} = 2$ for all $i \in I_n$, $M_{ij} \leq 0$ for all $i \neq j$ in I_n, and $M_{ij} = 0$ if and only if $M_{ji} = 0$. That is, M is a generalized Cartan matrix. $\square$

Combining both parts of the previous theorem we obtain:

Corollary 3.14 *Let $M = (M_{i,j})_{i,j \in I_n}$ be a real matrix. If there is a diamond-colored distributive lattice L with surjective edge-coloring function* $\mathbf{edgecolor}_L : \mathcal{E}(L) \to I_n$ *and having the M-structure property, then M must be a Cartan matrix.*

It is important to note that the condition "M is a Cartan matrix" in this corollary is necessary but not sufficient for there to be an M-structured diamond-colored distributive lattice.

Now return to the assumption that M is a Cartan matrix and $\mathfrak{g} = (\Gamma, M)$ is a Dynkin diagram. For a $\mathfrak{g}$-structured diamond-colored distributive lattice L, let λ be the weight of the unique maximal vertex of L. We say L is a $(\mathfrak{g}, \lambda)$-*structured* distributive lattice. A concept to be introduced in Chapter 3 (the 'distributive core') relates directly to the following question: For which Dynkin diagrams $\mathfrak{g}$ and weights λ is there a $(\mathfrak{g}, \lambda)$-structured distributive lattice? When a $(\mathfrak{g}, \lambda)$-structured distributive lattice exists, we have the following result concerning its unique minimal element. As this result can be demonstrated using facts about the 'numbers game' as in [Don6], we defer the proof to §3.11.

Proposition 3.15 *Let R be an M-structured poset with a unique maximal element of weight λ, necessarily dominant. Then R has a minimal element of weight $w_0.\lambda$. In particular, if L is a $(\mathfrak{g}, \lambda)$-structured distributive lattice for some dominant weight λ, then the unique minimal element of L has weight $w_0.\lambda$.*

§3.9 Weyl characters. See [Hum1], [FH], or [Stem3] for discussions of the basic theory of Weyl characters, which we outline here without much reference to Lie representation theory. Observe that Λ is an abelian subgroup of V. Let $\mathbb{Z}[\Lambda]$ be the group ring over Λ: that is, $\mathbb{Z}[\Lambda]$ consists of finite integral linear combinations of elements of the basis $\{e_\mu \,|\, \mu \in \Lambda\}$. Multiplication in $\mathbb{Z}[\Lambda]$ is given by $e_\mu e_\nu = e_{\mu+\nu}$. We sometimes use 1 to denote e_0. The Weyl group $\mathcal{W}$ acts on $\mathbb{Z}[\Lambda]$ by the rule $w.e_\mu := e_{w.\mu}$. The *character ring* $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ for $\mathfrak{g}$ is the ring of $\mathcal{W}$-invariant elements of $\mathbb{Z}[\Lambda]$; elements of $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ are *characters for* $\mathfrak{g}$. For any weight μ, let $A_\mu := \sum_{w \in \mathcal{W}} \det(\phi(w)) e_{w.\mu}$. Using the fact that $S_i = \phi(s_i)$ is a reflection and hence $\det(S_i) = -1$, it follows that $s_i.A_\mu = -A_\mu$. So, A_μ is not in the character ring. Let $\varrho := \omega_1 + \cdots + \omega_n$, the sum of the fundamental weights. Part (1) of the following well-known theorem is the famous Weyl character formula, due to H. Weyl.

Theorem 3.16 (Weyl) *(1) For each dominant $\lambda \in \Lambda^+$, there exists a unique $\chi_\lambda \in \mathbb{Z}[\Lambda]$ such that $A_\varrho \chi_\lambda = A_{\varrho+\lambda}$, and moreover $\chi_\lambda \in \mathbb{Z}[\Lambda]^{\mathcal{W}}$. (2) The characters $\{\chi_\lambda\}_{\lambda\in\Lambda^+}$ are a basis for the character ring $\mathbb{Z}[\Lambda]^{\mathcal{W}}$. (3) The characters $\{\chi_{\omega_i}\}_{i\in I_n}$ are an algebraic basis for the character ring $\mathbb{Z}[\Lambda]^{\mathcal{W}}$.*

Weyl characters are nonnegative integral linear combinations of the characters $\{\chi_\lambda\}_{\lambda\in\Lambda^+}$. Elements of the basis $\{\chi_\lambda\}_{\lambda\in\Lambda^+}$ for the character ring are *irreducible Weyl characters*, and elements of $\{\chi_{\omega_i}\}_{i\in I_n}$ are *fundamental characters*. At times we use the nomenclature '$\mathfrak{g}$-character' to emphasize the connection to the Dynkin diagram $\mathfrak{g}$. For each $i \in I_n$, set $z_i := e_{\omega_i}$. If $\mu = \sum m_i\omega_i \in \Lambda$, set $z^\mu := z_1^{m_1}\cdots z_n^{m_n}$. Then for any $\lambda \in \Lambda^+$ we can write $\chi_\lambda = \displaystyle\sum_{\mu\in\Lambda} c_{\lambda,\mu} z^\mu$ for some $c_{\lambda,\mu} \in \mathbb{Z}$. So we can think of an irreducible Weyl character as a Laurent polynomial in the variables $\{z_i\}_{i\in I_n}$. At times we will emphasize this viewpoint by writing $\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)$ in place of χ_λ. The following facts about irreducible Weyl characters can be proved using the representation theory of semisimple Lie algebras. We record these here for future use.

Facts 3.17 *Keep the notation of the previous paragraph. (1) Each coefficient $c_{\lambda,\mu}$ is nonnegative. (2) Moreover, $c_{\lambda,\lambda} = 1$ and $c_{\lambda,w_0.\lambda} = 1$. (3) Partially order the set $\Pi(\lambda) := \{\mu \in \Lambda \,|\, c_{\lambda,\mu} \neq 0\}$ by the rule $\mu \leq \nu$ if and only if $\nu - \mu = \sum k_i\alpha_i$ with each $k_i \geq 0$. Then $\Pi(\lambda)$ is a connected ranked poset with unique maximal element λ and unique minimal element $w_0.\lambda$. (4) Moreover, $\mu \to \nu$ in $\Pi(\lambda)$ if and only if $\mu + \alpha_i = \nu$ for some simple root α_i. Therefore by giving each such edge $\mu \to \nu$ the color $i \in I_n$ of the appropriate simple root α_i, $\Pi(\lambda)$ is a $\mathfrak{g}$-structured poset.*

For example, to see that each coefficient $c_{\lambda,\mu}$ is nonnegative, one observes that $c_{\lambda,\mu}$ counts the dimension of a certain subspace of the highest weight λ irreducible representation of the corresponding semisimple Lie algebra. Subsequently one can see that if we evaluate $\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)$ at $z_1 = \cdots = z_n = 1$ we obtain the number $\sum_{\mu\in\Lambda} c_{\lambda,\mu}$, which is the dimension of the representing space. For this reason we will refer to the nonnegative integer $\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)|_{z_1=\cdots=z_n=1}$ as the *dimension* of χ_λ. More generally, the dimension of a Weyl character $\chi = \sum_{\lambda\in\Lambda^+} m_\lambda\chi_\lambda$ is the nonnegative integer $\sum_{\lambda\in\Lambda^+} m_\lambda \mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)|_{z_1=\cdots=z_n=1}$.

Example 3.18: Adjoint characters. Assume $\mathfrak{g}$ is connected. The highest long root $\overline{\omega}$ and the highest short root $\overline{\omega}_{short}$ are dominant weights. From [Don5] (for example) it follows that $\chi_{\overline{\omega}} = n\, e_0 + \sum_{\alpha\in\Phi} e_\alpha$ and that $\chi_{\overline{\omega}_{short}} = m\, e_0 + \sum_{\alpha\in\Phi_{short}} e_\alpha$, where m is the number of short simple roots. To see that these are both in the character ring, it suffices to observe that $\mathcal{W}$ permutes Φ (resp. Φ_{short}). We call $\chi_{\overline{\omega}}$ and $\chi_{\overline{\omega}_{short}}$ the *adjoint* and *short adjoint* characters, respectively. $\square$

§3.10 Our main goal: 'splitting posets' as combinatorial models for Weyl characters. Let R be a ranked poset with edges colored by the set I_n. We say R is a *splitting poset* for a Weyl character χ if (1) R has the $\mathfrak{g}$-structure property and (2) the weight-generating function on R is the Weyl character χ in the following sense: $\chi = \sum_{\mathbf{t}\in R} z^{wt_R(\mathbf{t})}$. If R is a diamond-colored distributive lattice, then we say R is a *splitting distributive lattice* or *SDL*. The following is from Lemma 2.2 of [ADLMPPW].

Lemma 3.19 *Let $\lambda = \sum m_i\omega_i$ be dominant in the lattice of weights for $\mathfrak{g}$. Suppose R is a splitting poset for χ_λ. Then the dual R^* is a splitting poset for the irreducible $\mathfrak{g}$-Weyl character $\chi_{-w_0.\lambda}$. Given a one-to-one function $\sigma : I_n \to I_n$, the recolored poset R^σ is a splitting poset for the irreducible $\mathfrak{g}^\sigma$-Weyl character $\chi_{\sum m_i\omega_{\sigma(i)}}$. The σ_0-recolored dual $R^\triangle$ is also a splitting poset for the irreducible $\mathfrak{g}$-Weyl character χ_λ.*

If R is a connected splitting poset for an irreducible Weyl character χ_λ, then by Facts 3.17, R has a unique vertex **max** (respectively **min**) of maximal (resp. minimal) rank, and moreover we have $wt_R(\mathbf{max}) = \lambda$ and $wt_R(\mathbf{min}) = w_0.\lambda$. Set $\varrho^\vee := \sum_{i=1}^n \frac{2\omega_i}{\langle\alpha_i,\alpha_i\rangle}$. Observe that $\langle\alpha_i, \varrho^\vee\rangle = 1$ for $1 \le i \le n$. Using the vertices **max** and **min**, one now sees that the length of R is $\langle wt_R(\mathbf{max}) - wt_R(\mathbf{min}), \varrho^\vee\rangle = \langle\lambda - w_0.\lambda, \varrho^\vee\rangle$. This observation helps explain the appearance of the scaling factor $q^{-\langle w_0.\lambda,\varrho^\vee\rangle}$ in the next proposition, which shows how the rank generating function $RGF(R, q)$ for such a splitting poset R is obtained as a specialization of the irreducible Weyl character χ_λ.

Proposition 3.20 *Let R be a connected splitting poset for the irreducible Weyl character χ_λ. Then its rank generating function $RGF(R, q)$ can be obtained by specializing the Weyl character as follows:*

$$RGF(R, q) = q^{-\langle w_0.\lambda,\varrho^\vee\rangle}\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \dots, z_n)\big|_{z_i:=q^{\langle\omega_i,\varrho^\vee\rangle}}.$$

Proof. We use the notation of the paragraph preceding the proposition statement. Let $\mathbf{t} \in R$. Since R is connected, there is a path $\mathcal{P}$ from **min** to $\mathbf{t}$ in R. By applying the M-structure property along the edges of $\mathcal{P}$, we obtain that $\rho(\mathbf{t}) = \langle wt_R(\mathbf{t}) - wt_R(\mathbf{min}), \varrho^\vee\rangle$. In the computations that follow we use the fact that $wt_R(\mathbf{min}) = w_0.\lambda$.

$$\begin{aligned} RGF(R, q) = \sum_{\mathbf{t}\in R} q^{\rho(\mathbf{t})} &= \sum_{\mathbf{t}\in R} q^{\langle wt_R(\mathbf{t}) - wt_R(\mathbf{min}),\varrho^\vee\rangle} \\ &= q^{-\langle w_0.\lambda,\varrho^\vee\rangle}\sum_{\mathbf{t}\in R} q^{\langle wt_R(\mathbf{t}),\varrho^\vee\rangle} \\ &= q^{-\langle w_0.\lambda,\varrho^\vee\rangle}\sum_{\mathbf{t}\in R} q^{m_1(\mathbf{t})}q^{m_2(\mathbf{t})}\cdots q^{m_n(\mathbf{t})} \end{aligned}$$

$$= q^{-\langle w_0.\lambda, \varrho^\vee\rangle} \sum_{\mathbf{t}\in R} (q^{\langle \omega_1, \varrho^\vee\rangle})^{m_1(\mathbf{t})} (q^{\langle \omega_2, \varrho^\vee\rangle})^{m_2(\mathbf{t})} \cdots (q^{\langle \omega_n, \varrho^\vee\rangle})^{m_n(\mathbf{t})}$$
$$= q^{-\langle w_0.\lambda, \varrho^\vee\rangle} \mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)\big|_{z_i := q^{\langle \omega_i, \varrho^\vee\rangle}}.$$

This completes the proof. $\square$

In view of this result, we will use $\ell(\lambda)$ to denote the length $\langle \lambda - w_0.\lambda, \varrho^\vee\rangle$ of any connected splitting poset for χ_λ. The following result (appearing as Proposition 2.4 in [ADLMPPW], based on Proctor's work in Section 6 of [Pro3] with the M-structure poset context contributed by Donnelly) shows that connected splitting posets for irreducible Weyl characters have certain salient combinatorial features.

Theorem 3.21 *Let R be a connected splitting poset for the irreducible Weyl character χ_λ. Then R is rank symmetric, rank unimodal, and has rank generating function*

$$RGF(R, q) = \prod_{\alpha\in\Phi^+} \frac{1 - q^{\langle \lambda+\varrho, \alpha^\vee\rangle}}{1 - q^{\langle \varrho, \alpha^\vee\rangle}}$$

Letting $q \to 1$ in the above expression gives:

Corollary 3.22 (Weyl Dimension Formula) *The dimension of χ_λ is*

$$\prod_{\alpha\in\Phi^+} \frac{\langle \lambda + \varrho, \alpha^\vee\rangle}{\langle \varrho, \alpha^\vee\rangle}$$

Calculating the difference of the degrees of the numerator and denominator polynomials in Theorem 3.21 gives:

Corollary 3.23 *The length of any connected splitting poset for χ_λ is*

$$\ell(\lambda) = \sum_{\alpha\in\Phi^+} \langle \lambda, \alpha^\vee\rangle.$$

A crucial question at this point is: How does one obtain splitting posets? At present there are three general strategies. (1) *Impose 'natural' partial orders on combinatorial objects known to generate Weyl characters.* For example, the 'Littelmann' family of G_2-lattices shown in [Mc] to be SDL's for the irreducible G_2-characters were discovered by Donnelly by imposing a natural partial order on Littelmann's G_2 tableaux [Lit]. (2) *Apply Stembridge's product construction [Stem3].* For a given dominant weight λ, any resulting 'admissible system' is a 'minimal' splitting poset in the sense that it will not contain as a proper edge-colored subgraph a splitting poset for χ_λ. Further, one can sometimes show a given M-structure poset R is a splitting poset for χ_λ by locating an admissible system inside R as an edge-colored subgraph. This method is being employed right now by Alverson, Donnelly, Lewis, and Pervine to give another proof that the 'semistandard' lattices of [ADLMPPW] are SDL's for the irreducible Weyl characters for A_2, C_2, and G_2. (3) *Show that a given* $\mathfrak{g}$*-structured poset is a 'supporting graph' (cf. [Don4]) for a representation of the corresponding semisimple Lie algebra.* This method has been used in [Don3], [Don4], [Don5], and [DLP1] to produce/study many families of SDL's.

Example 3.24: The maximal splitting poset. Given an irreducible Weyl character $\chi_{_\lambda}$, consider the set of weights $\Pi(\lambda)$. By Facts 3.17, we may regard $\Pi(\lambda)$ as a ranked poset with edges colored by I_n, where $\mu \stackrel{i}{\rightarrow} \nu$ if and only if $\mu + \alpha_i = \nu$. We use $\Pi(\lambda)$ as the foundation for a new edge-colored ranked poset $\mathcal{M}(\lambda)$. As a set, we have

$$\mathcal{M}(\lambda) := \bigcup_{\mu \in \Pi(\lambda)} \left\{\mu^{(1)}, \ldots, \mu^{(c_{\lambda,\mu})}\right\},$$

where we have essentially extended each weight μ in $\Pi(\lambda)$ to a multiset of elements with weight μ using the coefficients $c_{\lambda,\mu}$. For $\mu^{(p)}$ and $\nu^{(q)}$ in $\mathcal{M}(\lambda)$, write $\mu^{(p)} \stackrel{i}{\rightarrow} \nu^{(q)}$ if and only if $\mu \stackrel{i}{\rightarrow} \nu$ in $\Pi(\lambda)$. In [Don4] it is observed that $\mathcal{M}(\lambda)$ is a supporting graph for the highest weight λ irreducible representation of the corresponding semisimple Lie algebra. In particular, $\mathcal{M}(\lambda)$ is a splitting poset for $\chi_{_\lambda}$. But this latter fact is easy enough to see directly from the definitions and Facts 3.17. It can be seen that $\mathcal{M}(\lambda)$ contains an isomorphic image of any other splitting poset R for $\chi_{_\lambda}$ as a weak subposet. In effect, such an R has the same vertices as $\mathcal{M}(\lambda)$ but only a subset of its edges. We call $\mathcal{M}(\lambda)$ the *maximal splitting poset* for $\chi_{_\lambda}$. $\square$

Example 3.25: Splitting posets for adjoint characters. Let $\mathfrak{g}$ be connected. Define $\mathcal{A}$ to be the set $\{(i,j) | \langle \alpha_i, \alpha_j^\vee \rangle < 0\}_{i,j \in I_n}$ modulo the equivalence $(i,j) \equiv (j,i)$. For $k \in I_n$ set $\mathcal{A}^{(k)} := \mathcal{A} \cup \{(k,k)\}$, so $|\mathcal{A}^{(k)}| = n$. Let $L^{(k)}$ be the set $\Phi^+ \cup \mathcal{A}^{(k)} \cup \Phi^-$. Place directed edges with colors from the set I_n between the elements of $L^{(k)}$ as follows: Write $\alpha \stackrel{i}{\rightarrow} \beta$ if α and β are both roots in Φ^+ (or are both in Φ^-) and $\alpha + \alpha_i = \beta$. For each pair (i,j) in $\mathcal{A}^{(k)}$, include edges $-\alpha_r \stackrel{r}{\rightarrow} (i,j) \stackrel{r}{\rightarrow} \alpha_r$ if and only if $r = i$ or $r = j$. It is a consequence of Facts 3.7 that $L^{(k)}$ is the Hasse diagram for a ranked poset. We call $\mathcal{A}^{(k)}$ the *middle rank* of $L^{(k)}$. For reasons explained by Theorem 1.2 of [Don5], we call $L^{(k)}$ the *kth extremal* splitting poset for the adjoint character $\chi_{_\varpi}$ for $\mathfrak{g}$. In that paper it is shown in Proposition 6.1 that the $L^{(k)}$'s are precisely the modular lattice supporting graphs for graphs for the adjoint representation of the simple Lie algebra $\mathfrak{g}$. It follows that each $L^{(k)}$ is a splitting modular lattice for the adjoint character $\chi_{_\varpi}$ for $\mathfrak{g}$ (cf. Example 3.18). In [Don5] Corollary 6.2, it is also observed that an extremal splitting poset $L^{(k)}$ is a distributive lattice if and only if $\mathfrak{g}$ is one of A_n, B_n, C_n, F_4, or G_2 and γ_k is one of the end nodes for $\mathfrak{g}$.

There are similar objects for the short adjoint characters $\chi_{\varpi_{short}}$, cf. Example 3.18, [Don5]. Modify the constructions of the previous paragraph using only Φ_{short}. This results in splitting modular lattices $L^{(k)}_{short}$ for the short adjoint character, where each index $k \in I_n$ is such that α_k is short. As with the extremal splitting posets for the adjoint characters, we see that $L^{(k)}_{short}$ is a distributive lattice if and only if $\mathfrak{g}$ is one of A_n, B_n, C_n, F_4, or G_2 and α_k is a short simple root with at most one adjacent short simple root in the Dynkin diagram $\mathfrak{g}$. $\square$

§3.11 The numbers game and computations related to Weyl groups and roots systems. This subsection applies recent results of [Don6] in studying the combinatorial 'numbers game' of Mozes [Moz] and Eriksson [Erik1], [Erik2], [Erik3].

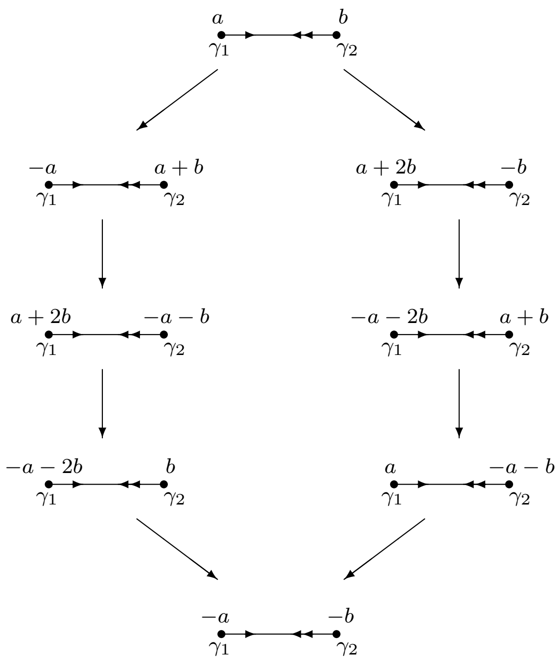
For the next two paragraphs, temporarily relax the finiteness hypothesis for $\mathcal{W} = \mathcal{W}_{\mathfrak{g}}$. For the game we describe next, a *position* λ is an assignment of numbers $(\lambda_i)_{i \in I_n}$ to the nodes of the GCM graph $\mathfrak{g} = (\Gamma, M)$. As with weights, say the position λ is *dominant* (respectively, *strongly dominant*) if $\lambda_i \geq 0$ (respectively $\lambda_i > 0$) for all $i \in I_n$; λ is *nonzero* if at least one $\lambda_i \neq 0$. Given a position λ on a GCM graph (Γ, M), to *fire* a node γ_i is to change the number at each node γ_j of Γ by the transformation

$$\lambda_j \longmapsto \lambda_j - M_{ij}\lambda_i,$$

provided the number at node γ_i is positive; otherwise node γ_i is not allowed to be fired. The *numbers game* is the one-player game on a GCM graph (Γ, M) in which the player (1) Assigns an initial position to the nodes of Γ; (2) Chooses a node with a positive number and fires the node to obtain a new position; and (3) Repeats step (2) for the new position if there is at least one node with a positive number.

Consider the GCM graph C_2. As we can see in Figure 3.4, the numbers game terminates in a finite number of steps for any initial position and any legal sequence of node firings, if it is understood that the player will continue to fire as long as there is at least one node with a positive number. In general, given a position λ, a *game sequence for* λ is the (possibly empty, possibly infinite) sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$, where γ_{i_j} is the jth node that is fired in some numbers game with initial position λ. More generally, a *firing sequence* from some position λ is an initial portion of some game sequence played from λ; the phrase *legal firing sequence* is used to emphasize that all node firings in the sequence are known or assumed to be possible. Note that a game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$ is of finite length l (possibly with $l = 0$) if the number is nonpositive at each node after the lth firing; in this case we say the game sequence is *convergent* and the resulting position is

Figure 3.4: The numbers game for the Dynkin diagram C_2.



the *terminal position* for the game sequence. We say a connected GCM graph (Γ, M) is *admissible* if there exists a nonzero dominant initial position with a convergent game sequence. Theorem 6.1 of [Don6] shows that a connected GCM graph is admissible if and only if it is a connected Dynkin diagram of finite type. In these cases, for any given initial position every game sequence will converge to the same terminal position in the same finite number of steps.

Return now to the assumption that $\mathcal{W} = \mathcal{W}_{\mathfrak{g}}$ is finite. The moves of the numbers game relate directly to the Euclidean representation $\phi : \mathcal{W}_{\mathfrak{g}} \to GL(V, \langle\cdot,\cdot\rangle)$, cf. §3.5. To see this, view a position $\lambda = (\lambda_i)_{i\in I_n}$ on $\mathfrak{g}$ as the weight $\sum \lambda_i\omega_i$. Now observe that firing node γ_i from weight λ on $\mathfrak{g}$ results in position $\phi(s_i)(\lambda)$: At each $j \in I_n$, $\langle s_i.\lambda, \alpha_j^\vee\rangle = \sum \lambda_k\langle\omega_k, \alpha_j^\vee\rangle - \lambda_i\langle\alpha_i, \alpha_j^\vee\rangle = \lambda_j - M_{ij}\lambda_i$. It follows from Eriksson's Reduced Word Result (see Theorem 2.8 of [Don6]) that $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$ is a game sequence for a numbers game played on $\mathfrak{g}$ from any given strongly dominant initial position if and only if $s_{i_l}\cdots s_{i_2}s_{i_1}$ is a reduced expression for w_0, the longest element of $\mathcal{W}$. For the rest of this subsection, let $s_{i_l}s_{i_{l-1}}\cdots s_{i_1}$ be a fixed reduced expression for w_0. The next result is an immediate application of Theorem 5.2 of [Don6] concerning the positive roots Φ^+.

Theorem 3.26 *For $1 \le j \le l$, set $\beta_j := s_{i_1}s_{i_2}\cdots s_{i_{j-1}}.\alpha_{i_j}$. Then $|\{\beta_j\}_{j=1}^l| = l$ and $\{\beta_j\} = \Phi^+$.*

Corollary 3.27 *Keep the notation of Theorem 3.26. Let $\lambda = \sum \lambda_i\omega_i \in \Lambda^+$. For $1 \le j \le l$ let c_j be the number at the i_jth node when we play the legal firing sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{j-1}})$ from the* *initial position* $(\lambda_i+1)_{i\in I_n}$ *on the Dynkin diagram* $\mathfrak{g}$. *Then* $\langle\lambda+\varrho,\beta_j^\vee\rangle = c_j$. *Moreover,*

$$\prod_{\alpha\in\Phi^+}(1-q^{\langle\lambda+\varrho,\alpha^\vee\rangle}) = \prod_{j=1}^{l}(1-q^{c_j}) \qquad \textit{and} \qquad \prod_{\alpha\in\Phi^+}\langle\lambda+\varrho,\alpha^\vee\rangle = \prod_{j=1}^{l} c_j.$$

Proof. It follows from Lemma 3.5 above that $\beta_j^\vee = s_{i_1}s_{i_2}\cdots s_{i_{j-1}}.\alpha_{i_j}^\vee$. We now have $\langle\lambda+\varrho,\beta_j^\vee\rangle = \langle\lambda+\varrho, s_{i_1}s_{i_2}\cdots s_{i_{j-1}}.\alpha_{i_j}^\vee\rangle = \langle s_{i_{j-1}}\cdots s_{i_2}s_{i_1}.(\lambda+\varrho),\alpha_{i_j}^\vee\rangle$, which is the i_jth coordinate of $s_{i_{j-1}}\cdots s_{i_2}s_{i_1}.(\lambda+\varrho)$. That is, $\langle\lambda+\varrho,\beta_j^\vee\rangle = c_j$. By the preceding theorem, for each positive root α there is one and only one j such that $\alpha=\beta_j$. Then we can index the products over the positive roots using $j=1,\ldots,l$ instead, which completes the proof. □

Proposition 3.28 *Keep the notation of Theorem 3.26. Assume* $\mathfrak{g}$ *is connected. Consider the transpose Euclidean representation* $\phi^{\mathsf{T}} : \mathcal{W}_{\mathfrak{g}} \to GL(V^{\mathsf{T}},\langle\cdot,\cdot\rangle_{\mathsf{T}})$ *as in §3.5, with simple roots* $\{\alpha_i^{\mathsf{T}}\}_{i\in I_n}$ *and fundamental weights* $\{\omega_i^{\mathsf{T}}\}_{i\in I_n}$. *Suppose* $\beta_j = s_{i_1}s_{i_2}\cdots s_{i_{j-1}}.\alpha_{i_j} = \sum k_i\alpha_i \in \Phi^+$ *is short (resp. long). Let* $\beta_j^{\mathsf{T}} := s_{i_1}s_{i_2}\cdots s_{i_{j-1}}.\alpha_i^{\mathsf{T}}$, *a root in* Φ^{T}, *with* $(\beta_j^{\mathsf{T}})^\vee = s_{i_1}s_{i_2}\cdots s_{i_{j-1}}.(\alpha_i^{\mathsf{T}})^\vee$ *the corresponding root in* $(\Phi^{\mathsf{T}})^\vee$, *cf. Lemma 3.5. (1) Then* $(\beta_j^{\mathsf{T}})^\vee$ *is positive and short (resp. long). (2) For a strongly dominant weight* $\mu=\sum\mu_i\omega_i^{\mathsf{T}}$, *let* d_j *denote the number at the* i_j*th node after playing the legal sequence* $(\gamma_{i_1},\gamma_{i_2},\ldots,\gamma_{i_{j-1}})$ *from initial position* $(\mu_i)_{i\in I_n}$ *on the transpose graph* $\mathfrak{g}^{\mathsf{T}}$. *Then* $\langle\mu,(\beta_j^{\mathsf{T}})^\vee\rangle_{\mathsf{T}} = d_j = \sum k_i\mu_i$.

Proof. By Proposition 3.8, $\mathcal{R}(\mathfrak{g},\phi^{\mathsf{T}},\{(\alpha_i^{\mathsf{T}})^\vee\}_{i\in I_n})$ is isomorphic to $\mathcal{R}(\mathfrak{g},(\phi^{\mathsf{T}})^{\mathsf{T}},\{(\alpha_i^{\mathsf{T}})^{\mathsf{T}}\}_{i\in I_n})$ via the identity bijection on I_n, which in turn is isomorphic to $\mathcal{R}(\mathfrak{g},\phi,\{\alpha_i\}_{i\in I_n})$ via the identity bijection on I_n. So given $\beta_j=\sum k_i\alpha_i$, it follows that $(\beta_j^{\mathsf{T}})^\vee = \sum k_i(\alpha_i^{\mathsf{T}})^\vee$. We see that $(\beta_j^{\mathsf{T}})^\vee$ is positive. From Proposition 3.8, it follows that $(\beta_j^{\mathsf{T}})^\vee$ is also short (resp. long). Now $\langle s_{i_{j-1}}\cdots s_{i_2}s_{i_1}.\mu,(\alpha_{i_j}^{\mathsf{T}})^\vee\rangle_{\mathsf{T}} = \langle\mu,(\beta_j^{\mathsf{T}})^\vee\rangle_{\mathsf{T}} = \langle\sum\mu_i\omega_i^{\mathsf{T}},\sum k_i(\alpha_i^{\mathsf{T}})^\vee\rangle_{\mathsf{T}} = \sum k_i\mu_i$. From the paragraph preceding Theorem 3.26, we see that $s_{i_{j-1}}\cdots s_{i_2}s_{i_1}.\mu$ is the position resulting from the firing sequence $(\gamma_{i_1},\gamma_{i_2},\ldots,\gamma_{i_{j-1}})$ played from initial position $(\mu_i)_{i\in I_n}$ on the transpose graph $\mathfrak{g}^{\mathsf{T}}$. Then $d_j = \langle s_{i_{j-1}}\cdots s_{i_2}s_{i_1}.\mu,(\alpha_{i_j}^{\mathsf{T}})^\vee\rangle_{\mathsf{T}}$ is the number at the i_jth node. □

Remark 3.29 In view of the preceding results, the numbers game gives us simple iterative procedures for producing data concerning roots and Weyl group actions needed for example for the following computations. To compute the rank generating function of Theorem 3.21 above, observe that by Corollary 3.27 the exponents of the numerator and denominator in that formula are numbers appearing in a numbers game played from initial positions $(\lambda_i+1)_{i\in I_n}$ and $(1)_{i\in I_n}$ on $\mathfrak{g}$ respectively. In combination, Theorem 3.26 and Proposition 3.28 show that if we play a numbers game on $\mathfrak{g}^{\mathsf{T}}$ from a generic strongly dominant position $(\mu_i)_{i\in I_n}$, then any positive root $\beta=\sum k_i\alpha_i$

in Φ will appear exactly once as the expression $\sum k_i\mu_i$ at node γ_{i_j} when it is fired. By Proposition 3.8, β will be short (resp. long) if and only if $\alpha_{i_j}^{\top}$ is long (resp. short) if and only if α_{i_j} is short (resp. long). Finally, to compute the action of w_0 on V, start with a generic strongly dominant weight $\lambda = \sum \lambda_i\omega_i$ as an initial position on $\mathfrak{g}$ and play the game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$. The terminal position is $s_{i_l}\cdots s_{i_2}s_{i_1}.\lambda = w_0.\lambda$. But since $w_0.\lambda = -\sum \lambda_i\omega_{\sigma_0(i)}$, one can now deduce how σ_0 permutes the elements of I_n. These techniques are applied to G_2 in the next subsection.

We close this subsection with proofs of two results stated in §3.8.

Proof of Theorem 3.13.2. First assume (Γ, M) is connected. In this case it is only required that R has at lease one edge. Choose a vertex $\mathbf{t}_0$ for which $\lambda^{(0)} := wt_R(\mathbf{t}_0)$ is dominant. (For example, take $\mathbf{t}_0$ to be any element of highest rank in R.) Since R has at least one edge, $\lambda^{(0)}$ is nonzero. Let $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$ be any game sequence played from initial position $\lambda^{(0)}$ on (Γ, M). For each $p \geq 1$, $\lambda^{(p)}$ is the position in the sequence just after node γ_{i_p} is fired. Next, we define by induction a special sequence of elements from R. For any $p \geq 1$, suppose we have a sequence $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{p-1}$ for which $wt_R(\mathbf{t}_q) = \lambda^{(q)}$ and $\rho(\mathbf{t}_q) < \rho(\mathbf{t}_{q-1})$ for all $1 \leq q \leq p-1$. We wish to show that we can extend this sequence by an element $\mathbf{t}_p$ so that $wt_R(\mathbf{t}_p) = \lambda^{(p)}$ and $\rho(\mathbf{t}_p) < \rho(\mathbf{t}_{p-1})$. Take $\mathbf{t}_p$ to be any element of $\mathbf{comp}_{i_p}(\mathbf{t}_{p-1})$ for which $\rho_{i_p}(\mathbf{t}_p) = l_{i_p}(\mathbf{t}_{p-1}) - \rho_{i_p}(\mathbf{t}_{p-1})$. Since firing node γ_{i_p} in the given numbers game is legal from position $\lambda^{(p-1)}$, then $\lambda_{i_p}^{(p-1)} > 0$. But $\lambda_{i_p}^{(p-1)} = 2\rho_{i_p}(\mathbf{t}_{p-1}) - l_{i_p}(\mathbf{t}_{p-1})$. So, $\rho_{i_p}(\mathbf{t}_p) = l_{i_p}(\mathbf{t}_{p-1}) - \rho_{i_p}(\mathbf{t}_{p-1}) < \rho_{i_p}(\mathbf{t}_{p-1})$. It follows that $\rho(\mathbf{t}_p) < \rho(\mathbf{t}_{p-1})$. Since R satisfies the M-structure condition, then $wt_R(\mathbf{t}_p) = wt_R(\mathbf{t}_{p-1}) - \lambda_{i_p}^{(p-1)}\alpha_{i_p}$. But $\lambda^{(p-1)} = wt_R(\mathbf{t}_{p-1})$ and $\lambda^{(p)} = \lambda^{(p-1)} - \lambda_{i_p}^{(p-1)}\alpha_{i_p}$. In other words, $wt_R(\mathbf{t}_p) = \lambda^{(p)}$. So we have extended our sequence as desired. But since R is finite, any such sequence must also be finite. Hence the game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$ is convergent. Then by Theorem 6.1 of [Don6], (Γ, M) must be a Dynkin diagram of finite type. Since every node must be fired in a convergent game sequence for a numbers game played on a connected GCM graph (Lemma 2.6 of [Don6]), then it follows that $\mathbf{edgecolor}_R$ is surjective.

In the general case, pick a connected component (Γ', M') of (Γ, M), and let $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$. Now pick a J-component $\mathcal{C}$ of R such that $\mathcal{C}$ contains at least one edge whose color is from J. The previous paragraph implies that (Γ', M') is a connected Dynkin diagram of finite type and that for every color in J there is an edge in $\mathcal{C}$ having that color. Applying this reasoning to each connected component of (Γ, M), we see that (Γ, M) is a Dynkin diagram of finite type and that $\mathbf{edgecolor}_R$ is surjective. $\square$

Proof of Proposition 3.15. The second assertion of the proposition, which concerns $(\mathfrak{g},\lambda)$-structured distributive lattices, is an immediate consequence of the first assertion concerning M-structured posets with unique maximal elements. So we only need to prove the first assertion. Then let R be as in the first assertion of the proposition statement. Let $\mathbf{t}_0$ be the unique maximal element of R. Then $wt_R(\mathbf{t}_0) = \lambda$ is dominant. If λ is the trivial zero weight, then $R = \{\mathbf{t}_0\}$, an equality of sets. That is, $\mathbf{t}_0$ is also the unique minimal element of R. Note that then $w_0.\lambda = \lambda = wt_R(\mathbf{t}_0)$, so the proposition statement is true in this case.

Now suppose λ is nonzero. Proceeding exactly as in the previous proof, we can construct a sequence of elements of R and their associated weights corresponding to a numbers game played on $\mathfrak{g}$ from initial position λ. We get a game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_p})$ such that the corresponding numbers game positions are $\lambda^{(0)} = \lambda, \lambda^{(1)}, \ldots, \lambda^{(p)}$. We also have these positions as weights of certain vertices $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_p$ from R: $wt_L(\mathbf{t}_q) = \lambda^{(q)}$ for $1 \leq q \leq p$.

We now invoke results from §3 of [Don6]. Let λ be J^c-dominant, for $J \subseteq I_n$. We write $(w_0)_J$ for the longest element of $\mathcal{W}_J$ (the subgroup of $\mathcal{W}$ generated by $\{s_j\}_{j\in J}$), and $(w_0)^J$ denotes the minimal length coset representative of w_0, cf. §3 of [Don6]. Then by Corollary 3.4 of [Don6], we must have $(w_0)^J = s_{i_p} \cdots s_{i_2} s_{i_1}$. So the terminal position must be $(w_0)^J.\lambda$. But $(w_0)_J$ stablizes λ, so $w_0.\lambda = (w_0)^J (w_0)_J.\lambda = (w_0)^J.\lambda$ is the terminal position for the game. That is, $\lambda^{(p)} = wt_R(\mathbf{t}_p) = w_0.\lambda$.

Let $\Pi(R) := \{wt_R(\mathbf{s}) \,|\, \mathbf{s} \in R\}$. We claim that $\Pi(R) \subseteq \Pi(\lambda)$. That is, we claim that for all $\mathbf{s} \in R$, $wt_R(\mathbf{s}) \in \Pi(\lambda)$. (For a definition and some results about $\Pi(\lambda)$, see Facts 3.17 above and Chapter 13 of [Hum1].) We prove our claim by induction on the depth of elements in R. When $\delta(\mathbf{s}) = 0$, then $\mathbf{s} = \mathbf{t}_0$. Then $wt_R(\mathbf{s}) = \lambda \in \Pi(\lambda)$. For our induction hypothesis, assume that for some positive integer k and for all $\mathbf{x} \in R$ with $\delta(\mathbf{x}) < k$, it is the case that $wt_R(\mathbf{x}) \in \Pi(\lambda)$. If $\delta(\mathbf{s}) = k$, then $\mathbf{s}$ is not maximal in R, so $\mathbf{s} \xrightarrow{i} \mathbf{t}$ for some $\mathbf{t} \in R$. Let $\mathbf{u} \in \mathbf{comp}_i(\mathbf{s})$ such that $\delta_i(\mathbf{u}) = 0$. Let $\mu := wt_R(\mathbf{u})$. We have $0 < \delta_i(\mathbf{s}) \leq l_i(\mathbf{s})$, and $wt_R(\mathbf{s}) = \mu - \delta_i(\mathbf{s})\alpha_i$. By the induction hypothesis, $\mu \in \Pi(\lambda)$. Now $\langle \mu, \alpha_i^\vee \rangle = \rho_i(\mathbf{u}) - \delta_i(\mathbf{u}) = \rho(\mathbf{u}) = l_i(\mathbf{u}) = l_i(\mathbf{s})$. By Chapter 13 of [Hum1], we know that $\Pi(\lambda)$ is 'saturated'. In particular, this means that $\mu - p\alpha_i \in \Pi(\lambda)$ for all $0 \leq p \leq l_i(\mathbf{s})$. So $wt_R(\mathbf{s}) \in \Pi(\lambda)$. This completes the induction step, and the proof of our claim.

So now if $\mathbf{s} \xrightarrow{i} \mathbf{t}_p$ for some $\mathbf{s} \in R$, then $wt_R(\mathbf{s}) = wt_R(\mathbf{t}_p) - \alpha_i = w_0.\lambda - \alpha_i$. But $\Pi(R) \subseteq \Pi(\lambda)$, in which case $wt_R(\mathbf{s}) < w_0.\lambda$ in the partial order on $\Pi(\lambda)$. But by Facts 3.17, $w_0.\lambda$ is the unique minimal element in $\Pi(\lambda)$. So there can be no such $\mathbf{s}$. In particular, $\mathbf{t}_p$ is a minimal element of R with weight $w_0.\lambda$, as desired. $\square$

§3.12 An extended example: G_2. We now illustrate the main ideas of the preceding subsections with an example. We work with $\mathfrak{g} = \mathsf{G}_2$, which has Cartan matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ with inverse $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

$\boxed{\text{§3.2}}$ The Weyl group $\mathcal{W}$ is $\langle s_1, s_2 \,|\, s_1^2 = s_2^2 = (s_1s_2)^6 = (s_2s_1)^6 = \varepsilon\rangle$. This is easily seen to be the 12-element dihedral group. Its elements are $\{\varepsilon, s_1,\ s_2,\ s_1s_2,\ s_2s_1,\ s_1s_2s_1,\ s_2s_1s_2,\ s_1s_2s_1s_2,$ $s_2s_1s_2s_1,\ s_1s_2s_1s_2s_1,\ s_2s_1s_2s_1s_2,\ s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1\}$.

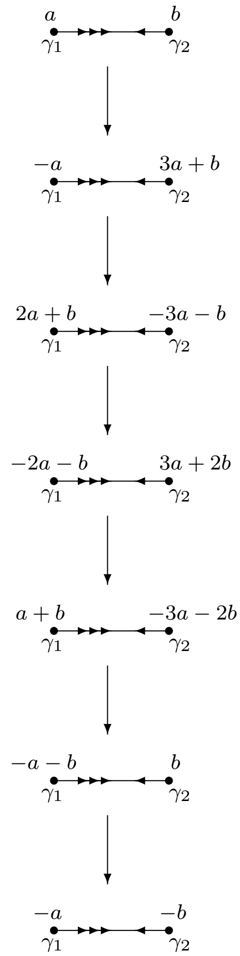
$\boxed{\text{§3.4}}$ Let α_1 and α_2 be simple roots for the $\mathcal{W}$-module $V = \mathrm{span}_{\mathbb{R}}(\alpha_1, \alpha_2)$. We have $s_i.\alpha_j = \alpha_j - M_{ji}\alpha_i$ for $i, j = 1, 2$. Set $\langle\alpha_1, \alpha_1\rangle = 2$. Then $\langle\alpha_2, \alpha_2\rangle = \frac{M_{21}}{M_{12}}\langle\alpha_1, \alpha_1\rangle = 3 \cdot 2 = 6$. So α_1 is short and α_2 is long. Also, $\langle\alpha_1, \alpha_2\rangle = \frac{1}{2}\langle\alpha_2, \alpha_2\rangle M_{12} = \frac{1}{2} \cdot 6 \cdot (-1) = -3$. Similarly see that $\langle\alpha_2, \alpha_1\rangle = -3$ as well. Then relative to the basis $\{\alpha_1, \alpha_2\}$ for V, the inner product $\langle\cdot,\cdot\rangle$ is represented by the matrix $\begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}$.

$\boxed{\text{§3.5}}$ Using $\mathfrak{g}^{\top}$, we compute the short and long roots in Φ^+. For the game sequence $(\gamma_1,$ $\gamma_2,\ \gamma_1,\ \gamma_2,\ \gamma_1,\ \gamma_2)$ played in Figure 3.5 from a generic strongly dominant initial position (a, b) on $\mathfrak{g}^{\top}$, observe that the numbers at the fired nodes are a, $3a + b$, $2a + b$, $3a + 2b$, $a + b$, and b respectively. Using Remark 3.29, it follows that $\Phi^+ = \{\alpha_1, 3\alpha_1{+}\alpha_2, 2\alpha_1{+}\alpha_2, 3\alpha_1{+}2\alpha_2, \alpha_1{+}\alpha_2, \alpha_2\}$, $\Phi^+_{short} = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2\}$, and $\Phi^+_{long} = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_2\}$. Recall that α_1 corresponds to the first row of the Cartan matrix and α_2 corresponds to the second, relative to the basis of fundamental weights. That is, $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = -3\omega_1 + 2\omega_2$. Note that $3\alpha_1 + 2\alpha_2 = \omega_2$ is the highest root $\overline{\omega}$, and that $2\alpha_1 + \alpha_2 = \omega_1$ is the highest short root $\overline{\omega}_{short}$. (Alternatively, these calculations are easily confirmed by directly computing the actions of the 12 elements of $\mathcal{W}$ on the simple roots α_1 and α_2.)

$\boxed{\text{§3.6}}$ At this point, we could use Theorem 3.11 to confirm that $|\mathcal{W}| = 12$, if we did not already know this by other means. The highest short root $\overline{\omega}_{short} = \omega_1$ is J^c-dominant for $J = \{2\}$. Then we have $|\Phi_{short}| = 2|\Phi^+_{short}| = 2 \cdot 3 = 6$, and $|\mathcal{W}_J| = |\mathcal{W}_{\{2\}}| = 2$. Then $|\mathcal{W}| = 6 \cdot 2 = 12$.

$\boxed{\text{§3.7}}$ From the numbers game played on $\mathfrak{g}^{\top}$ from the generic strongly dominant position (a, b), we see in Figure 3.5 that the terminal position is $(-a, -b)$. That is, $w_0.(a\omega_1^{\top} + b\omega_2^{\top}) = -a\omega_1^{\top} - b\omega_2^{\top}$. Since $\mathfrak{g}^{\top} \cong \mathfrak{g}$, we obtain that $w_0.(a\omega_1 + b\omega_2) = -a\omega_1 - b\omega_2$ for a generic strongly dominant weight

Figure 3.5: The game sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ played on $\mathsf{G}_2^{\mathsf{T}}$ from a generic strongly dominant position (a, b).



$a\omega_1 + b\omega_2$. Then $w_0.\omega_1 = -\omega_1$ and $w_0.\omega_2 = -\omega_2$. In particular, the symmetry σ_0 of the Dynkin diagram $\mathfrak{g}$ is the identity.

$\boxed{\S 3.9}$ From Proposition 3.9 it follows that $s_1.\omega_1 = \omega_1 - \alpha_1 = -\omega_1 + \omega_2$, $s_1.\omega_2 = \omega_2$, $s_2.\omega_1 = \omega_1$, and $s_2.\omega_2 = \omega_2 - \alpha_2 = 3\omega_1 - \omega_2$. Let z_1 and z_2 denote the elements e_{ω_1} and e_{ω_2} of the group ring $\mathbb{Z}[\Lambda]$. In this notation, $s_1.z_1 = z_1^{-1}z_2$, $s_1.z_2 = z_2$, $s_2.z_1 = z_1$, and $s_2.z_2 = z_1^3 z_2^{-1}$.

Following Example 3.18 the adjoint and short adjoint characters are:

$$\begin{aligned}
\chi_{\varpi} = \chi_{\omega_2} = \mathbf{char}(\omega_2; z_1, z_2) &= z_2 + z_1^3 z_2^{-1} + z_1 + z_1^{-1} z_2 + z_1^{-3} z_2^2 + z_1^2 z_2^{-1} \\
&\quad + 2 + z_1^{-2} z_2 + z_1^3 z_2^{-2} + z_1 z_2^{-1} + z_1^{-1} + z_1^{-3} z_2 + z_2^{-1} \\
\chi_{\varpi_{short}} = \chi_{\omega_1} = \mathbf{char}(\omega_1; z_1, z_2) &= z_1 + z_1^{-1} z_2 + z_1^2 z_2^{-1} + 1 + z_1^{-2} z_2 + z_1 z_2^{-1} + z_1^{-1}
\end{aligned}$$

One can verify by hand that these polynomials are $\mathcal{W}$-invariant by using the prescribed action to see that s_1 and s_2 preserve each polynomial. For example,

$$\begin{aligned}
s_2.\chi_{\overline{\omega}_{short}} = s_2.\chi_{\omega_1} &= s_2.(z_1 + z_1^{-1}z_2 + z_1^2 z_2^{-1} + 1 + z_1^{-2}z_2 + z_1 z_2^{-1} + z_1^{-1}) \\
&= z_1 + z_1^{-1}(z_1^3 z_2^{-1}) + z_1^2(z_1^3 z_2^{-1})^{-1} + z_1^{-2}(z_1^3 z_2^{-1}) + z_1(z_1^3 z_2^{-1})^{-1} + z_1^{-1} \\
&= z_1 + z_1^2 z_2^{-1} + z_1^{-1}z_2 + 1 + z_1 z_2^{-1} + z_1^{-2}z_2 + z_1^{-1}
\end{aligned}$$

We note for the record that the alternating sums A_ϱ and $A_{\varrho+\lambda}$ can be written down directly using the definitions *since the Weyl group* $\mathcal{W}$ *is small for* G_2:

$$\begin{aligned}
A_\varrho &= z_1 z_2 - z_1^{-1}z_2^2 - z_1^4 z_2^{-1} + z_1^{-4}z_2^3 + z_1^5 z_2^{-2} - z_1^{-5}z_2^3 - z_1^5 z_2^{-3} + z_1^{-5}z_2^2 + z_1^4 z_2^{-3} \\
&\quad - z_1^{-4}z_2 - z_1 z_2^{-2} + z_1^{-1}z_2^{-1} \\
A_{\varrho+\lambda} &= z_1^{a+1}z_2^{b+1} - z_1^{-(a+1)}z_2^{a+b+2} - z_1^{a+3b+4}z_2^{-(b+1)} + z_1^{-(a+3b+4)}z_2^{a+2b+3} \\
&\quad + z_1^{2a+3b+5}z_2^{-(a+b+2)} - z_1^{-(2a+3b+5)}z_2^{a+2b+3} - z_1^{2a+3b+5}z_2^{-(a+2b+3)} \\
&\quad + z_1^{-(2a+3b+5)}z_2^{a+b+2} + z_1^{a+3b+4}z_2^{-(a+2b+3)} - z_1^{-(a+3b+4)}z_2^{b+1} \\
&\quad - z_1^{a+1}z_2^{-(a+b+2)} + z_1^{-(a+1)}z_2^{-(b+1)}
\end{aligned}$$

At this point, one could use a computer algebra system to quickly confirm that $A_\varrho \chi_{\omega_i} = A_{\varrho+\omega_i}$ for each $i = 1, 2$.

$\boxed{\S 3.10}$ We can compute the q-specialization of Proposition 3.20 for an irreducible $\mathfrak{g}$-character χ_λ as follows. Take $\lambda = a\omega_1 + b\omega_2 \in \Lambda^+$. Note that $\langle \omega_1, \varrho^\vee \rangle = \langle 2\alpha_1 + \alpha_2, \varrho^\vee \rangle = 3$ and $\langle \omega_2, \varrho^\vee \rangle = \langle 3\alpha_1 + 2\alpha_2, \varrho^\vee \rangle = 5$. Also, $-\langle w_0.\lambda, \varrho^\vee \rangle = -\langle -a\omega_1 - b\omega_2, \varrho^\vee \rangle = 3a + 5b$. Then for any connected splitting poset for χ_λ we have

$$RGF(R, q) = q^{3a+5b}\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, z_2)|_{z_1=q^3, z_2=q^5}$$

In the case of $\lambda = \omega_2$, we have

$$\begin{aligned}
RGF(R, q) &= q^5\Big(q^5 + q^9 q^{-5} + q^3 + q^{-3}q^5 + q^{-9}q^{10} + q^6 q^{-5} \\
&\qquad +2 + q^{-6}q^5 + q^9 q^{-10} + q^3 q^{-5} + q^{-3} + q^{-9}q^5 + q^{-5}\Big) \\
&= q^{10} + q^9 + q^8 + q^7 + 2q^6 + 2q^5 + 2q^4 + q^3 + q^2 + q + 1
\end{aligned}$$

In the case of $\lambda = \omega_1$, we have

$$\begin{aligned}
RGF(R, q) &= q^3(q^3 + q^{-3}q^5 + q^6 q^{-5} + 1 + q^{-6}q^5 + q^3 q^{-5} + q^{-3}) \\
&= q^6 + q^5 + q^4 + q^3 + q^2 + q + 1
\end{aligned}$$

Now on $\mathfrak{g}$ play the numbers game from initial position $(a{+}1, b{+}1)$, where a and b are nonnegative. For the game sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ played from this position, the numbers at the fired nodes are $a{+}1$, $a{+}b{+}2$, $2a{+}3b{+}5$, $a{+}2b{+}3$, $a{+}3b{+}4$, and $b{+}1$ respectively. See Figure 3.6. Then using Remark 3.29 together with Theorem 3.21, we have the following formula for the rank generating function for any connected splitting poset R for the $\mathfrak{g}$-character χ_λ with $\lambda = a\omega_1 + b\omega_2 \in \Lambda^+$:

$$RGF(R, q) = \frac{(1-q^{2a+3b+5})(1-q^{a+3b+4})(1-q^{a+2b+3})(1-q^{a+b+2})(1-q^{b+1})(1-q^{a+1})}{(1-q^5)(1-q^4)(1-q^3)(1-q^2)(1-q)(1-q)}$$

It follows from Corollary 3.22 that the dimension of χ_λ is

$$\frac{(2a+3b+5)(a+3b+4)(a+2b+3)(a+b+2)(b+1)(a+1)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}$$

and from Corollary 3.23 that the length of R is

$$\ell(\lambda) = (2a+3b) + (a+3b) + (a+2b) + (a+b) + a + b = 6a + 10b.$$

Now consider the short adjoint character, which is the fundamental character χ_{ω_1}. In this case, note from our computation above that each coefficient $c_{\omega_1,\mu}$ in the character polynomial is unity. From Facts 3.17 and Example 3.24 it follows that the maximal splitting poset $\mathcal{M}(\omega_1)$ coincides with $\Pi(\omega_1)$, as depicted in Figure 3.7. Check that in this case, no edges can be removed from $\mathcal{M}(\omega_1)$ without violating the $\mathfrak{g}$-structure property. Thus $\mathcal{M}(\omega_1)$ is the unique splitting poset for χ_{ω_1}. In particular, $\mathcal{M}(\omega_1)$ coincides with the SDL built from Φ_{short} in Example 3.25. The vertex-colored poset of irreducibles P_{ω_1} is also depicted in Figure 3.7. Next we consider the adjoint character χ_{ω_2}. In this case, we can build two SDL's using Example 3.25. These are depicted in Figures 3.8 and 3.9, along with their vertex-colored posets of irreducibles. The poset of irreducibles depicted in Figure 3.8 is designated as P_{ω_2} for reasons explained in the next paragraph.

Certain distributive lattice orderings of Littelmann's G_2-tableaux [Lit] were found by Donnelly. The main result of [Mc] was to confirm Donnelly's conjecture that these lattices are SDL's for the irreducible $\mathfrak{g}$-characters. Using ideas related to [DW], these G_2 lattices are constructed in [ADLMPPW] by 'stacking' the posets of irreducibles denoted P_{ω_1} and P_{ω_2}. For a dominant weight $\lambda = a\omega_1 + b\omega_2$, one 'stacks' a copies of P_{ω_1} 'on top of' b copies of P_{ω_2}, or alternatively one stacks b copies of P_{ω_2} on top of a copies of P_{ω_1}. (See Figures 3.10 and 3.11 for the $a = 2, b = 2$ cases.) These are posets of irreducibles for two 'G_2-semistandard' lattices denoted $L^{\beta\alpha}_{\mathsf{G}_2}(\lambda)$ and $L^{\alpha\beta}_{\mathsf{G}_2}(\lambda)$. These SDL's for χ_λ are related by the recolored dual: $L^{\alpha\beta}_{\mathsf{G}_2}(\lambda) \cong (L^{\beta\alpha}_{\mathsf{G}_2}(\lambda))^{\triangle}$.

Figure 3.6: The game sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ played on G_2 from a position $(a+1, b+1)$ with a and b nonnegative.

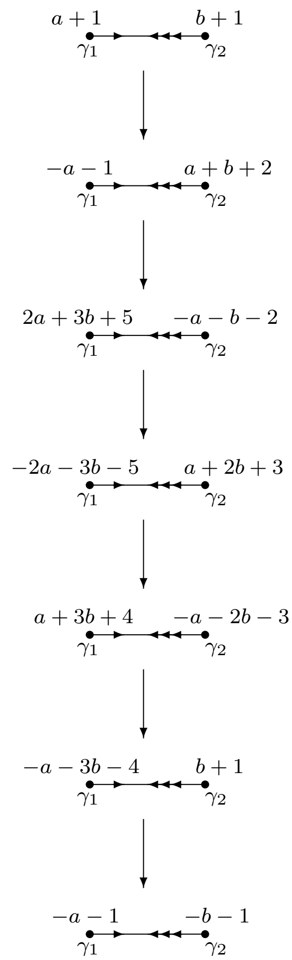


Figure 3.7: $\mathcal{M}(\omega_1) = \Pi(\omega_1)$ is edge-color isomorphic to $\mathrm{J}_{color}(P_{\omega_1})$.
Order ideals are notated as in Figure 2.7. A weight $p\omega_1 + q\omega_2$ is denoted (p, q).

Vertex	$\mathbf{t}_0$	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$	$\mathbf{t}_5$	$\mathbf{t}_6$
Weight	$(1,0)$	$(1,-1)$	$(-2,1)$	$(0,0)$	$(2,-1)$	$(-1,1)$	$(-1,0)$
Root	$2\alpha_1+\alpha_2$	$\alpha_1+\alpha_2$	α_1	NA	$-\alpha_1$	$-\alpha_1-\alpha_2$	$-2\alpha_1-\alpha_2$

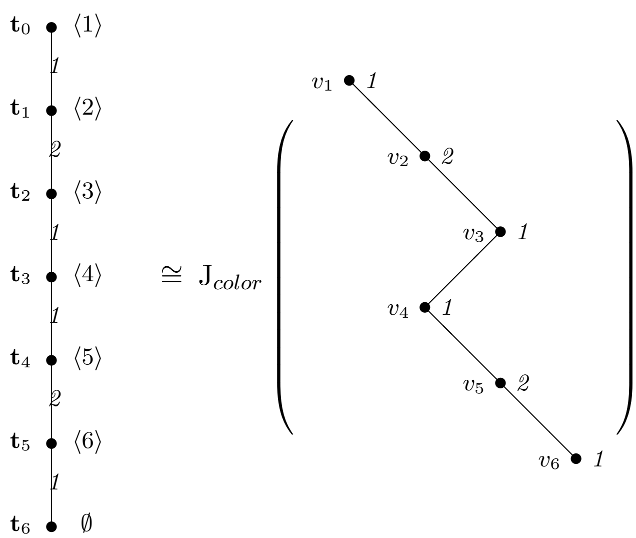


Figure 3.8: An SDL for χ_{ω_2} identified as $\mathrm{J}_{color}(P_{\omega_2})$ for a vertex-colored poset P_{ω_2}. Order ideals are notated as in Figure 2.7. A weight $p\omega_1 + q\omega_2$ is denoted (p, q).

Vertex	$\mathbf{t}_0$	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$	$\mathbf{t}_5$	$\mathbf{t}_6$
Weight	$(0,1)$	$(3,-1)$	$(1,0)$	$(-1,1)$	$(-3,2)$	$(2,-1)$	$(0,0)$
Root	$3\alpha_1+2\alpha_2$	$3\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	$\alpha_1+\alpha_2$	α_2	α_1	NA

Vertex	$\mathbf{t}_7$	$\mathbf{t}_8$	$\mathbf{t}_9$	$\mathbf{t}_{10}$	$\mathbf{t}_{11}$	$\mathbf{t}_{12}$	$\mathbf{t}_{13}$
Weight	$(0,0)$	$(3,-2)$	$(-2,1)$	$(1,-1)$	$(-1,0)$	$(-2,1)$	$(0,-1)$
Root	NA	$-\alpha_2$	$-\alpha_1$	$-\alpha_1-\alpha_2$	$-2\alpha_1-\alpha_2$	$-3\alpha_1-\alpha_2$	$-3\alpha_1-2\alpha_2$

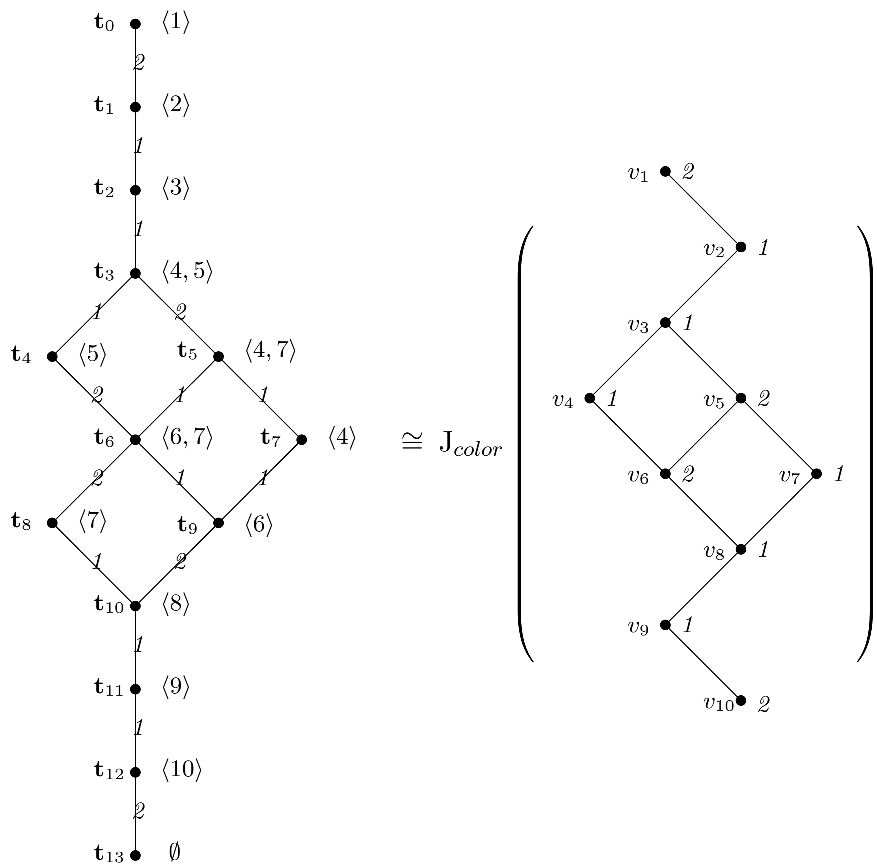


Figure 3.9: An SDL for χ_{ω_2} identified as $\mathrm{J}_{color}(Q)$ for a vertex-colored poset Q. Order ideals are notated as in Figure 2.7. A weight $p\omega_1 + q\omega_2$ is denoted (p,q).

Vertex	$\mathbf{t}_0$	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$	$\mathbf{t}_5$	$\mathbf{t}_6$
Weight	$(0,1)$	$(3,-1)$	$(1,0)$	$(-1,1)$	$(-3,2)$	$(2,-1)$	$(0,0)$
Root	$3\alpha_1+2\alpha_2$	$3\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	$\alpha_1+\alpha_2$	α_2	α_1	NA

Vertex	$\mathbf{t}_7$	$\mathbf{t}_8$	$\mathbf{t}_9$	$\mathbf{t}_{10}$	$\mathbf{t}_{11}$	$\mathbf{t}_{12}$	$\mathbf{t}_{13}$
Weight	$(0,0)$	$(3,-2)$	$(-2,1)$	$(1,-1)$	$(-1,0)$	$(-2,1)$	$(0,-1)$
Root	NA	$-\alpha_2$	$-\alpha_1$	$-\alpha_1-\alpha_2$	$-2\alpha_1-\alpha_2$	$-3\alpha_1-\alpha_2$	$-3\alpha_1-2\alpha_2$

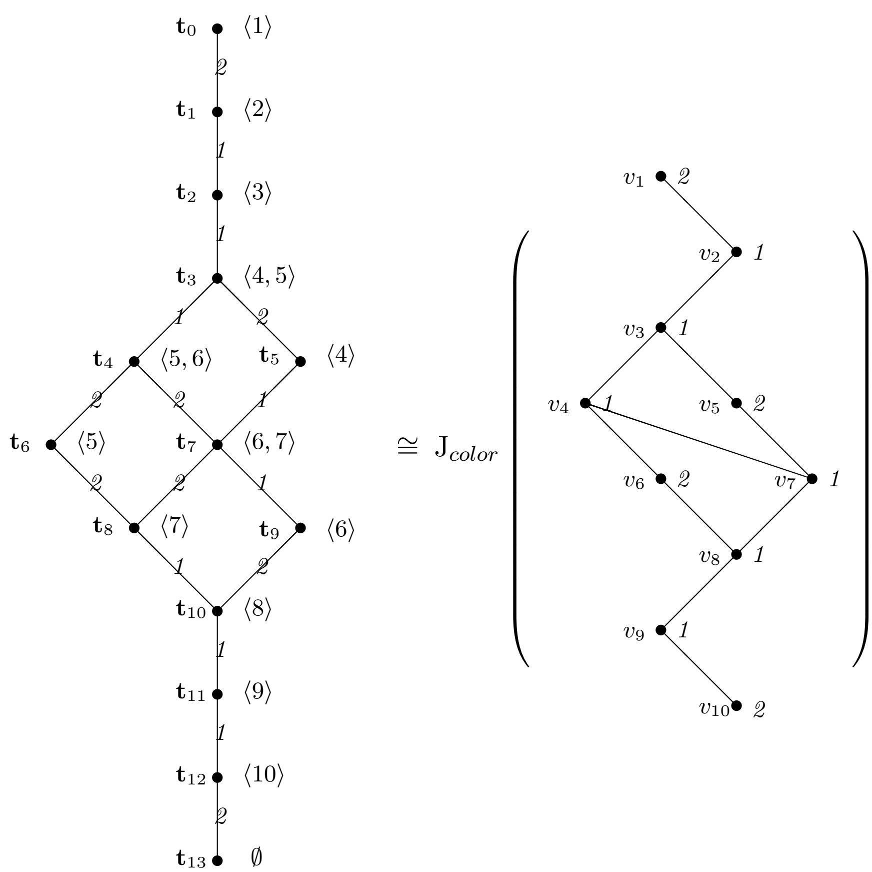


Figure 3.10: The stacking $P := P_{\omega_2} \triangleleft P_{\omega_2} \triangleleft P_{\omega_1} \triangleleft P_{\omega_1}$ of fundamental posets P_{ω_1} and P_{ω_2}. Theorem 5.3 of [ADLMPPW] shows that $\mathrm{J}_{color}(P)$ is an SDL for the G_2-character $\chi_{2\omega_1+2\omega_2}$.

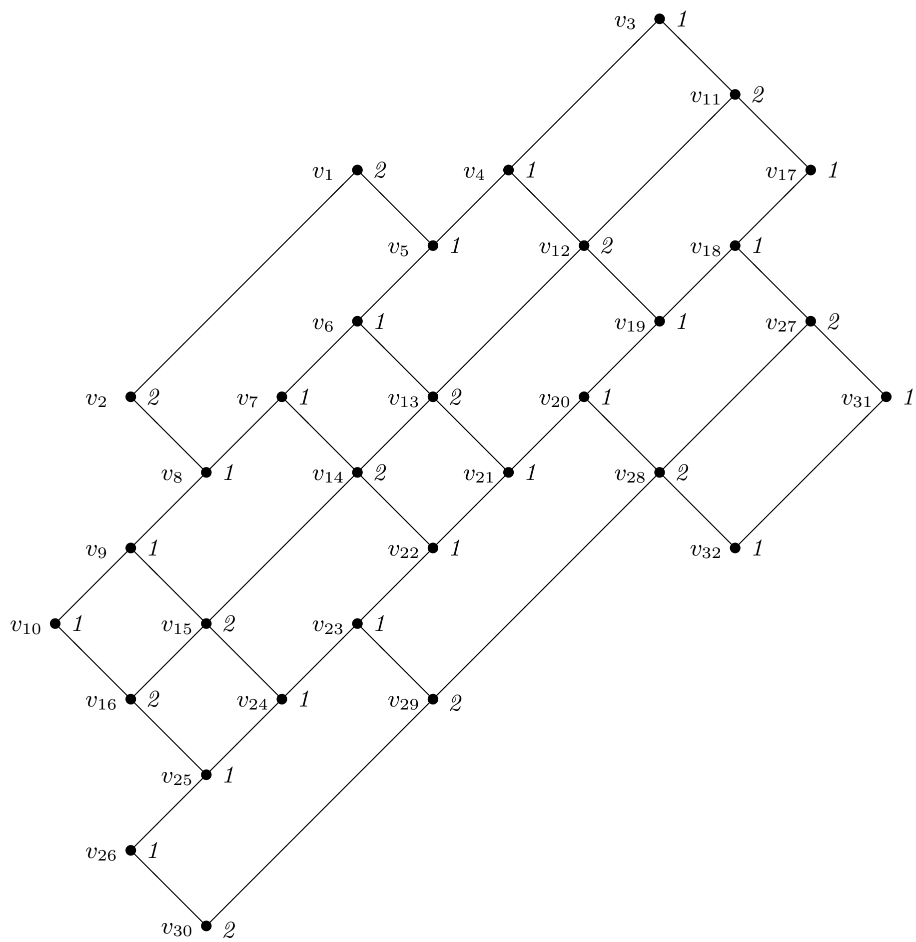
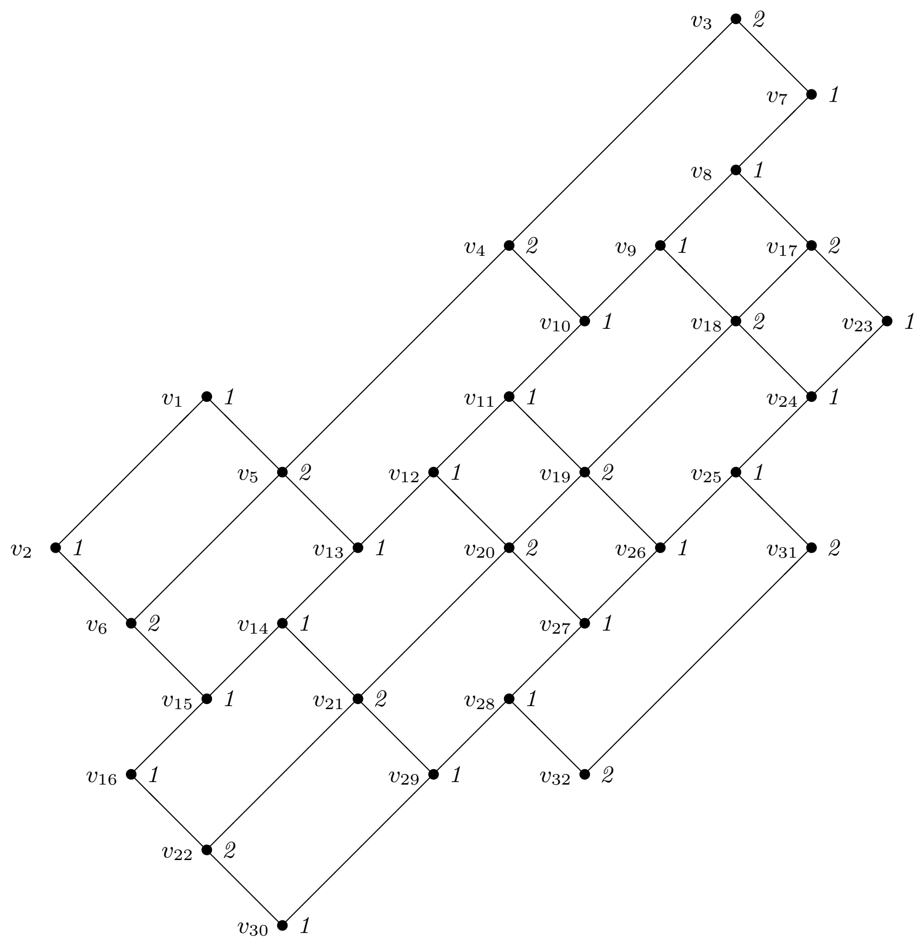


Figure 3.11: The stacking $Q := P_{\omega_1} \triangleleft P_{\omega_1} \triangleleft P_{\omega_2} \triangleleft P_{\omega_2}$ of fundamental posets P_{ω_1} and P_{ω_2}. Theorem 5.3 of [ADLMPPW] shows that $\mathrm{J}_{color}(Q)$ is an SDL for the G_2-character $\chi_{2\omega_1+2\omega_2}$.



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