October 29, 2008

The numbers game and Dynkin diagram classification results

Robert G. Donnelly[1] and Kimmo Eriksson[2]
[1]Department of Mathematics and Statistics, Murray State University, Murray, KY 42071
[2]Department of Mathematics and Physics, Mälardalen University, Västerås, Sweden

Abstract

The numbers game is a one-player game played on a finite simple graph with certain "amplitudes" assigned to its edges and with an initial assignment of real numbers to its nodes. The moves of the game successively transform the numbers at the nodes using the amplitudes in a certain way. Combinatorial reasoning is used to show that those connected graphs with negative integer amplitudes for which the numbers game meets a certain finiteness requirement are precisely the Dynkin diagrams associated with the finite-dimensional complex simple Lie algebras. This strengthens a result originally due to the second author. A more general result is obtained when certain real number amplitudes are allowed. The resulting graphs are in families, each family corresponding to a finite irreducible Coxeter group. These results are used to demonstrate that the only generalized Cartan matrices for which there exist finite edge-colored ranked posets enjoying a certain structure property are the Cartan matrices for the finite-dimensional complex semisimple Lie algebras. In this setting, classifications of the finite-dimensional Kac–Moody algebras and of the finite Coxeter and Weyl groups are re-derived.

Keywords: numbers game, generalized Cartan matrix, Dynkin diagram, Coxeter/Weyl group, semisimple Lie algebra, Kac–Moody algebra, edge-colored ranked poset

1. Introduction

Dynkin diagrams are certain finite simple graphs whose edges carry certain information. For some examples, see the connected Dynkin diagrams of "finite type" depicted in Figure 2.1 below. Many mathematical objects are classified by Dynkin diagrams, perhaps most famously the finite-dimensional complex semisimple Lie algebras. For examples of other Dynkin diagram classifications, see [HHSV], [Pro2], and [Pro3]. The main results of this paper (Theorems 2.1, 4.1, and 5.2) are Dynkin diagram classifications obtained in the context of the so-called "numbers game." These results can be viewed as classifications of certain combinatorial finiteness phenomena which are related to Coxeter/Weyl groups, Kac-Moody Lie algebras, and their representations. For example, in §6 we recapitulate observations made by the second author in [Erik2] to say how the Dynkin diagram classifications of the finite-dimensional Kac–Moody algebras (the finite-dimensional complex semisimple Lie algebras cf. [Hum1], [Kac]) and of the finite Coxeter and Weyl groups (see [Hum2]) can be re-derived in this context. See Theorem 6.2. For further connections, see §5 and §7 below or [Don2].

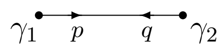
The numbers game is a one-player game played on a finite simple graph with certain real number weights (which we call "amplitudes") on its edges and with an initial assignment of real numbers to its nodes. The move a player can make is to "fire" one of the nodes with a positive number. This move transforms the number at the fired node by changing its sign, and it also transforms the number at each adjacent node in a certain way using an amplitude along the incident edge. The player fires the nodes in some sequence of the player's choosing, continuing until no node has a positive number.

The first main results of this paper (Theorems 2.1 and 4.1) address the question: for which such graphs does there exist a nontrivial initial assignment of nonnegative numbers such that the numbers game terminates in a finite number of steps? For graphs with negative integer amplitudes, we use combinatorial methods to show in Theorem 2.1 that the only such connected graphs are the Dynkin diagrams of Figure 2.1. These Dynkin diagrams are in one-to-one correspondence with the finite-dimensional complex simple Lie algebras. More generally, allowing for certain real number amplitudes we show in Theorem 4.1 that the resulting graphs are the "E-Coxeter graphs" of Figure 4.1, which are related to the finite irreducible Coxeter groups. Our proof of this latter result borrows extensively from the second author's thesis [Erik2] and applies the Perron–Frobenius theory for nonnegative real matrices. A different proof of Theorem 4.1 that depends on the classification of finite Coxeter groups and on results concerning certain geometric representations of Coxeter groups is given in [Don2]. All of these approaches build on the second author's work in [Erik2], [Erik5], and [Erik6].

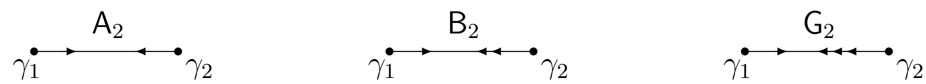
Our third main result (Theorem 5.2) is an application of Theorem 2.1 (or Theorem 4.1) to edge-colored ranked posets. In connection with a poset-theoretic study of the finite-dimensional representations of a given finite-dimensional complex semisimple Lie algebra, the first author observed in [Don1] that its "supporting graphs" are finite edge-colored ranked posets possessing a certain structure property relative to the Cartan matrix for the Lie algebra. "Crystal graphs" (see for example [Stem]) and "splitting posets" (see [ADLMPPW]) are other edge-colored ranked posets associated to semisimple Lie algebra representations which also possess this structure property. Here we begin to address the problem of classifying those matrices for which there are finite edge-colored ranked posets possessing the associated structure property. In Theorem 5.2 we show that among the "generalized Cartan matrices" (defined below), the only such matrices are those arising as Cartan matrices for semisimple Lie algebras. While our motivation for studying this property comes from Lie-theoretic considerations, the statement and proof of Theorem 5.2 are purely combinatorial. It is hoped that, as in [Don4], this structure property will be part of combinatorial characterizations of families of nice supporting graphs or splitting posets.

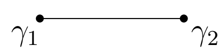
The numbers game as formulated by Mozes [Moz] has also been studied by Proctor [Pro1], [Pro4], Björner [Björ], Eriksson [Erik1], [Erik2], [Erik3], [Erik4], [Erik5], [Erik6], and Wildberger [Wil1], [Wil2], [Wil3], and Donnelly [Don2]. Wildberger studies a dual version which he calls the "mutation game." See Alon *et al* [AKP] for a brief and readable treatment of the numbers game on "unweighted" cyclic graphs. Much of the numbers game discussion in Chapter 4 of the book [BB] by Björner and Brenti can be found in [Erik2] and [Erik5]. Proctor developed this process in [Pro1] to compute Weyl group orbits of weights with respect to the fundamental weight basis. Here we use his perspective of firing nodes with positive, as opposed to negative, numbers. Mozes studied numbers games on graphs for which the matrix M of integer amplitudes is *symmetrizable* (i.e. there is a nonsingular diagonal matrix D such that $D^{-1}M$ is symmetric); in [Moz] he obtained "strong convergence" results and a geometric characterization of the initial positions for which the game terminates. Our main results make no symmetrizable assumption.

2. Definitions and statement of first main result

Fix a positive integer n and a totally ordered set I_n with n elements (usually $I_n := \{1 < \ldots < n\}$). A *generalized Cartan matrix* (or *GCM*) is an $n \times n$ matrix $M = (M_{ij})_{i,j \in I_n}$ with integer entries satisfying the requirements that each main diagonal matrix entry is 2, that all other matrix entries are nonpositive, and that if a matrix entry M_{ij} is nonzero then its transpose entry M_{ji} is also nonzero. Generalized Cartan matrices are the starting point for the study of Kac–Moody algebras: beginning with a GCM, one can write down a list of the defining relations for a Kac–Moody algebra as well as the associated Weyl group (see [Kac], [Kum], or §6 below). To an $n \times n$ generalized Cartan matrix $M = (M_{ij})_{i,j \in I_n}$ we associate a finite graph Γ (which has undirected edges, no loops, and no multiple edges) as follows: The nodes $(\gamma_i)_{i \in I_n}$ of Γ are indexed by the set I_n, and an edge is placed between nodes γ_i and γ_j if and only if $i \neq j$ and the matrix entries M_{ij} and M_{ji} are nonzero. Call the pair (Γ, M) a *GCM graph*. We consider two GCM graphs $(\Gamma, M = (M_{ij})_{i,j \in I_n})$ and $(\Gamma', M' = (M'_{pq})_{p,q \in I'_n})$ to be the same if under some bijection $\sigma : I_n \to I'_n$ we have nodes γ_i and γ_j in Γ adjacent if and only if $\gamma'_{\sigma(i)}$ and $\gamma'_{\sigma(j)}$ are adjacent in Γ' with $M_{ij} = M'_{\sigma(i),\sigma(j)}$. We depict a generic connected two-node GCM graph as 

, where $p = -M_{12}$ and $q = -M_{21}$. We use special names and notation to refer to two-node GCM graphs which have $p = 1$ and $q = 1$, 2, or 3 respectively:



When $p = 1$ and $q = 1$ it is convenient to use the graph 

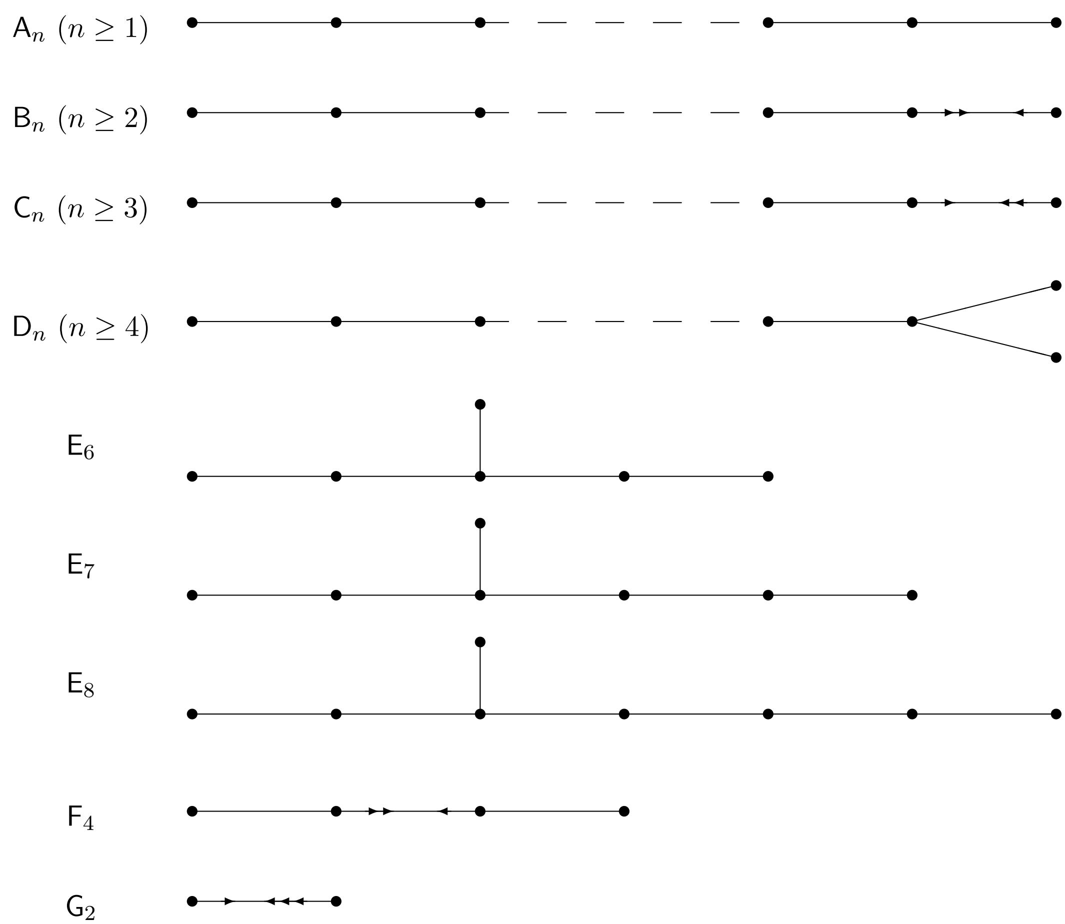
to represent the GCM graph A_2. A GCM graph (Γ, M) is a *Dynkin diagram of finite type* if each connected component of (Γ, M) is one of the graphs of Figure 2.1. Number the nodes as in §11.4 of [Hum1]. In these cases the GCMs are "Cartan" matrices.

A *position* $\lambda = (\lambda_i)_{i \in I_n}$ is an assignment of real numbers to the nodes of the GCM graph (Γ, M). The position λ is *dominant* (respectively, *strongly dominant*) if $\lambda_i \geq 0$ (respectively $\lambda_i > 0$) for all $i \in I_n$; λ is *nonzero* if at least one $\lambda_i \neq 0$. For $i \in I_n$, the *fundamental position* ω_i is the assignment of the number 1 at node γ_i and the number 0 at all other nodes. Given a position λ on a GCM graph (Γ, M), to *fire* a node γ_i is to change the number at each node γ_j of Γ by the transformation

$$\lambda_j \longmapsto \lambda_j - M_{ij}\lambda_i,$$

provided the number at node γ_i is positive; otherwise node γ_i is not allowed to be fired. Since the generalized Cartan matrix M assigns a pair of *amplitudes* (M_{ij} and M_{ji}) to each edge of the graph Γ, we sometimes refer to GCMs as *amplitude matrices*. The *numbers game* is the one-player game on a GCM graph (Γ, M) in which the player (1) Assigns an initial position to the nodes of Γ; (2) Chooses a node with a positive number and fires the node to obtain a new position; and (3) Repeats step (2) for the new position if there is at least one node with a positive number. Consider now the GCM graph B_2. As one can see in Figure 2.2, the numbers game terminates in a finite number of steps for any initial position and any legal sequence of node firings, if it is understood that the player will continue to fire as long as there is at least one node with a positive number. In general, given a position λ, a *game sequence for* λ is the (possibly empty, possibly infinite) sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$, where γ_{i_j} is the jth node that is fired in some numbers game with initial position λ. More generally, a *firing sequence* from some position λ is an initial portion of some game sequence played from λ; the phrase *legal firing sequence* is used to emphasize that all node firings in the sequence are known or assumed to be possible. Note that a game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$ is of finite length l (possibly with $l = 0$) if the number is nonpositive at each node after the lth firing; in this case the game sequence is *convergent* and the resulting position is the *terminal position* for the

Figure 2.1: Connected Dynkin diagrams of finite type.



game sequence. A connected GCM graph (Γ, M) is *admissible* if there exists a nonzero dominant initial position with a convergent game sequence.

The first main result of this paper is:

Theorem 2.1 *A connected GCM graph is admissible if and only if it is a connected Dynkin diagram of finite type. In these cases, for any given initial position every game sequence will converge to the same terminal position in the same finite number of steps.*

Our combinatorial proof of this result is given in §3. Theorem 4.1 is a more general result.

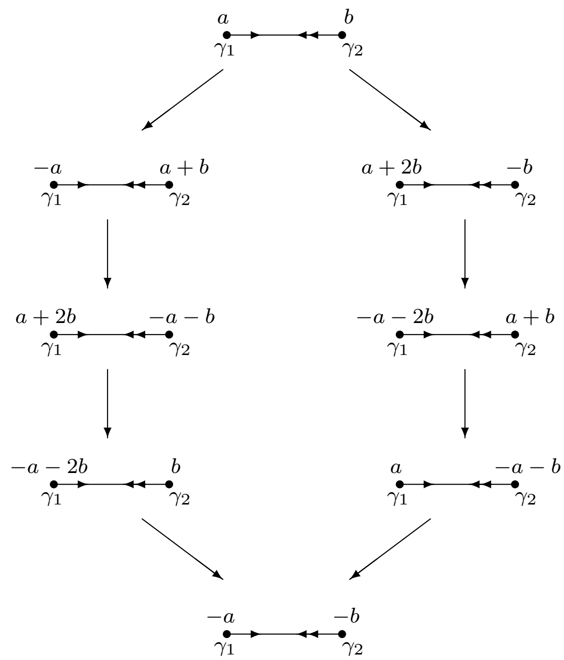
3. Proof of first main result

Lemmas 3.2, 3.4, 3.5, 3.7, and 3.8 are stated and used in [Don2]. Proofs or references for these results are given here as well. Our proof of the "only if" direction of the first claim of Theorem 2.1 uses a series of reductions that are typical in Dynkin diagram classification arguments. These reductions are implemented using two key results (Theorems 3.1 and 3.3) due to the second author. A proof of the remaining assertions of Theorem 2.1 is given at the end of the section.

Proof of the "only if" direction of the first claim of Theorem 2.1:

Step 1: Strong convergence. Following [Erik2] and [Erik6], we say the numbers game on a GCM graph (Γ, M) is *strongly convergent* if given any initial position, any two game sequences either

Figure 2.2: The numbers game for the GCM graph B_2.



both diverge or both converge to the same terminal position in the same number of steps. The next result follows from Theorem 3.1 of [Erik6] (or see Theorem 3.6 of [Erik2]).

Theorem 3.1 (Strong Convergence Theorem) *The numbers game on a connected GCM graph is strongly convergent.*

For this part of our proof of Theorem 2.1, we only require the following weaker result, which also applies when the GCM graph is not connected:

Lemma 3.2 *For any GCM graph, if a game sequence for an initial position λ diverges, then all game sequences for λ diverge.*

Step 2: Comparison. The next result is an immediate consequence of Theorem 4.3 of [Erik2] or Theorem 4.5 of [Erik5]. The proof of this result in [Erik2] uses only combinatorial and linear algebraic methods.

Theorem 3.3 (Comparison Theorem) *Given a GCM graph, suppose that a game sequence for an initial position $\lambda = (\lambda_i)_{i \in I_n}$ converges. Suppose that a position $\lambda' := (\lambda'_i)_{i \in I_n}$ has the property that $\lambda'_i \leq \lambda_i$ for all $i \in I_n$. Then some game sequence for the initial position λ' also converges.*

Let r be a positive real number. Observe that if $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ is a convergent game sequence for an initial position $\lambda = (\lambda_i)_{i \in I_n}$, then $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ is a convergent game sequence for the initial position $r\lambda := (r\lambda_i)_{i \in I_n}$. This observation and Theorem 3.3 imply the following result:

Lemma 3.4 *Let $\lambda = (\lambda_i)_{i \in I_n}$ be a dominant initial position such that $\lambda_j > 0$ for some $j \in I_n$. Suppose that a game sequence for λ converges. Then some game sequence for the fundamental position ω_j also converges.*

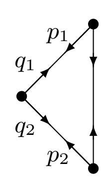
Step 3: A catalog of connected GCM graphs that are not admissible. The following immediate consequence of Lemmas 3.2 and 3.4 is useful in the proof of Lemma 3.6:

Lemma 3.5 *A GCM graph is not admissible if for each fundamental position there is a divergent game sequence.*

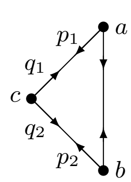
Lemma 3.6 *The connected GCM graphs of Figure 3.1 are not admissible.*

Outline of proof. By Lemma 3.5 it suffices to show that for each graph in Figure 3.1 and for each fundamental position, there is a divergent game sequence. In each case we can exhibit a divergent game sequence which is a simple pattern of node firings. Remarkably, in all cases trial and error quickly lead us to these patterns. Our goal in this proof sketch is not to develop any general theory for finding divergent game sequences for these cases, but rather to show that such game sequences can be found and presented in an elementary (though sometimes tedious) manner. For complete details see [Don3]. The $\widetilde{\mathsf{A}}$, $\widetilde{\mathsf{B}}$, $\widetilde{\mathsf{C}}$, $\widetilde{\mathsf{D}}$, $\widetilde{\mathsf{E}}$, and $\widetilde{\mathsf{F}}$ cases are handled using a common line of reasoning: A finite sequence of legal node firings is applied to a position whose numbers are linear expressions in an index variable k, and it is shown that the numbers for the resulting position are linear expressions of the same form with respect to the variable $k+1$. The $\widetilde{\mathsf{G}}$ cases and the families of small cycles are handled using a variation of this kind of argument: A finite sequence of legal node firings is applied to a generic position satisfying certain inequalities, and it is shown that the resulting position also satisfies these inequalities. The two paragraphs that follow borrow from [Don3] and demonstrate inadmissibility for some selected graphs from our list.

In the $\widetilde{\mathsf{C}}$ family, we show why is not admissible when the graph has $n \geq 3$ nodes. (Since firing the middle node in the $n = 3$ case is comparable to firing γ_2 in the $n \geq 4$ cases, then the $n = 3$ case does not need to be considered separately here.) Label the nodes as $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$, and γ_n from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(2k+1, -k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)$. This sequence results in the position $(2(k+1)+1, -(k+1), 0, \ldots, 0)$. For $2 \leq i \leq n-1$, any fundamental position $\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(0, \ldots, 0, 2k+1, -2k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_i, \gamma_{i-1}, \ldots, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+2}, \gamma_{i+1})$. This sequence results in the position $(0, \ldots, 0, 2(k+1)+1, -2(k+1), 0, \ldots, 0)$. The fundamental position $\omega_n = (0, \ldots, 0, 1)$ is the $k = 0$ version of the position $(0, \ldots, 0, -2k, 2k+1)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_n, \gamma_{n-1}, \ldots, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_{n-2}, \gamma_{n-1})$. This sequence results in the position $(0, \ldots, 0, -2(k+1), 2(k+1)+1)$.

In the families of small cycles, we show why GCM graphs of the form 

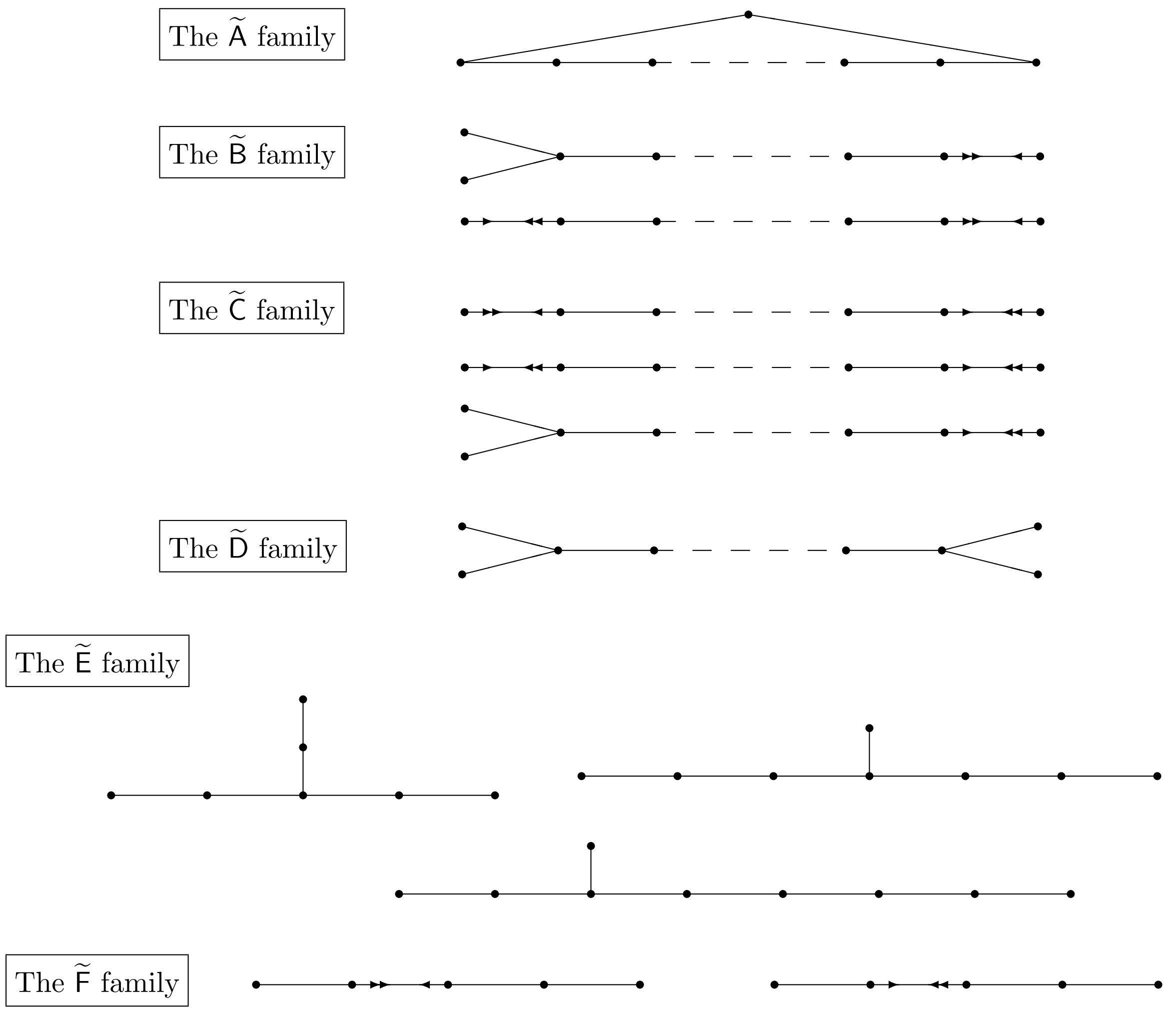
are not admissible.

Assign numbers a, b, and c as follows: 

Set $\kappa := (p_1 + p_2 - \frac{1}{q_2})a + (p_1 + p_2 - \frac{1}{q_1})b + c$.

Assume for now that $a \geq 0$, $b \geq 0$, $c \leq 0$, and $\kappa > 0$; when these inequalities hold we will say

Figure 3.1: Some connected GCM graphs that are not admissible.
(Figure continues on the next page.)



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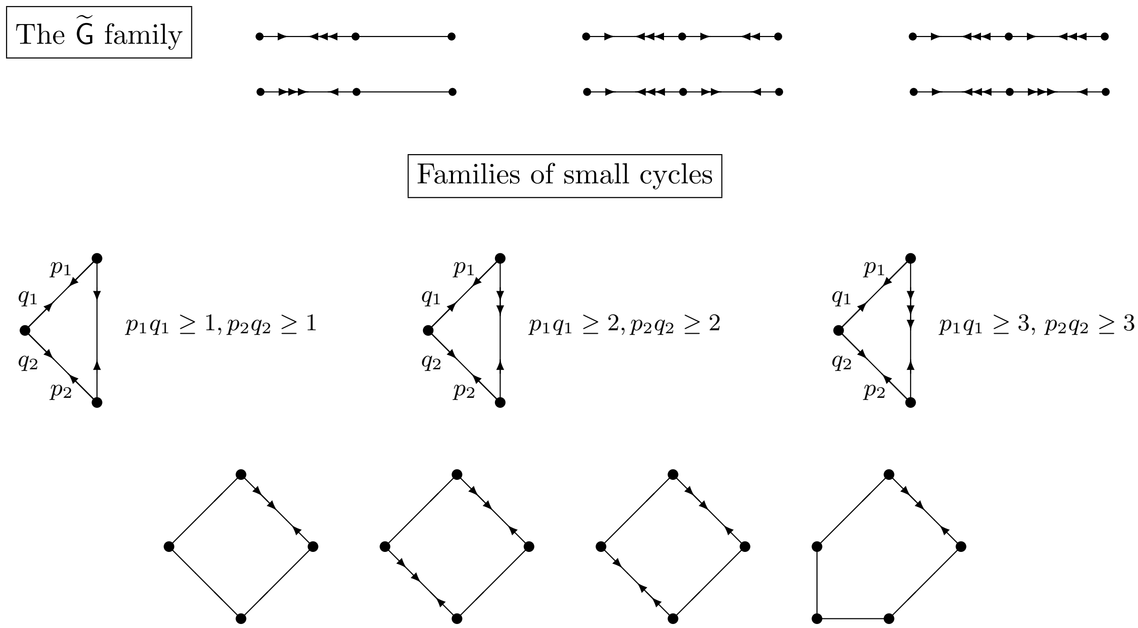
the position (a, b, c) meets condition $(*)$. Under condition $(*)$ notice that a and b cannot both be zero. Begin by firing only at the two rightmost nodes. When this is no longer possible, fire at the leftmost node. The resulting corresponding numbers are $a_1 = q_1(\kappa + \frac{1}{q_2}a)$, $b_1 = q_2(\kappa + \frac{1}{q_1}b)$, and $c_1 = -\kappa - \frac{1}{q_2}a - \frac{1}{q_1}b$. In particular, $a_1 > 0$, $b_1 > 0$, and $c_1 < 0$. Next we check that $\kappa_1 := (p_1 + p_2 - \frac{1}{q_2})a_1 + (p_1 + p_2 - \frac{1}{q_1})b_1 + c_1$ is also positive. Now

$$\kappa_1 = Q\kappa + Q_1 a + Q_2 b,$$

where $Q = q_1(p_2 - \frac{1}{q_2}) + q_2(p_1 - \frac{1}{q_1}) + (p_1q_1 + p_2q_2 - 1)$, $Q_1 = \frac{1}{q_2}[q_1(p_2 - \frac{1}{q_2}) + (p_1q_1 - 1)]$, and $Q_2 = \frac{1}{q_1}[q_2(p_1 - \frac{1}{q_1}) + (p_2q_2 - 1)]$. Since each parenthesized quantity in our expression for Q is nonnegative and the last of these is positive, then $Q > 0$. Similar reasoning shows that each bracketed quantity in our expressions for Q_1 and Q_2 is nonnegative, hence $Q_1 \geq 0$ and $Q_2 \geq 0$. Since $\kappa > 0$ by hypothesis, it now follows that $\kappa_1 > 0$. Then (a_1, b_1, c_1) meets condition $(*)$, so we can legally repeat the above firing sequence from position (a_1, b_1, c_1) to obtain another position (a_2, b_2, c_2) that meets condition $(*)$, etc. Since the fundamental positions $(a, b, c) = (1, 0, 0)$ and

Figure 3.1 (continued): Some connected GCM graphs that are not admissible.



$(a, b, c) = (0, 1, 0)$ meet condition $(*)$, then we see that the indicated legal firing sequence can be repeated indefinitely from these positions. For the fundamental position $(a, b, c) = (0, 0, 1)$, begin by firing at the leftmost node to obtain the position $(q_1, q_2, -1)$. This latter position meets condition $(*)$ with $\kappa = Q$, and so the legal firing sequence indicated above can be repeated indefinitely from this position.

Step 4: Every node is fired. The following is proved easily with an induction argument on the number of nodes.

Lemma 3.7 *Suppose (Γ, M) is connected with nonzero dominant position λ. Then in any convergent game sequence for λ, every node of Γ is fired at least once.*

Step 5: Subgraphs. If I'_m is a subset of the node set I_n of a GCM graph (Γ, M), then let Γ' be the subgraph of Γ with node set I'_m and the induced set of edges, and let M' be the corresponding submatrix of the amplitude matrix M; we call (Γ', M') a *GCM subgraph* of (Γ, M). In light of Lemmas 3.2 and 3.7, the following result amounts to an observation.

Lemma 3.8 *If a connected GCM graph is admissible, then any connected GCM subgraph is also admissible.*

Step 6: Amplitude products must be 1, 2, or 3.

Lemma 3.9 *If γ_i and γ_j are adjacent nodes in a connected admissible GCM graph (Γ, M), then the product of the amplitudes $M_{ij}M_{ji}$ is 1, 2, or 3. That is, the GCM subgraph of (Γ, M) with nodes γ_i and γ_j is in this case one of A_2, B_2, or G_2.*

Proof. By Lemma 3.8 we may restrict attention to the admissible GCM subgraph (Γ', M') with node set $\{i, j\}$. A nonzero dominant position with a convergent game sequence might not begin with positive numbers at both nodes; nonetheless, by examining the proof one sees that Lemma

3.7 of [Erik6] still applies to show that the product $M_{ij}M_{ji}$ of amplitudes in the admissible GCM graph (Γ', M') is 1, 2, or 3.

Conclusion of the "only if" part of the first claim of Theorem 2.1. Putting Steps 1 through 6 together, we see that the only possible connected admissible GCM graphs are the Dynkin diagrams of finite type.

Proof of the remaining claims of Theorem 2.1:

Let (Γ, M) be a connected Dynkin diagram of finite type. The Strong Convergence Theorem shows that if a game sequence for some initial position λ converges, then all game sequences from λ converge to the same terminal position in the same finite number of steps. Then in light of the Comparison Theorem, it suffices to show that for any strongly dominant initial position on (Γ, M), there is a convergent game sequence. Complete details for a case-by-case argument are given in [Don3]. We summarize the work done there as follows. For the exceptional graphs (E_6, E_7, E_8, F_4, and G_2) this can be checked by hand (requiring 36, 63, 120, 24, and 6 firings respectively). For each of the four infinite families of Dynkin diagrams of finite type, the proof of the next result given in [Don3] is straightforward and uses induction on the number of nodes.

Lemma 3.10 *For any positive integer n (respectively, any integer $n \geq 2$, $n \geq 3$, $n \geq 4$) and for any strongly dominant position $(a_1, \ldots, a_n)$ on A_n (respectively, B_n, C_n, D_n), one can obtain the terminal position $(-a_n, \ldots, -a_2, -a_1)$ (respectively $(-a_1, -a_2, \ldots, -a_n)$, $(-a_1, -a_2, \ldots, -a_n)$, $(-a_1, -a_2, \ldots, -a_{n-2}, -b_{n-1}, -b_n)$ where $b_{n-1} := a_{n-1}$ and $b_n := a_n$ when n is even and where $b_{n-1} := a_n$ and $b_n := a_{n-1}$ when n is odd) by a sequence of $\frac{n(n+1)}{2}$ (resp. n^2, n^2, $n(n-1)$) node firings.*

This completes the proof of Theorem 2.1. □

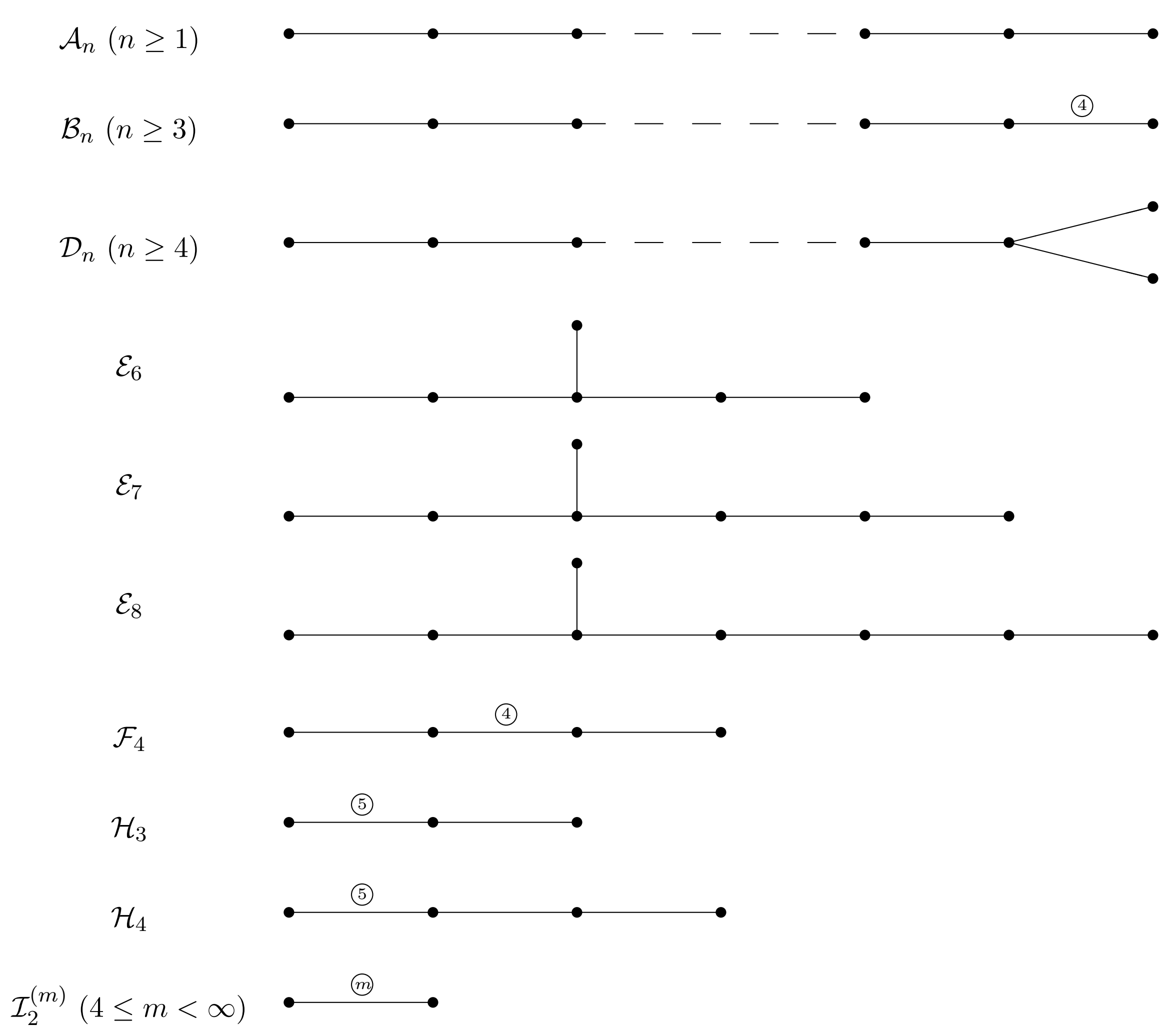
4. A more general admissibility result

In this section we prove a generalization (Theorem 4.1) of Theorem 2.1: Allowing for certain real number amplitudes, we show that the connected admissible graphs are precisely those depicted in Figure 4.1. The proof of Theorem 4.1 uses reasoning from Ch. 5 and 6 of [Erik2]. The key is the application of the Perron–Frobenius theory of nonnegative real matrices to a matrix closely related to the amplitude matrix for the graph.

Readers familiar with Coxeter groups will notice that the families of graphs of Figure 4.1 correspond to the finite irreducible Coxeter groups. However, the proof of Theorem 4.1 requires no Coxeter group theory. We will see in §6 how the classification of finite irreducible Coxeter groups can be deduced from Theorem 4.1.

Now we play the numbers game in the following more general environment. An *E-generalized Cartan matrix* or *E-GCM* is a real $n \times n$ matrix $M = (M_{ij})_{i,j \in I_n}$ satisfying the requirements that each main diagonal matrix entry is 2, that all other matrix entries are nonpositive, that if a matrix entry M_{ij} is nonzero then its transpose entry M_{ji} is also nonzero, and that if $M_{ij}M_{ji}$ is nonzero then $M_{ij}M_{ji} \geq 4$ or $M_{ij}M_{ji} = 4\cos^2(\pi/k_{ij})$ for some integer $k_{ij} \geq 3$. We apply the language of GCM graphs in this E-GCM setting, so an *E-GCM graph* is a pair (Γ, M) where M is an E-GCM, etc. For the remainder of this section, (Γ, M) denotes an E-GCM graph. Examining the proofs and references given in §3, one sees that Lemmas 3.2, 3.4, 3.5, 3.7, and 3.8 as well as the Strong Convergence and Comparison Theorems stated for GCM graphs also hold for E-GCM

Figure 4.1: Families of connected E-Coxeter graphs.
(For adjacent nodes, the notation $\textcircled{m}$ means that the amplitude product on the edge is $4\cos^2(\pi/m)$; for an unlabelled edge take $m = 3$.)



graphs. Any particular two-node E-GCM graph is depicted in the same way as a two-node GCM graph. We use $\gamma_1 \overset{\textcircled{m}}{\bullet\!\!-\!\!-\!\!-\!\!\bullet} \gamma_2$ for the collection of all two-node E-GCM graphs for which $M_{12}M_{21} = pq = 4\cos^2(\pi/m)$ for an integer $m > 3$; we use $m = \infty$ if $M_{12}M_{21} = pq \geq 4$. When $m = 3$ (i.e. $pq = 1$), we use an unlabelled edge $\gamma_1 \bullet\!\!-\!\!-\!\!-\!\!\bullet \gamma_2$. An *E-Coxeter graph* will be any E-GCM graph whose connected components come from one of the collections of Figure 4.1.

The peculiar constraints on products of transpose pairs of E-GCM entries are precisely those required in order to guarantee strong convergence for these "E-games" (see [Erik6]). These constraints also afford a precise connection between E-games and certain geometric representations of Coxeter groups. This connection was developed by Vinberg [Vin] and the second author [Erik2], [Erik5], and it has also been studied in [BB], [Don2], and [Pro5].

The main result of this section is:

Theorem 4.1 *A connected E-GCM graph is admissible if and only if it is a connected E-Coxeter graph. In these cases, for any given initial position every game sequence will converge to the same terminal position in the same finite number of steps.*

The proof is at the end of the section and requires the following linear algebra set-up. A matrix (or vector) of real numbers is *nonnegative* if all its entries are nonnegative. For real matrices $X = (X_{ij})$ and $Y = (Y_{ij})$, say $X \geq Y$ if $X_{ij} \geq Y_{ij}$ for all i, j. With respect to this partial ordering, we let $X > Y$ mean $X \geq Y$ but $X \neq Y$. A square matrix $X = (X_{ij})_{i,j \in I_n}$ is *indecomposable* if there is no permutation of I_n such that we get a block matrix $\begin{pmatrix} X^{(1)} & X^{(2)} \\ O & X^{(3)} \end{pmatrix}$ where $X^{(1)}$ and $X^{(3)}$ are square and O denotes a zero matrix. A *principal submatrix* is a submatrix obtained by deleting some rows and the corresponding columns. Let $\varrho(X)$ denote the spectral radius, i.e. the modulus of the largest eigenvalue, of a square matrix X. The well-known results comprising the following theorem can be found for example in [Minc].

Theorem (Perron–Frobenius) *A nonnegative square indecomposable matrix X has one eigenvector, unique up to multiplication with a scalar, with all elements positive. Its corresponding eigenvalue is $\varrho(X) > 0$, and it is simple. No other eigenvector is nonnegative. If Y is a principal submatrix of X, then $\varrho(X) > \varrho(Y)$. If $X > Y$, then again $\varrho(X) > \varrho(Y)$.*

For $J \subseteq I_n$ and $m = |J|$, let $E(J)$ denote the $m \times m$ identity matrix $(\delta_{ij})_{i,j \in J}$. Let $E := E(I_n)$. Let Γ be a finite simple graph with node set $\{\gamma_x\}_{x \in I_n}$. Let A be a nonnegative matrix such that $A_{ij} > 0$ if and only if $A_{ji} > 0$ if and only if γ_i and γ_j are adjacent nodes in Γ. Let A^{sym} denote the symmetric nonnegative matrix for which $A^{\mathrm{sym}}_{ij} A^{\mathrm{sym}}_{ji} = A_{ij} A_{ji}$ for all i, j.

Lemma 4.2 *Keep the notation of the preceding paragraph. (1) Γ is connected if and only if A is indecomposable. (2) If Γ is acyclic, then A has the same characteristic polynomial as A^{sym}, and all eigenvalues of A are real. (3) Let $B := A^{\mathsf{T}} - 2E$. If Γ is connected and acyclic and if $\varrho(A) < 2$, then there is a positive definite diagonal matrix D such that DB^{-1} is symmetric and positive definite.*

Proof. For (1), A is indecomposable if and only if there is no partitioning $I_n = J_1 \cup J_2$ such that $A_{ij} = 0$ when $i \in J_1$ and $j \in J_2$. That is, A is indecomposable iff Γ is connected. For (2), induct on the number of nodes. Without loss of generality assume node γ_1 has at most one adjacent node in Γ. Let Γ' be the subgraph obtained by removing node γ_1. Let A' be the corresponding principal submatrix of A, and set $J' := I_n \setminus \{1\}$. If a node γ_2 is adjacent to γ_1, let Γ'' be the subgraph obtained by removing γ_1 and γ_2, and let A'' be the corresponding principal submatrix. In any case, set $J'' := I_n \setminus \{1, 2\}$. Let x be an indeterminate. It is easy to check that

$$\det(xE - A) = x \det(xE(J') - A') - A_{12}A_{21} \det(xE(J'') - A'').$$

Assuming the result holds for such graphs on fewer than n nodes, it follows from this computation that $\det(xE - A) = \det(xE - A^{\mathrm{sym}})$. Since a symmetric real matrix has real eigenvalues, the result follows. For (3), note that all eigenvalues for B are real and positive. By §1.5 of [Kum], B^{T} is symmetrizable and there is a positive definite diagonal matrix D such that $D^{-1}B^{\mathsf{T}}$ is symmetric. (See remarks following Definition 1.5.1 as well as Exercise 1.5.1 in [Kum].) Then $(D^{-1}B^{\mathsf{T}})^{-1}$ is symmetric and positive definite, hence DB^{-1} is symmetric and positive definite. □

We apply these linear algebra facts to E-games as follows. Think of a position $\lambda = (\lambda_i)_{i \in I_n}$ on (Γ, M) as an $n \times 1$ column vector. Firing node γ_i from λ results in the position $\lambda^{\mathrm{new}} = \lambda - \lambda_i M^{\mathsf{T}} \omega_i$.

For any connected E-GCM subgraph (Γ', M') with nodes indexed by the set $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$, set $A := A_{\Gamma',M'} := 2E(J) - M'$. Then A is a nonnegative real matrix. Connectedness of Γ' implies that A is indecomposable. By Perron–Frobenius, there is a vector $\nu := \nu_{\Gamma',M'} = (\nu_i)_{i \in I_n}$ (unique up to a positive scalar multiple) with $\nu_i > 0$ for all i and $A\nu = \varrho(A)\nu$. A *looping game* is a nonempty legal sequence of node firings from some position λ that returns to position λ.

Lemma 4.3 *Suppose (Γ, M) is connected. Let $A := A_{\Gamma,M}$ and $\nu := \nu_{\Gamma,M}$. (1) Suppose that A has largest eigenvalue $\varrho(A) \geq 2$, that from a position λ some node is legally fired to obtain the position λ^{new}, and that $\nu^{\mathsf{T}}\lambda > 0$. Then $\nu^{\mathsf{T}}\lambda^{\text{new}} > 0$, and some node may be legally fired from position λ^{new}. (2) If there is a looping game on (Γ, M), then $\varrho(A) = 2$.*

Proof. For (1) we have $\nu^{\mathsf{T}}\lambda^{\text{new}} = \nu^{\mathsf{T}}\lambda - \lambda_i(M\nu)^{\mathsf{T}}\omega_i = \nu^{\mathsf{T}}\lambda + (\varrho(A) - 2)\lambda_i\nu_i$, assuming γ_i is the fired node of the hypothesis. Since $\nu^{\mathsf{T}}\lambda > 0$, $\lambda_i > 0$, $\nu_i > 0$, and $\varrho(A) - 2 \geq 0$, then $\nu^{\mathsf{T}}\lambda^{\text{new}} > 0$. That some some number λ_j^{new} is positive follows from the facts that $\nu^{\mathsf{T}}\lambda^{\text{new}} > 0$ and $\nu_k > 0$ for all k. For (2), let $(\gamma_{i_1}, \ldots, \gamma_{i_j})$ be the sequence of nodes fired for a looping game played from position $\lambda^{(1)}$. Let $\lambda^{(1)}$, $\lambda^{(2)}$, …, $\lambda^{(j)}$, $\lambda^{(j+1)}$ be the sequence of positions for the looping game, with $\lambda^{(j+1)} = \lambda^{(1)}$. As in the proof of (1), we compute that $\nu^{\mathsf{T}}\lambda^{(j+1)} = \nu^{\mathsf{T}}\lambda^{(1)} + (\varrho(A) - 2)\sum_{k=1}^{j} \lambda_{i_k}^{(k)}\nu_{i_k}$. Then $(\varrho(A) - 2)\sum_{k=1}^{j} \lambda_{i_k}^{(k)}\nu_{i_k} = 0$. Since $\sum_{k=1}^{j} \lambda_{i_k}^{(k)}\nu_{i_k} > 0$, then $\varrho(A) = 2$. □

We apply part (1) of the preceding lemma to obtain the following criterion for inadmissibility.

Lemma 4.4 *Suppose (Γ, M) is connected, and suppose (Γ', M') is a connected E-GCM subgraph with $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$. Suppose $A' := A_{\Gamma',M'}$ has largest eigenvalue $\varrho(A') \geq 2$. Then (Γ, M) is not admissible.*

Proof. We prove the contrapositive. Suppose (Γ, M) is admissible. Then for some nonzero dominant position λ, there is a convergent game sequence. The Strong Convergence Theorem guarantees that all game sequences played from λ are convergent. In particular, we may choose a convergent game sequence that starts by firing only nodes indexed by the set $I_n \setminus J$. When there are no positive numbers on nodes for $I_n \setminus J$, we then have a dominant position λ' on (Γ', M'). Since every node must be fired in a convergent game sequence played from λ (Lemma 3.7), then λ' must be nonzero. Using Perron–Frobenius, take a vector $\nu' = (\nu'_j)_{j \in J}$ such that $\nu'_j > 0$ for all j and $A'\nu' = \varrho(A')\nu'$. Since λ' is nonzero and dominant for (Γ', M') and since $\nu'_j > 0$ for all j, it follows that $(\nu')^{\mathsf{T}}\lambda' > 0$. If $\varrho(A') \geq 2$, then Lemma 4.3 shows we will have a divergent game sequence on (Γ', M') played from λ', and hence a divergent game sequence on (Γ, M) from λ. So it must be the case that $\varrho(A') < 2$. □

Lemma 4.5 *Suppose (Γ, M) is a connected E-GCM graph for which $\varrho(A_{\Gamma,M}) < 2$. Let λ be any position on (Γ, M). Then there is a convergent game sequence played from λ.*

Proof. Set $A := A_{\Gamma,M}$ and $B := -M^{\mathsf{T}}$. From Lemma 4.2.3 it follows that there is a positive definite diagonal matrix D such that DB^{-1} is symmetric and positive definite. Suppose that a position μ^{new} is obtained from some position μ by legally firing γ_i. We claim that $(\mu^{\text{new}})^{\mathsf{T}}DB^{-1}\mu^{\text{new}} = \mu^{\mathsf{T}}DB^{-1}\mu$. To see this, make the substitution $\mu^{\text{new}} = \mu + \mu_i B\omega_i$ on the left-hand side. Then the claim is equivalent to the statement that $\mu_i\mu^{\mathsf{T}}D\omega_i + \mu_i\omega_i^{\mathsf{T}}B^{\mathsf{T}}DB^{-1}\mu + \mu_i^2\omega_i^{\mathsf{T}}B^{\mathsf{T}}D\omega_i = 0$. Now $\mu_i\mu^{\mathsf{T}}D\omega_i$ is just $\mu_i^2 D_{ii}$. Since DB^{-1} is symmetric, then $\mu_i\omega_i^{\mathsf{T}}B^{\mathsf{T}}DB^{-1}\mu$ is also $\mu_i^2 D_{ii}$. Finally, $\mu_i^2\omega_i^{\mathsf{T}}B^{\mathsf{T}}D\omega_i = \mu_i^2 D_{ii}B_{ii} = -2\mu_i^2 D_{ii}$.

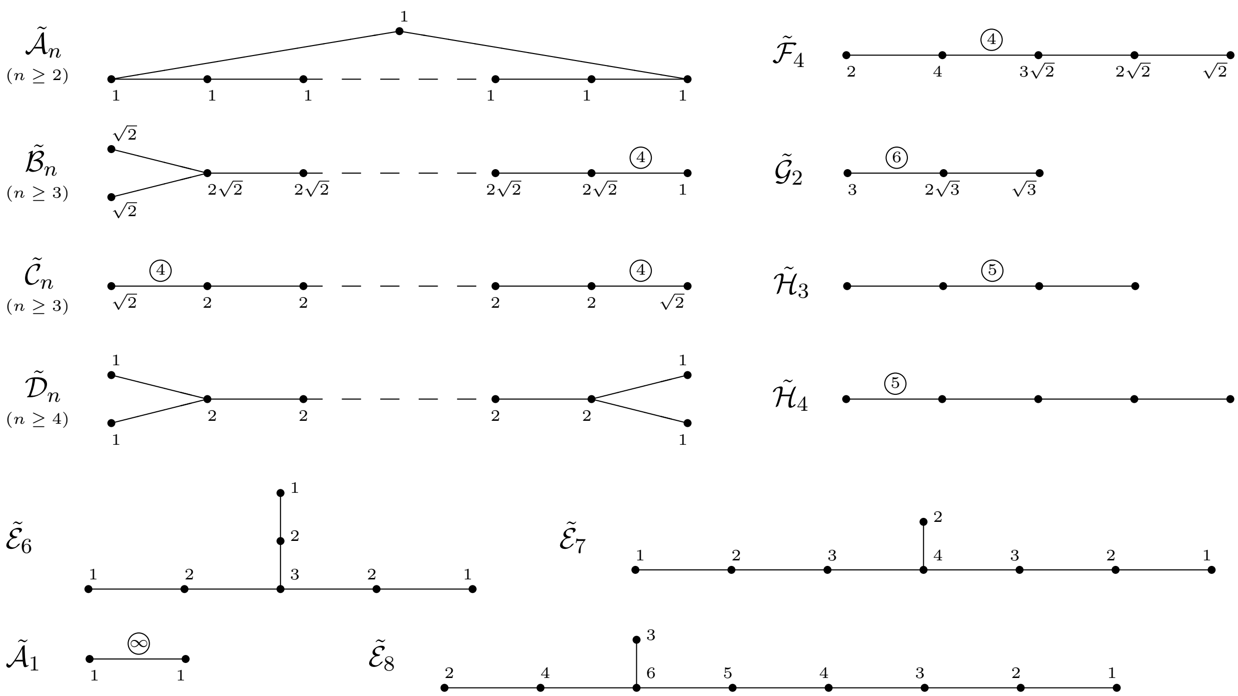
Topologize the space of positions by identifying it with $\mathbb{R}^n$. Now let λ be any position. Let $\mathfrak{P}(\lambda)$ be the set of all positions that are reachable from position λ by legal firing sequences. Note that $\mathfrak{P}(\lambda)$ is a discrete subset of the space of all positions. The preceding paragraph shows that $\mathfrak{P}(\lambda) \subseteq \{\text{positions } \mu \,|\, \mu^{\mathsf{T}} DB^{-1}\mu = \lambda^{\mathsf{T}} DB^{-1}\lambda\}$. The latter set is compact since DB^{-1} is positive definite. Since $\mathfrak{P}(\lambda)$ is a discrete subset of a compact set, then $\mathfrak{P}(\lambda)$ is finite. By Lemma 4.3.2 there are no looping games on (Γ, M). Since there are no repeated positions in E-game play from λ and since the set of positions reachable from λ is finite, it follows that there must be a convergent game sequence from λ. $\square$

Lemma 4.6 *Let (Γ, M) be an E-GCM graph from one of the families of Figure 4.2. If (Γ, M) is from one of the $\tilde{\mathcal{H}}_3$ or $\tilde{\mathcal{H}}_3$ families, then $\varrho(A_{\Gamma,M}) > 2$. If (Γ, M) is in the $\tilde{\mathcal{A}}_n$ family ($n \geq 1$), then $\varrho(A_{\Gamma,M}) \geq 2$. If (Γ, M) is from any other family of Figure 4.2, then $\varrho(A_{\Gamma,M}) = 2$.*

Figure 4.2: Some connected E-GCM graphs for Lemma 4.6

(Notation: An E-GCM graph in family $\tilde{\mathcal{X}}_n$ below has $n+1$ nodes.

The numbers assigned to the nodes of some of these graphs pertain to the proof of Lemma 4.6.)



Proof. Consider an E-GCM graph (Γ, M) from one of the $\tilde{\mathcal{B}}_n$, $\tilde{\mathcal{C}}_n$, $\tilde{\mathcal{D}}_n$, $\tilde{\mathcal{E}}_6$, $\tilde{\mathcal{E}}_7$, $\tilde{\mathcal{E}}_8$, $\tilde{\mathcal{F}}_4$, or $\tilde{\mathcal{G}}_2$ families of Figure 4.2. Take M to be symmetric. Regard the numbers on the nodes as coordinates for a vector ν. One can confirm in each case that $A_{\Gamma,M}\nu = 2\nu$, and hence that $\varrho(A_{\Gamma,M}) = 2$ and $\nu = \nu_{\Gamma,M}$. By Lemma 4.2.2, it follows that $\varrho(A_{\Gamma',M'}) = 2$ for any E-GCM graph (Γ', M') in one of these families. Now take (Γ, M) in the $\tilde{\mathcal{A}}_1$ family with M symmetric and $M_{12}M_{21} = 4$. With ν as prescribed in Figure 4.2, we get $A_{\Gamma,M}\nu = 2\nu$, and hence $\varrho(A_{\Gamma,M}) = 2$ and $\nu = \nu_{\Gamma,M}$. By Lemma 4.2.2, it follows that $\varrho(A_{\Gamma',M'}) = 2$ for any E-GCM graph (Γ', M') in the $\tilde{\mathcal{A}}_1$ family for which $M'_{12}M'_{21} = 4$. If (Γ'', M'') is any E-GCM graph from the $\tilde{\mathcal{A}}_1$ family, then we can find a graph (Γ', M') from this same family such that $M'_{12} \geq M''_{12}$, $M'_{21} \geq M''_{21}$, and $M'_{12}M'_{21} = 4$. Then $A_{\Gamma',M'} \leq A_{\Gamma'',M''}$, and hence by Perron–Frobenius $\varrho(A_{\Gamma'',M''}) \geq 2$. We used a computer algebra system to calculate the eigenvalues for $A_{\Gamma,M}$ when (Γ, M) is in one of the $\tilde{\mathcal{H}}_3$ or $\tilde{\mathcal{H}}_4$ families with M symmetric. In these cases one finds that $\varrho(A_{\Gamma,M}) > 2$. It follows from Lemma 4.2.2 that $\varrho(A_{\Gamma,M}) > 2$ for any E-GCM graph (Γ, M) in the $\tilde{\mathcal{H}}_3$ or $\tilde{\mathcal{H}}_4$ families. Now consider an E-GCM graph (Γ, M) in the $\tilde{\mathcal{A}}_n$ family ($n \geq 2$). Assume the nodes are numbered consecutively around the cycle. Set $\Pi := (-1)^n M_{12}M_{23}\cdots M_{n-1,n}M_{n,1}$, so $\Pi > 0$. One can show by a computation that the characteristic polynomial of $A := A_{\Gamma,M}$ is the characteristic polynomial of A^{sym} plus the constant $2 - \Pi - \frac{1}{\Pi}$. We have $A^{\mathrm{sym}}\nu = 2\nu$ for the vector ν identified in Figure 4.2, so 2 is an eigenvalue for A^{sym}. Since $2 - \Pi - \frac{1}{\Pi}$ is nonpositive, it follows that A has a real eigenvalue no less than 2. □

Remark In [Erik2], the second author classifies the "E-loopers," i.e. those connected E-GCM graphs with looping games. It is shown that an E-looper must be one of: an E-GCM graph in one of the $\tilde{\mathcal{B}}_n$, $\tilde{\mathcal{C}}_n$ $\tilde{\mathcal{D}}_n$, $\tilde{\mathcal{E}}_6$, $\tilde{\mathcal{E}}_7$, $\tilde{\mathcal{E}}_8$, $\tilde{\mathcal{F}}_4$, or $\tilde{\mathcal{G}}_2$ families; an E-GCM graph (Γ, M) in the $\tilde{\mathcal{A}}_1$ family such that $-M_{12} = -M_{21} = 2$; or an E-GCM graph (Γ, M) in the $\tilde{\mathcal{A}}_n$ family ($n \geq 2$) such that $(-1)^n M_{12}M_{23}\cdots M_{n-1,n}M_{n,1} = 1$, where the nodes of Γ are are numbered consecutively around the cycle.

Lemma 4.7 *If (Γ, M) is a connected E-Coxeter graph, then $\varrho(A_{\Gamma,M}) < 2$.*

Proof. Let (Γ, M) be the E-GCM graph with symmetric M from the $\mathcal{I}_2(m)$ family. Let $(\tilde{\Gamma}, \tilde{M})$ be the E-GCM graph from the $\tilde{\mathcal{A}}_1$ family with symmetric $\tilde{M}$ satisfying $\tilde{M}_{12}\tilde{M}_{21} = 4$. Check that $\varrho(A_{\tilde{\Gamma},\tilde{M}}) = 2$ (see e.g. the proof of Lemma 4.6). Since $A_{\Gamma,M} < A_{\tilde{\Gamma},\tilde{M}}$, then by Perron–Frobenius $\varrho(A_{\Gamma,M}) < 2 = \varrho(A_{\tilde{\Gamma},\tilde{M}})$. By Lemma 4.2.2, $\varrho(A_{\Gamma,M}) < 2$ for any E-GCM graph (Γ, M) in the $\mathcal{I}_2(m)$ family. Use similar reasoning to see that other E-Coxeter graphs from Figure 4.1 have dominant eigenvalue less than 2. In particular, let (Γ, M) be the E-Coxeter graph from family $\mathcal{X}_n \in \{\mathcal{A}_n, \mathcal{B}_n, \mathcal{D}_n, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8, \mathcal{F}_4\}$ such that M is symmetric. Let $(\tilde{\Gamma}, \tilde{M})$ be the respective E-Coxeter graph from family $\tilde{\mathcal{X}}_n \in \{\tilde{\mathcal{A}}_n, \tilde{\mathcal{B}}_n, \tilde{\mathcal{D}}_n, \tilde{\mathcal{E}}_6, \tilde{\mathcal{E}}_7, \tilde{\mathcal{E}}_8, \tilde{\mathcal{F}}_4\}$ such that $\tilde{M}$ is symmetric. By Perron–Frobenius and the fact that $A_{\Gamma,M}$ is a principal submatrix of $A_{\tilde{\Gamma},\tilde{M}}$ we get $\varrho(A_{\Gamma,M}) < 2 = \varrho(A_{\tilde{\Gamma},\tilde{M}})$. By Lemma 4.2.2, $\varrho(A_{\Gamma,M}) < 2$ for any E-GCM graph (Γ, M) in the $\mathcal{X}_n$ family. We used a computer algebra system to calculate the eigenvalues for $A_{\Gamma,M}$ when (Γ, M) is in one of the $\mathcal{H}_3$ or $\mathcal{H}_4$ families with M symmetric. In these cases one finds that $\varrho(A_{\Gamma,M}) < 2$ as well. It follows from Lemma 4.2.2 that $\varrho(A_{\Gamma,M}) < 2$ for any E-Coxeter graph (Γ, M) in the $\mathcal{H}_3$ or $\mathcal{H}_4$ families. □

Proof of Theorem 4.1. Lemmas 4.5 and 4.7 demonstrate admissibility for any connected E-Coxeter graph. In such a case it follows from the Strong Convergence and Comparison Theorems that for any given initial position any two game sequences will converge to the same terminal position in the same finite number of steps. Now suppose a connected E-GCM graph (Γ, M) is admissible. First, we show that (Γ, M) has no cycles. If an E-GCM subgraph (Γ', M') is a cycle, then we can let M'' be an E-GCM for which $M''_{ij} \geq M'_{ij}$ for all appropriate i, j and such that (Γ'', M'') is in the $\tilde{\mathcal{A}}_k$ family. Then $A_{\Gamma'',M''} < A_{\Gamma',M'}$, and by Perron–Frobenius, $2 \leq \varrho(A_{\Gamma'',M''}) < \varrho(A_{\Gamma',M'})$. But then Lemma 4.4 implies that (Γ, M) is not admissible. Hence (Γ, M) has no cycles. Similar reasoning using inadmissibility of E-GCM graphs in the $\tilde{\mathcal{B}}_k$ family shows that (Γ, M) cannot have a node with more than two neighbors if $M_{ij}M_{ji} > 1$ for some pair of adjacent nodes γ_i and γ_j, i.e. $m_{ij} > 3$. Assume (Γ, M) has a pair of adjacent nodes γ_i and γ_j with $M_{ij}M_{ji} > 1$. Inadmissibility of E-GCM graphs in the $\tilde{\mathcal{A}}_1$ family shows that for all such pairs of nodes $M_{ij}M_{ji} < 4$. Now using inadmissibility of E-GCM graphs in the $\tilde{\mathcal{C}}_k$ family shows that (Γ, M) can have at most one such pair of adjacent nodes. Using inadmissibility of E-GCM graphs in the $\tilde{\mathcal{F}}_4$ family shows that if neither γ_i nor γ_j is a leaf, then (Γ, M) must be in the $\mathcal{F}_4$ family. If at least one of γ_i or γ_j is a leaf, then inadmissibility of E-GCM graphs in the $\tilde{\mathcal{G}}_2$, $\tilde{\mathcal{H}}_3$, and $\tilde{\mathcal{H}}_4$ families shows that (Γ, M) must be in the $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{H}_3$, $\mathcal{H}_4$, or $\mathcal{I}_2(m)$ family. Now assume that for each pair of adjacent nodes γ_i and γ_j in

(Γ, M) we have $M_{ij}M_{ji} = 1$, i.e. $m_{ij} = 3$. Using inadmissibility of E-GCM graphs in the $\tilde{\mathcal{D}}_k$ family shows that (Γ, M) can have at most one node with more than two neighbors and that such a node can have at most three neighbors. Using inadmissibility of E-GCM graphs in the $\tilde{\mathcal{E}}_6$, $\tilde{\mathcal{E}}_7$, and $\tilde{\mathcal{E}}_8$ families we conclude if (Γ, M) has a node with three neighbors such that at most one of them is a leaf in the tree, then (Γ, M) must be in one of the $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{E}_8$ families. Otherwise, (Γ, M) is in the $\mathcal{A}_n$ or $\mathcal{D}_n$ family. $\square$

5. A structure property for edge-colored ranked posets

The so-called structure property we study in this section is motivated by a certain Lie-theoretic phenomenon. "Crystal graphs" (see [Stem]), "supporting graphs" (see [Don1]), and "splitting posets" (see [ADLMPPW]) are edge-colored directed graphs which encode certain information about the finite-dimensional representations of a given finite-dimensional complex simple Lie algebra $\mathfrak{g}$. Ignoring edge colors, any such graph is the Hasse diagram for a ranked partially ordered set (defined below). The number n of edge colors is the rank of $\mathfrak{g}$, i.e. the dimension of a Cartan subalgebra. The edges and edge colors for such a poset determine a "weight" rule, that is, an assignment of an integer n-tuple to each vertex of the Hasse diagram. The weight rule has the following property: the difference of the weights for two vertices in such a graph is the ith row of the Cartan matrix for $\mathfrak{g}$ if the vertices form an edge whose color corresponds to i. The results of this section begin to address the question: If an edge-colored ranked poset possesses such a property relative to some matrix M, what can be said about M?

The set-up of this paragraph follows Section 2 of [ADLMPPW]. Identify a partially ordered set R with its *Hasse diagram*, that is, the directed graph whose edges depict the *covering relations* for the poset: for elements $\mathbf{s}$ and $\mathbf{t}$ in R the directed edge $\mathbf{s} \rightarrow \mathbf{t}$ means that $\mathbf{s} < \mathbf{t}$ and if $\mathbf{s} \leq \mathbf{x} \leq \mathbf{t}$ then $\mathbf{s} = \mathbf{x}$ or $\mathbf{x} = \mathbf{t}$. The edge set $\mathcal{E}(R)$ is the set of all covering relations in R. Given an n-element set I_n, a function $\mathbf{edgecolor}_R : \mathcal{E}(R) \longrightarrow I_n$ is an *edge coloring function*, in which case we say the edges of R are colored by the set I_n. The notation $\mathbf{s} \xrightarrow{i} \mathbf{t}$ means that $\mathbf{edgecolor}_R(\mathbf{s} \rightarrow \mathbf{t}) = i$. If J is a subset of I_n, remove all edges from R whose colors are not in J; connected components of the resulting edge-colored poset are called *J-components* of R. For any $\mathbf{x}$ in R, we let $\mathbf{comp}_J(\mathbf{x})$ denote the J-component of R containing $\mathbf{x}$. When R is finite we say it is *ranked* if there exists a surjective function $\rho : R \longrightarrow \{0, 1, \ldots, l\}$ such that $\rho(\mathbf{s}) + 1 = \rho(\mathbf{t})$ whenever $\mathbf{s} \rightarrow \mathbf{t}$; in this case the number l is the *length* of R, ρ is a *rank function*, and $\rho(\mathbf{x})$ is the *rank* of any element $\mathbf{x}$ in R. With respect to an edge coloring function $\mathbf{edgecolor}_R : \mathcal{E}(R) \longrightarrow I_n$ on our finite ranked poset R, we let $\rho_i(\mathbf{x})$ denote the rank of an element $\mathbf{x}$ in R within its i-component $\mathbf{comp}_i(\mathbf{x})$ and we let $l_i(\mathbf{x})$ denote the length of $\mathbf{comp}_i(\mathbf{x})$. For any $\mathbf{x} \in R$, let $wt_R(\mathbf{x})$ be the n-tuple $(\, m_i(\mathbf{x})\,)_{i \in I_n}$, where $m_i(\mathbf{x}) := 2\rho_i(\mathbf{x}) - l_i(\mathbf{x})$ for each $i \in I_n$. Now let $M = (M_{i,j})_{i,j \in I_n}$ be a GCM, and for each $i \in I_n$ let α_i denote the ith row of M: $\alpha_i = (M_{i,j})_{j \in I_n}$. An *$M$-structure poset* $(R, \mathbf{edgecolor}_R)$ is a finite ranked poset R together with an edge coloring function $\mathbf{edgecolor}_R : \mathcal{E}(R) \longrightarrow I_n$ satisfying the following *M-structure property*: $wt_R(\mathbf{s}) + \alpha_i = wt_R(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in R. Note the finiteness requirement of the definition.

Proposition 5.1 *Suppose $M = (M_{i,j})_{i,j \in I_n}$ is a generalized Cartan matrix with connected GCM graph (Γ, M), and suppose there is an M-structure poset $(R, \mathbf{edgecolor}_R)$ with at least one edge. Then (Γ, M) is a connected Dynkin diagram of finite type, and $\mathbf{edgecolor}_R$ is surjective.*

Proof. Choose a vertex $\mathbf{t}_0$ for which $\lambda^{(0)} := wt_R(\mathbf{t}_0)$ is dominant. (For example, take $\mathbf{t}_0$ to be any element of highest rank in R.) Since R has at least one edge, $\lambda^{(0)}$ is nonzero. Let $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$ be any game sequence played from initial position $\lambda^{(0)}$ on (Γ, M). For each $p \geq 1$, $\lambda^{(p)}$ is the position in the sequence just after node γ_{i_p} is fired. Next, we define by induction a special sequence of elements from R. For any $p \geq 1$, suppose we have a sequence $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{p-1}$ for which $wt_R(\mathbf{t}_q) = \lambda^{(q)}$ and $\rho(\mathbf{t}_q) < \rho(\mathbf{t}_{q-1})$ for all $1 \leq q \leq p-1$. We wish to show that we can extend this sequence by an element $\mathbf{t}_p$ so that $wt_R(\mathbf{t}_p) = \lambda^{(p)}$ and $\rho(\mathbf{t}_p) < \rho(\mathbf{t}_{p-1})$. Take $\mathbf{t}_p$ to be any element of $\mathbf{comp}_{i_p}(\mathbf{t}_{p-1})$ for which $\rho_{i_p}(\mathbf{t}_p) = l_{i_p}(\mathbf{t}_{p-1}) - \rho_{i_p}(\mathbf{t}_{p-1})$. Since firing node γ_{i_p} in the given numbers game is legal from position $\lambda^{(p-1)}$, then $\lambda_{i_p}^{(p-1)} > 0$. But $\lambda_{i_p}^{(p-1)} = 2\rho_{i_p}(\mathbf{t}_{p-1}) - l_{i_p}(\mathbf{t}_{p-1})$. So, $\rho_{i_p}(\mathbf{t}_p) = l_{i_p}(\mathbf{t}_{p-1}) - \rho_{i_p}(\mathbf{t}_{p-1}) < \rho_{i_p}(\mathbf{t}_{p-1})$. It follows that $\rho(\mathbf{t}_p) < \rho(\mathbf{t}_{p-1})$. Since R satisfies the M-structure condition, then $wt_R(\mathbf{t}_p) = wt_R(\mathbf{t}_{p-1}) - \lambda_{i_p}^{(p-1)}\alpha_{i_p}$. But $\lambda^{(p-1)} = wt_R(\mathbf{t}_{p-1})$ and $\lambda^{(p)} = \lambda^{(p-1)} - \lambda_{i_p}^{(p-1)}\alpha_{i_p}$. In other words, $wt_R(\mathbf{t}_p) = \lambda^{(p)}$. So we have extended our sequence as desired. But since R is finite, any such sequence must also be finite. Hence the game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$ is convergent. Then by Theorem 2.1, (Γ, M) must be a Dynkin diagram of finite type. Since every node must be fired in a convergent game sequence for a numbers game played on a connected GCM graph (Lemma 3.7), it follows that $\mathbf{edgecolor}_R$ is surjective. $\square$

Given an M-structure poset $(R, \mathbf{edgecolor}_R)$ for a generalized Cartan matrix $M = (M_{i,j})_{i,j \in I_n}$, we say $\mathbf{edgecolor}_R$ is *sufficiently surjective* if in each connected component of the GCM graph (Γ, M) there is some node γ_j such that $j \in \mathbf{edgecolor}_R(\mathcal{E}(R))$.

Theorem 5.2 *Let $(\Gamma, M = (M_{i,j})_{i,j \in I_n})$ be a GCM graph. Suppose there is an M-structure poset $(R, \mathbf{edgecolor}_R)$ with sufficiently surjective edge coloring function $\mathbf{edgecolor}_R$. Then (Γ, M) is a Dynkin diagram of finite type, and $\mathbf{edgecolor}_R$ is surjective.*

Proof. Pick a connected component (Γ', M') of (Γ, M), and let $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$. Now pick a J-component $\mathcal{C}$ of R such that $\mathcal{C}$ contains at least one edge whose color is from J. Proposition 5.1 implies that (Γ', M') is a connected Dynkin diagram of finite type and that for every color in J there is an edge in $\mathcal{C}$ having that color. Applying this reasoning to each connected component of (Γ, M), we see that (Γ, M) is a Dynkin diagram of finite type and that $\mathbf{edgecolor}_R$ is surjective. $\square$

Remark The existence of M-structure posets meeting the hypotheses of Theorem 5.2 can be seen as follows. Given a Dynkin diagram (Γ, M) of finite type, let $\mathfrak{g}$ be the finite-dimensional complex semisimple Lie algebra with Cartan matrix M. Let V be any finite-dimensional irreducible representation of $\mathfrak{g}$ whose highest weight has the following property: it is a nonnegative integer linear combination of fundamental weights such that for each connected component (Γ', M') of (Γ, M) there is a fundamental weight appearing nontrivially in the linear combination whose corresponding fundamental position is in (Γ', M'). Then any supporting graph for V is a connected M-structure poset satisfying the hypotheses of Theorem 5.2, cf. Lemmas 3.1 and 3.2 of [Don1]. From §3 of [Stem], one can similarly see that any "admissible system" for V (e.g. a crystal graph) is a connected M-structure poset meeting the hypotheses of Theorem 5.2.

6. Classifications of finite-dimensional Kac–Moody algebras and finite Coxeter and Weyl groups

Here we re-derive some well-known Dynkin diagram classification results using an argument from §8.4 of [Erik2]. Since the classifications of the finite-dimensional Kac–Moody algebras and of the finite Coxeter and Weyl groups are not used in the proofs of Theorems 2.1 and 4.1, we can use these theorems to obtain these classification results. This is recorded below as Theorem 6.2. These classifications are obtained in [Kac] and [Hum2] respectively by carefully studying properties of the generalized Cartan matrix (or a closely related matrix). The key observation is that a divergent game sequence requires an infinite group: any finite firing sequence corresponds to an element of the same finite length in a corresponding Coxeter group. From this reasoning we obtain for free the classification of the finite-dimensional Kac–Moody algebras.

Given an E-GCM graph (Γ, M), define the associated Coxeter group $W = W(\Gamma, M)$ to be the Coxeter group with identity denoted ε, generators $\{s_i\}_{i\in I_n}$, and defining relations $s_i^2 = \varepsilon$ for $i \in I_n$ and $(s_i s_j)^{m_{ij}} = \varepsilon$ for all $i \neq j$, where the m_{ij} are determined as follows:

$$m_{ij} = \begin{cases} k_{ij} & \text{if } M_{ij}M_{ji} = 4\cos^2(\pi/k_{ij}) \text{ for some integer } k_{ij} \geq 2 \\ \infty & \text{if } M_{ij}M_{ji} \geq 4 \end{cases}$$

(Conventionally, $m_{ij} = \infty$ means there is no relation between generators s_i and s_j.) One can think of the E-GCM graph as a refinement of the information from the Coxeter graph for the associated Coxeter group. Observe that any Coxeter group on a finite set of generators is isomorphic to the Coxeter group associated to some E-GCM graph. The Coxeter group W is *irreducible* if Γ is connected. If the graph Γ has connected components $\Gamma_1, \ldots, \Gamma_k$ with corresponding amplitude matrices $M_1, \ldots, M_k$, then $W(\Gamma, M) \approx W(\Gamma_1, M_1) \times \cdots \times W(\Gamma_k, M_k)$. Let ℓ denote the length function for W. An expression $s_{i_p} \cdots s_{i_2} s_{i_1}$ for an element of W is *reduced* if $\ell(s_{i_p} \cdots s_{i_2} s_{i_1}) = p$. An empty product in W is taken as ε. For a firing sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_p})$ from some initial position on (Γ, M), the corresponding element of W is taken to be $s_{i_p} \cdots s_{i_2} s_{i_1}$. The next result follows from Propositions 4.1 and 4.2 of [Erik5] and is the key step in the proof of Theorem 6.2.

Proposition 6.1 *(1) If $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_p})$ is a legal sequence of node firings in a numbers game played from some initial position on an E-GCM graph (Γ, M), then $s_{i_p} \cdots s_{i_2} s_{i_1}$ is a reduced expression for the corresponding element of $W(\Gamma, M)$. (2) If $s_{i_p} \cdots s_{i_2} s_{i_1}$ is a reduced expression for an element of $W(\Gamma, M)$, then $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_p})$ is a legal sequence of node firings in a numbers game played from any strongly dominant position on E-GCM graph (Γ, M).*

The Lie algebra that is constructed next does not depend on the specific choices made. The definitions we use here basically follow [Kum] (but see also [Kac]). Given a GCM graph (Γ, M) with n nodes, choose a complex vector space $\mathfrak{h}$ of dimension $n + \mathrm{corank}(M)$. Choose n linearly independent vectors $\{\beta_i^\vee\}_{1\leq i\leq n}$ in $\mathfrak{h}$, and find n linearly independent functionals $\{\beta_i\}_{1\leq i\leq n}$ in $\mathfrak{h}^*$ satisfying $\beta_j(\beta_i^\vee) = M_{ij}$. The *Kac–Moody algebra* $\mathfrak{g} = \mathfrak{g}(\Gamma, M)$ is the Lie algebra over $\mathbb{C}$ generated by the set $\mathfrak{h} \cup \{x_i, y_i\}_{i\in I_n}$ with relations $[\mathfrak{h}, \mathfrak{h}] = 0$; $[h, x_i] = \beta_i(h)x_i$ and $[h, y_i] = -\beta_i(h)y_i$ for all $h \in \mathfrak{h}$ and $i \in I_n$; $[x_i, y_j] = \delta_{i,j}\beta_i^\vee$ for all $i, j \in I_n$; $(\mathrm{ad}x_i)^{1-M_{ji}}(x_j) = 0$ for $i \neq j$; and $(\mathrm{ad}y_i)^{1-M_{ji}}(y_j) = 0$ for $i \neq j$, where $(\mathrm{ad}z)^k(w) = [z, [z, \cdots, [z, w] \cdots]]$. If the graph Γ has connected components $\Gamma_1, \ldots, \Gamma_k$ with corresponding amplitude matrices $M_1, \ldots, M_k$, then $\mathfrak{g}(\Gamma, M) \approx \mathfrak{g}(\Gamma_1, M_1) \oplus \cdots \oplus \mathfrak{g}(\Gamma_k, M_k)$. Note that if $M_{ij}M_{ji} < 4$, then $M_{ij}M_{ji} \in \{0, 1, 2, 3\}$, and so $m_{ij} \in \{2, 3, 4, 6\}$. It is known (see for example Proposition 1.3.21 of [Kum]) that the associated *Weyl group* is $W(\Gamma, M)$.

Theorem 6.2 *(1) Given a generalized Cartan matrix, the associated Weyl group is finite if and only if the associated Kac–Moody algebra is finite-dimensional if and only if the associated GCM graph is a Dynkin diagram of finite type. (2) Given an E-GCM, the associated Coxeter group is finite if and only if the associated E-GCM graph is an E-Coxeter graph.*

Proof. For (2) consider an E-GCM graph (Γ, M). If $W(\Gamma, M)$ is finite, then it follows from Proposition 6.1.1 above that every game sequence played from some given strongly dominant position will converge. Then (Γ, M) is admissible and by Theorem 4.1 must be an E-Coxeter graph. Conversely, suppose (Γ, M) is an E-Coxeter graph. Pick a strongly dominant position λ. Since every game sequence played from λ converges to the same terminal position in the same finite number of steps, then by Proposition 6.1.2 we have an upper bound on the lengths of reduced expressions in $W(\Gamma, M)$. Then $W(\Gamma, M)$ is finite. For (1) observe that this same reasoning together with Theorem 2.1 shows that for a GCM graph (Γ, M), the associated Weyl group $W(\Gamma, M)$ is finite if and only if (Γ, M) is a Dynkin diagram of finite type. By the root space decomposition of $\mathfrak{g}(\Gamma, M)$ ([Kum] Theorem 1.2.1) and Proposition 1.4.2 of [Kum], we have that $\mathfrak{g}(\Gamma, M)$ is finite-dimensional if and only if $W(\Gamma, M)$ is finite. $\square$

It is well known that the Kac–Moody algebras associated to the Dynkin diagrams of Figure 2.1 are the complex finite-dimensional simple Lie algebras (see for example [Hum1] §18). It is also well known that Lie algebras corresponding to distinct Dynkin diagrams of Figure 2.1 are non-isomorphic. For the associated Weyl groups, the only redundancy is that the groups corresponding to the B_n and C_n graphs for $n \geq 3$ are the same. The irreducible Coxeter groups associated to two connected E-Coxeter graphs are isomorphic if and only if the graphs are of the same type $\mathcal{X}_n \in \{\mathcal{A}_n, \mathcal{B}_n, \mathcal{D}_n, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8, \mathcal{F}_4, \mathcal{H}_3, \mathcal{H}_4, \mathcal{I}_2(m)\}$.

7. Comments on connections with other work

Say an E-GCM graph is *strongly admissible* if every nonzero dominant position has a convergent game sequence. In [Erik2], it is shown that: *A connected E-GCM graph is strongly admissible if and only if it is a connected E-Coxeter graph.* This statement essentially combines Theorems 6.5 and 6.7 of [Erik2]. Wildberger re-derives this result in [Wil3]. See [Erik1] for an "A-D-E" version.

For a Cartan matrix M, the M-structure property of §5 is a necessary condition for an edge-colored ranked poset to carry certain information about some finite-dimensional representation of the corresponding semisimple Lie algebra. Indeed, identifying such combinatorial properties is part of a program described in [Don1] for obtaining combinatorial models for Lie algebra representations. For example, in [ADLMPPW], four families of finite edge-colored distributive lattices are introduced, one family for each of the four rank two semisimple Lie algebras. Each lattice possesses the M-structure property for a Cartan matrix M corresponding to the appropriate rank two semisimple Lie algebra. The "weight-generating functions" on these lattices are Weyl characters for the irreducible representations of the rank two semisimple Lie algebras. In [Don4], the posets of join irreducibles (cf. [Sta]) for these distributive lattices are shown to be characterized by a short list of combinatorial properties. These are called "semistandard posets" in [ADLMPPW]; the smallest of these posets are called "fundamental posets." In [DW], we will say how these fundamental and semistandard posets can be constructed from information obtained by playing the numbers game on two-node Dynkin diagrams of finite type. More generally, for an n-node Dynkin diagram of finite type we will show how to construct other fundamental posets from numbers games played from certain special fundamental positions, namely the "adjacency-free" fundamental positions classified in [Don2]. These fundamental posets can be combined to obtain semistandard posets whose corresponding distributive lattices produce the Weyl characters (as above) for certain irreducible representations of the corresponding semisimple Lie algebra. It is natural to ask to what extent such distributive lattice models for Weyl characters can be characterized combinatorially as in [Don4].

Acknowledgments We thank John Eveland for stimulating discussions during his work on an undergraduate research project [Eve] at Murray State University; this helped lead to the questions answered by our main results. We thank Norman Wildberger for sharing his perspective on the numbers game; this helped us formulate the question as well as our proof of our first main result. We thank Bob Proctor for many helpful communications during the preparation of this paper.

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