PICTURING REPRESENTATIONS OF SIMPLE LIE ALGEBRAS OF RANK TWO

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ABSTRACT

The two known families of supporting graphs for the Gelfand-Tsetlin bases for the irreducible representations of the rank two simple Lie algebra A_2 are presented as a guide for producing analogous distributive lattice supports for the irreducible representations of shape λ of the simple Lie algebra G_2. Littelmann produced a set of tableaux of shape "$6 \times \lambda$" that had the correct numbers of elements; i.e., the number of elements in his set of tableaux equaled the dimension of the corresponding representation of the Lie algebra G_2. A translation of these objects into tableaux of shape λ is made and a method of constructing distributive lattices with several important combinatorial qualities from these translated G_2 tableaux of shape λ is presented. Strong evidence is provided to support the claim that these Littelmann lattices are indeed supporting graphs for the irreducible representations of the Lie algebra G_2.

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Chapter 1

Introduction

There are several technical words in the title of this thesis that warrant an explanation at the outset. For now, it is safe to think of a "representation of a simple Lie algebra" as a collection of square matrices, or better yet as a collection of linear operators acting on some vector space (henceforth, the vector space being acted on will be referred to as a representing space). The goal of the program introduced in [Don] is to find the "nicest" possible bases for presenting the actions of the elements of simple Lie algebras on representing spaces. The approach is combinatorial and visual: the idea is to associate a certain directed graph with colored edges to each basis for the representing space.

These pictures have connections with many well-known combinatorial problems. In [Pr], for instance, Proctor shows how Erdős' "subset sum problem" can be solved by analyzing pictures for certain representations of the odd orthogonal Lie algebra $so(2n+1,\mathbb{C})$. Erdős' subset sum problem can be described as follows: Find a set of n positive numbers that has the most subsets having the same sum. For example, $\{1,2,3,4,5\}$ has three subsets whose sum is 7, and it can be shown that there is not another set of five numbers that will have more than three subsets with the same sum. More generally, it can be shown that $\{1,2,\ldots,n\}$ is an "optimal" set in this same sense: no other set of n positive numbers has more subsets which have the same sum. So pictures of representations of Lie algebras have intrinsic combinatorial, as well as algebraic, interest.

In this thesis, we will be primarily interested in representations of two “small” Lie algebras, the rank two Lie algebras A_2 and G_2. (These Lie algebras are small in the sense that the dimension of A_2 is 8 and the dimension of G_2 is 14.) For A_2, “nice” pictures for its representations are known, and these will be presented in this thesis. The main problem we will address is to produce similarly “nice” pictures for representations of G_2. There are two phases of this problem: the first phase is to produce pictures which could possibly be used to picture representations of G_2, and the second phase is to confirm that these pictures actually do arise from the action of G_2 on certain vector spaces. This thesis (mostly) completes the first phase by producing a family of pictures which satisfy some rare necessary conditions, thus making them good candidates for picturing the representations of G_2. In completing the first phase, we provide a suitably nice combinatorial environment for addressing the second phase of this problem.

The thesis is organized as follows: In the next chapter, we will provide some combinatorial background to our problem. We will present several definitions of terms that appear throughout the text, along with some results that are applied to our main problem later in this thesis. In Chapter 3, we will look at the algebraic context of our problem. Therein, we will formally define what a Lie algebra is and then describe in some detail the four “classical Lie algebras.” We will also look at substructures of Lie algebras and homomorphisms between Lie algebras, and then make some comparisons between these ideas and their analogous counterparts for more familiar algebraic structures like groups and rings. In Section 3.4, we define the main algebraic object of interest to our problem. We will say what a representation of a Lie algebra is and then examine some representations of a specific Lie algebra. In Chapter 4, we will link the two settings together; that is, we will make a connection between the algebraic world of representations and the combinatorial world of partially ordered sets and directed graphs. There we will learn how to produce a graph from a Lie algebra representation and how we can know that these graphs are the kind for which we are looking. In Chapter 5, we will describe the “nice” pictures that are known for the irreducible representations for the simple Lie algebra A_2. Those pictures will serve as a guide for the main problem we address in this thesis: In the final chapter, we offer a construction for what we believe to be “nice” pictures for the irreducible representations of the rank two simple Lie algebra G_2 that are analogous to those that are known for A_2. We will end this thesis by providing strong evidence for our claim about the G_2 pictures.

Chapter 2

Combinatorics Background

This section can be skipped on the first reading, and consulted for definitions as needed. However, results 2.2.2 and 2.2.3 will be referred to in later sections. For more on these definitions, see [Sta].

2.1 Posets and Directed Graphs. A *partially ordered set* (or *poset*) P is a set together with a partial order, $\leq$, that satisfies these three axioms:

(1) $x \leq x$ for all $x \in P$ (reflexive property),

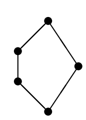
(2) $x \leq y$ and $y \leq x \Rightarrow x = y$ for $x, y \in P$ (antisymmetric property),

(3) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ for $x, y \in P$ (transitive property).

Let P be a poset and let $\mathbf{s}$ and $\mathbf{t}$ be elements of P. We write $\mathbf{s} \rightarrow \mathbf{t}$ if $\mathbf{s}$ is *covered* by $\mathbf{t}$ in P (i.e. $\mathbf{s} \leq \mathbf{r} \leq \mathbf{t}$ in P implies that $\mathbf{s} = \mathbf{r}$ or $\mathbf{r} = \mathbf{t}$). The *order diagram* of a poset P is the directed graph whose nodes or vertices are the elements of P and whose directed edges are given by the covering relations in P. We will not usually distinguish a poset from its order diagram. When we depict the order diagram for a poset, arrows on the edges will not be drawn; the direction of these edges is taken to be "up." An element $\mathbf{t}$ in a poset P is *maximal* (respectively, *minimal*) if there are no elements above (respectively, below) $\mathbf{t}$ in the order diagram for P. We will only be using finite posets and directed graphs, and we will allow directed graphs to have at most one edge between any two vertices. A *chain* C in a poset P is a totally ordered subset; i.e., it is a subset for which any two elements are comparable, so if $\mathbf{s}$ and $\mathbf{t}$ are in C, then $\mathbf{s} \leq \mathbf{t}$ or $\mathbf{s} \geq \mathbf{t}$. An *antichain* in P is a subset of P whose elements are pairwise incomparable.

A *path* $\mathcal{P}$ from $\mathbf{s}$ to $\mathbf{t}$ in a directed graph P is a sequence $\mathcal{P} = (\mathbf{s} = \mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_p = \mathbf{t})$ such that either $\mathbf{s}_{j-1} \to \mathbf{s}_j$ or $\mathbf{s}_j \to \mathbf{s}_{j-1}$ for $1 \leq j \leq p$. A directed graph (or poset) P is *connected* if any two elements in P can be joined by a path.

We can "color" the edges of a directed graph (or poset) P with elements from a set I by assigning an element of I to each edge of P. We call P an *edge-colored directed graph* (with edges colored by the set I). Now fix an edge color i. An *i-component* of P is a connected component that is leftover whenever all edges of colors other than i are removed. Two edge-colored directed graphs are isomorphic if there is a bijection between them that preserves edges and edge colors. The *dual* P^* is the set $\{\mathbf{t}^*\}_{\mathbf{t} \in P}$ together with colored edges $\mathbf{t}^* \stackrel{i}{\to} \mathbf{s}^*$ $(i \in I)$ if and only if $\mathbf{s} \stackrel{i}{\to} \mathbf{t}$ in P. A poset P is *self-dual* if P and P^* are isomorphic as directed graphs.

For a directed graph (or poset) P, a *rank function* is a surjective function $\rho : P \longrightarrow \{0, \ldots, l\}$ (where $l \geq 0$) with the property that if $\mathbf{s} \to \mathbf{t}$ in P, then $\rho(\mathbf{s}) + 1 = \rho(\mathbf{t})$. We call l the *length* of P with respect to ρ, and the set $\rho^{-1}(i)$ is the i*th rank* of P. (Not every directed graph, or even every poset, has a rank function, as  shows.) If a directed graph P has a rank function ρ, the directed graph is the order diagram for some poset (in which case we call P a *ranked poset*).

A ranked poset P is *rank symmetric* if $|\rho^{-1}(i)| = |\rho^{-1}(l-i)|$ for $0 \leq i \leq l$. A ranked poset is *rank unimodal* if there is an m such that $|\rho^{-1}(0)| \leq |\rho^{-1}(1)| \leq \cdots \leq |\rho^{-1}(m)| \geq$

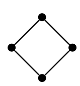
$|\rho^{-1}(m+1)| \geq \cdots \geq |\rho^{-1}(l)|$. It is *strongly Sperner* if for every $k \geq 1$, the largest union of k antichains is no larger than the largest union of k ranks.

2.2 Lattices. The posets that are the main focus in this thesis are actually *distributive lattices*. A *lattice* L is a poset in which any two elements x and y of L have a unique least upper bound (called the *join or sup* of x and y, and denoted $x \vee y$) and a unique greatest lower bound (called the *meet or inf* of x and y, and denoted $x \wedge y$).

A *distributive lattice* L is a lattice in which the meets and joins satisfy the "distributive laws":

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ and}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ for all } x, y, z \in L.$$

These technical conditions imply (among other things) that distributive lattices are built up out of "diamonds" . It can be shown that any distributive lattice is a ranked poset.

Any totally ordered set is easily seen to be a distributive lattice. But perhaps the most canonical example of a distributive lattice is the Boolean lattice $\mathcal{B}_n$. As a set $\mathcal{B}_n$ consists of all 2^n subsets of the n-element set $\{1, 2, \ldots, n\}$. The partial order is set containment. The meet (respectively, join) of two elements in $\mathcal{B}_n$ is just their intersection (respectively, union). By the well-known fact that set intersection distributes over set union, we see that $\mathcal{B}_n$ is a distributive lattice. As an example, the Boolean lattice $\mathcal{B}_3$ is shown below in Figure 2.2.1. Note that the set $\mathcal{B}_3$ consists of the eight subsets of $\{1, 2, 3\}$.

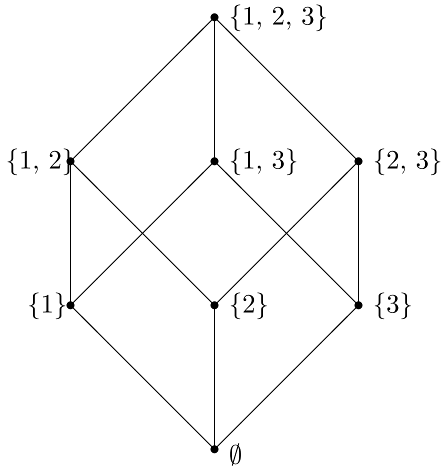


Figure 2.2.1 The Boolean Lattice $\mathcal{B}_3$

Proposition 2.2.2 *The dual of a distributive lattice is distributive.*

Proof. For any u, v in a distributive lattice, let u^*, v^* be their respective vertices in the dual of that distributive lattice. Note that $u \vee v = u^* \wedge v^*$ and $u \wedge v = u^* \vee v^*$. Then for any x, y, z in the distributive lattice, $x \wedge (y \vee z) = x^* \vee (y^* \wedge z^*)$ and $(x \wedge y) \vee (x \wedge z) = (x^* \vee y^*) \wedge (x^* \vee z^*)$. Since we know that the left sides of those two equations are equal, we can equate the right sides. Thus, $x^* \vee (y^* \wedge z^*) = (x^* \vee y^*) \wedge (x^* \vee z^*)$. Similarly, $x \vee (y \wedge z) = x^* \wedge (y^* \vee z^*)$ and $(x \vee y) \wedge (x \vee z) = (x^* \wedge y^*) \vee (x^* \wedge z^*)$. This means that $x^* \wedge (y^* \vee z^*) = (x^* \wedge y^*) \vee (x^* \wedge z^*)$. Thus, the dual of a distributive lattice is distributive. □

Lemma 2.2.3 *Let P_i be a distributive lattice for each i ($1 \leq i \leq k$), and let $L \subseteq P_1 \times \cdots \times P_k$. Suppose L is closed under component-wise max and component-wise min. (That is, for $s = (s_1, \ldots, s_k)$ and $t = (t_1, \ldots, t_k)$ in L, both $(s_1 \vee t_1, \ldots, s_k \vee t_k)$ and $(s_1 \wedge t_1, \ldots, s_k \wedge t_k)$ are in L.) Then L is a distributive lattice.*

Proof. Let $P_1, \ldots, P_k$ be distributive lattices and let $L \subseteq P_1 \times \cdots \times P_k$. Let $s = (s_1, \ldots, s_k) \in L$ and $t = (t_1, \ldots, t_k) \in L$. Set $x := (s_1 \vee t_1, \ldots, s_k \vee t_k)$ and $y := (s_1 \wedge t_1, \ldots, s_t \wedge t_k)$. First, we need to show that x is the least upper bound (or sup) in L for s and t, and that y is the greatest lower bound (or inf) in L for s and t.

To show that x is the least upper bound in L for s and t, we first need to show that x is an upper bound of s and t. Since $x_i = s_i \vee t_i$ for $i = 1, \ldots, k, x_i$ is an upper bound for s and t in each coordinate $i = 1, \ldots, k$. So x is an upper bound for s and t.

Now, we need to show that x is the least upper bound for s and t. Suppose w is an upper bound of s and t in L. Since $s \leq w$ and $t \leq w$ (in component-wise ordering), we see that $s_i \leq w_i$ and $t_i \leq w_i$ for $i = 1, \ldots, k$. Thus, $s_i \vee t_i \leq w_i$. But by the definition of $x, x_i = s_i \vee t_i$. Thus, $x_i \leq w_i$ for each $i = 1, \ldots, k$. Hence, x is the least upper bound of s and t in L. (A similar argument can be made to show that $y = s \wedge t = (s_1 \wedge t_1, \ldots, s_k \wedge t_k)$ is the greatest lower bound in L for s and t.) In particular, L is a lattice.

Finally, we need to show that for $a, b, c \in L, a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Let $a = (a_1, \ldots, a_k), b = (b_1, \ldots, b_k)$, and $c = (c_1, \ldots, c_k)$. To show that these distributive laws hold, we need to show that they hold in each coordinate a_i, b_i, and c_i for $i = 1, \ldots k$. Since $a_i, b_i, c_i \in P_i$ (a distributive lattice), these equalities hold for each $i = 1, \ldots k$. Thus, they hold for $a, b, c \in L$. Therefore, L is a distributive lattice. $\square$

Chapter 3

Introduction to Lie Algebras and Their Representations

3.1 Preliminaries. Much of the exposition in this chapter follows from [Hum], but our goal is to present those aspects of the theory that are needed to understand the connections between the combinatorial results of this thesis and the algebraic setting which gives context to these results.

Abstractly, a Lie algebra is just a vector space equipped with a certain non-associative multiplication. Lie algebras were originally invented in the late nineteenth and early twentieth centuries to provide a purely algebraic tool for studying Lie groups and their representations. Lie algebras have since become objects of mathematical interest in their own right because of close connections to other branches of mathematics including combinatorics, the theory of finite simple groups, and knot theory. The goal of Lie algebra representation theory is to see how Lie algebras arise as vector spaces of linear transformations. Concretely, this just means we hope to identify each element of the Lie algebra with a matrix.

Definition 3.1.1 A vector space $\mathcal{L}$ over an arbitrary field $\mathbb{F}$, with an operation $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by $(x, y) \mapsto [xy]$ and called the bracket or commutator of x and y, is called a *Lie algebra* over $\mathbb{F}$ if it satisfies the following axioms:

(*L1*) The bracket is bilinear.

(*L2*) $[xx] = 0$ for every $x \in L$.

(*L3*) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for every $x, y, z \in \mathcal{L}$ (Jacobi identity).

One very natural and familiar example of a Lie algebra is $(\mathbb{R}^3, \times)$; that is, the set of vectors in Euclidean 3-space having the operation of cross product of vectors. One can easily check that the cross product on $\mathbb{R}^3$ satisfies $(L1), (L2)$, and $(L3)$.

Let V be a complex, n-dimensional vector space. Consider the set End V (actually a vector space) of all linear transformations $V \rightarrow V$ (they are all endomorphisms). For x and y in End V, define the operation $[x, y] = xy - yx$, where the operation xy on the right side of the equation just means composition of linear transformations. With this bracket operation, the set of all linear transformations is a Lie algebra over $\mathbb{C}$. Viewed as a Lie algebra, the set of all linear transformations is called the *general linear algebra* and is denoted by $gl(V)$. If we fix a basis for V, $gl(V)$ can be thought of as the set of all $n \times n$ matrices with complex entries, denoted $gl(n, \mathbb{C})$. Then the bracket operation on $gl(n, \mathbb{C})$ would be $[S, T] = ST - TS$, where S, T are $n \times n$ matrices (and the product ST is just matrix multiplication).

Definition 3.1.2 A subspace $\mathcal{K}$ of a Lie algebra $\mathcal{L}$ is a *Lie subalgebra* if $\mathcal{K}$ is closed under the bracket operation. That is, we have $[xy] \in \mathcal{K}$ whenever $x, y \in \mathcal{K}$.

In particular, if $x \in \mathcal{K}$, then $kx \in \mathcal{K}$ for an arbitrary constant k and if $x, y \in \mathcal{K}$, then $x + y \in \mathcal{K}$. Note also that $\mathcal{K}$ is a Lie algebra relative to the bracket operation inherited from $\mathcal{L}$. The definition and notion of a Lie subalgebra is analogous to that of a subgroup in group theory and that of a subring in ring theory.

3.2 The Classical Lie Algebras. There are four families of matrix algebras called the classical algebras, and these are indexed by "type": $A_n(n \geq 1), B_n(n \geq 3), C_n(n \geq 2)$, and $D_n(n \geq 4)$. These are arguably the most important Lie algebras, so we will take some time to describe each of these families.

Type A_n: Let V be an $(n+1)$-dimensional complex vector space. Let $sl(n+1, \mathbb{C})$ denote the set of all linear transformations $V \to V$ (endomorphisms) that have trace 0 or equivalently, the set of all $(n+1) \times (n+1)$ matrices with zero trace. Recall that the trace of a matrix, $Tr(M)$, is the sum of all entries on the main diagonal of any matrix for T. Note that for any matrices, S and T, $Tr(ST) = Tr(TS)$ and $Tr(S) + Tr(T) = Tr(S+T)$. Note also that $sl(n+1, \mathbb{C})$ is a vector subspace of $gl(n+1, \mathbb{C})$. Let $S, T \in sl(n+1, \mathbb{C})$. Then $Tr([S,T]) = Tr(ST - TS) = Tr(ST) - Tr(TS) = 0$. Thus, $sl(n+1, \mathbb{C})$ is closed under the bracket operation. Therefore, $sl(n+1, \mathbb{C})$ is a Lie subalgebra of $gl(n+1, \mathbb{C})$. This Lie subalgebra is called the *special linear algebra*, and its dimension is $(n+1)^2 - 1$.

Type C_n: Let dim $V = 2n$. Define the matrix $B = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is an $n \times n$ identity matrix and 0 is an $n \times n$ zero matrix. Define a map $f : V \times V \to R$ by $f(v, w) = v^T B w$. (Think of the vectors v and w as $2n \times 1$ column vectors.) Using the properties of matrices, we can see that this is a bilinear map: $f(u+v, w) = (u+v)^T Bw = (u^T + v^T)Bw = u^T Bw + v^T Bw = f(u, w) + f(v, w)$ and $f(kv, w) = (kv)^T Bw = kv^T Bw = kf(v, w)$, where k is an arbitrary constant. The map f is linear in the second variable since f is also skew-symmetric: $f(v, w) = v^T Bw = -v^T B^T w = -v^T B^T (w^T)^T = -(w^T Bv)^T = -w^T Bv = -f(w, v)$. So f is a skew-symmetric, non-degenerate bilinear form on V.

Define $sp(2n,\mathbb{C}) = \{2n \times 2n \text{ matrices } X | f(X(v),w) = -f(v,X(w))\}$. Using the fact that $f(v,w) = v^T Bw$, we can see that $(Xv)^T Bw = -v^T BXw$, which means $v^T X^T Bw = v^T(-BX)w$ for all v, w , and it follows that $X^T B = -BX$.

Lemma 3.2.1 *$sp(2n,\mathbb{C})$ is a Lie subalgebra of $gl(2n,\mathbb{C})$.*

Proof. We first need to check that $sp(2n,\mathbb{C})$ is a vector subspace of $gl(2n,\mathbb{C})$. Let $X, Y \in sp(2n,\mathbb{C})$, and let k be any complex constant. Consider kX. Then $f(kX(v),w) = (kX(v))^T Bw = kv^T X^T Bw = kv^T(-BX)w = -v^T BkXw = -f(v,kX(w))$. Thus, kX is in $sp(2n,\mathbb{C})$. Now, consider $X+Y$. Note that since $X \in sp(2n,\mathbb{C})$, $X^T B = -BX$ and $Y \in sp(2n,\mathbb{C})$ implies that $Y^T B = -BY$. Then $f((X+Y)(v),w) = [(X+Y)(v)]^T Bw = (Xv + Yv)^T Bw = [(Xv)^T + (Yv)^T]Bw = v^T X^T Bw + v^T Y^T Bw = v^T(-BX)w + v^T(-BY)w = -v^T B(Xw + Yw) = -v^T B(X+Y)(w) = -f(v,(X+Y)(w))$, so $X+Y$ is in $sp(2n,\mathbb{C})$. Thus, $sp(2n,\mathbb{C})$ is a subspace of $gl(2n,\mathbb{C})$.

Now, to finish showing $sp(2n,\mathbb{C})$ is a subalgebra of $gl(2n,\mathbb{C})$, we need to show that it is closed under the bracket operation. That is, if $X, Y \in sp(2n,\mathbb{C})$, then $[XY] = XY - YX \in sp(2n,\mathbb{C})$. Consider $f((XY-YX)(v),w) = [(XY-YX)(v)]^T Bw = [XYv - YXv]^T Bw = [(XYv)^T - (YXv)^T]Bw = (v^T Y^T X^T - v^T X^T Y^T)Bw = v^T Y^T X^T Bw - v^T X^T Y^T Bw = -v^T(X^T Y^T Bw - Y^T X^T Bw) = -v^T(X^T(-BY)w - Y^T(-BX)w) = -v^T(BXYw - BYXw) = -v^T B[XYw - YXw] = -v^T B(XY - YX)(w) = -f(v,(XY - YX)(w))$. Thus, $sp(2n,\mathbb{C})$ is closed under the bracket operation. Therefore, $sp(2n,\mathbb{C})$ is a Lie subalgebra of $gl(2n,\mathbb{C})$. □

For $X = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$ (each block M, N, P, Q is an $n \times n$ matrix), we need to have $-BX = X^T B$. Recall that $B = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Consider

$$-BX = -\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} M & N \\ P & Q \end{bmatrix} = -\begin{bmatrix} P & Q \\ -M & -N \end{bmatrix} = \begin{bmatrix} -P & -Q \\ M & N \end{bmatrix} \text{ and}$$

$$X^T B = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}^T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} = \begin{bmatrix} M^T & P^T \\ N^T & Q^T \end{bmatrix}\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} = \begin{bmatrix} -P^T & M^T \\ -Q^T & N^T \end{bmatrix}.$$

Now, equating these two matrices, we have that $P = P^T$, $-Q = M^T$, and $N = N^T$. Note that $-Q = M^T$ and $M = -Q^T$ are equivalent conditions. See Appendix A for the C_2 matrices.

Lemma 3.2.2 *dim* $sp(2n, \mathbb{C}) = 2n^2 + n$.

Proof. Let $X = \begin{bmatrix} M & N \\ P & Q \end{bmatrix} \in sp(2n, \mathbb{C})$. Above, we determined that the following relations must hold: $P = P^T, -Q = M^T$, and $N = N^T$. Consider the block P. Since the matrix P is an $n \times n$ matrix, there are n choices on the diagonal. Now we can just consider the upper triangular part of P for counting the remaining possible choices, since the lower triangular part is determined because of the relation $P = P^T$. In the upper triangular part, on the ith row there are $n - i$ choices. So there are $(n-1) + (n-2) + \cdots + (1) = \binom{n}{2} = \frac{n(n-1)}{2}$ total choices in the upper triangular part. This means that there are $n + \frac{n(n-1)}{2}$ possible matrices P. The basis for all such matrices P is $\{e_{n+i,i}\}_{i=1}^{n} \cup \{e_{n+i,j} + e_{n+j,i}\}_{i<j}$. Since the matrix N has the same condition as P, the same counting arguments apply. So there are $n + \frac{n(n-1)}{2}$ such matrices N. The basis for all such matrices N is $\{e_{i,n+i}\}_{i=1}^{n} \cup \{e_{j,n+i} + e_{i,n+j}\}_{i<j}$. For

the blocks M and Q, they must satisfy $-Q = M^T$ or $Q = -M^T$. There are no conditions placed on the entries of M itself, so there are n^2 possible matrices for block M. Once a matrix M is chosen, Q is determined. Now counting up the total possible choices in the matrix X, we have $n + \frac{n(n-1)}{2} + n + \frac{n(n-1)}{2} + n^2 = 2n + n(n-1) + n^2 = 2n + n^2 - n + n^2 = 2n^2 + n$. Thus, dim $sp(2n, \mathbb{C}) = 2n^2 + n$. □

Type B_n: Let dim $V = 2n + 1$. Define the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}$, where I_n is an $n \times n$ identity matrix and all the other entries, except the $(1,1)$-entry, are 0. As in the C_n case, we define the bilinear form f by the rule $f(v, w) = v^T B w$, for $v, w \in V$. Analogous to the form for type C_n, this map is bilinear and symmetric.

Define $so(2n+1, \mathbb{C}) = \{(2n+1) \times (2n+1) \text{ matrices } X | f(X(v), w) = -f(v, X(w))\}$, the same requirement on X as for $sp(2n, \mathbb{C})$. So we have the same equality, $X^T B = -BX$.

Lemma 3.2.3 *$so(2n+1, \mathbb{C})$ is a Lie subalgebra of $gl(2n+1, \mathbb{C})$.*

Proof. This proof is analogous to the proof for the C_n case. □

The algebra $so(2n+1, \mathbb{C})$ is called the *odd orthogonal algebra*. For $X = \begin{bmatrix} R & S_1 & S_2 \\ T_1 & M & N \\ T_2 & P & Q \end{bmatrix}$, we have $X^T B = -BX$. Consider $-BX = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} R & S_1 & S_2 \\ T_1 & M & N \\ T_2 & P & Q \end{bmatrix} =$

$$-\begin{bmatrix} R & S_1 & S_2 \\ T_2 & P & Q \\ T_1 & M & N \end{bmatrix} = \begin{bmatrix} -R & -S_1 & -S_2 \\ -T_2 & -P & -Q \\ -T_1 & -M & -N \end{bmatrix} \text{ and } X^T B = \begin{bmatrix} R & S_1 & S_2 \\ T_1 & M & N \\ T_2 & P & Q \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} =$$

$$\begin{bmatrix} R^T & {T_1}^T & {T_2}^T \\ {S_1}^T & M^T & P^T \\ {S_2}^T & N^T & Q^T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} = \begin{bmatrix} R^T & {T_2}^T & {T_1}^T \\ {S_1}^T & P^T & M^T \\ {S_2}^T & Q^T & N^T \end{bmatrix}.$$

Then if we equate these two matrices, and thus their corresponding entries, we have $-R = R^T, -S_1 = {T_2}^T$ (equivalently, $-T_2 = {S_1}^T$), $-S_2 = {T_1}^T$ (equivalently, $-T_1 = {S_2}^T$), $-P = P^T, -Q = M^T$ (equivalently, $Q^T = -M$), and $-N = N^T$. Since R is actually a 1×1 matrix (and thus, just a complex number), $-R = R^T$ implies that $R = 0$. See Appendix A for the B_2 matrices.

Lemma 3.2.4 *dim* $so(2n+1, \mathbb{C}) = 2n^2 + n$.

Proof. A similar counting argument to the one described in detail for the C_n case can be applied. □

Type D_n: Let dim $V = 2n$. Define the matrix $B = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$, where I_n is an $n \times n$ identity matrix. Define the map f as before, with $f(v, w) = v^T B w$. Define $so(2n, \mathbb{C}) = \{2n \times 2n \text{ matrices } X | f(X(v), w) = -f(v, X(w))\}$.

Lemma 3.2.5 $so(2n, \mathbb{C})$ *is a Lie subalgebra of* $gl(2n, \mathbb{C})$.

Proof. This proof is analogous to the proof for the C_n case. □

The Lie algebra $so(2n, \mathbb{C})$ is the *even orthogonal algebra*. For $X = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$, it must also be true that $X^T B = -BX$. So $X^T B = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}^T \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} =$

$\begin{bmatrix} M^T & P^T \\ N^T & Q^T \end{bmatrix} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} P^T & M^T \\ Q^T & N^T \end{bmatrix}$ and $-BX = -\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} M & N \\ P & Q \end{bmatrix} =$

$-\begin{bmatrix} P & Q \\ M & N \end{bmatrix} = \begin{bmatrix} -P & -Q \\ -M & -N \end{bmatrix}$.

By equating these matrices and their corresponding entries, we have $P^T = -P, M^T = -Q$, and $-N = N^T$.

Lemma 3.2.6 *dim* $so(2n, \mathbb{C}) = 2n^2 - n$.

Proof. A counting argument similar to the one described in detail in the C_n case can be applied. □

3.3 Ideals, Homomorphisms, and Classifications. As with other algebraic objects (like finite groups, vector spaces, or rings), we would like to understand the substructures of a given Lie algebra as well as homomorphisms between Lie algebras. We would also like to know if it is possible to classify Lie algebras (or some large set of Lie algebras) in the same way that finite simple groups have been classified, for example.

Definition 3.3.1 Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Lie algebras over $\mathbb{C}$. A linear map $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ is a *Lie algebra homomorphism* if $\psi([x, y]_{\mathcal{L}_1}) = [\psi(x), \psi(y)]_{\mathcal{L}_2}$.

We can see that this definition is very analogous to that for group homomorphisms and rings homomorphisms. The idea of a homomorphism is the same, but for Lie algebras, we require that the map preserves the bracket operation.

Definition 3.3.2 A subspace I of a Lie algebra $\mathcal{L}$ is an *ideal* if $[x, y] \in I$, whenever $x \in \mathcal{L}, y \in I$.

Note that the axioms $(L1)$ and $(L2)$ in the definition of a Lie algebra imply that the bracket operation is anticommutative; i.e., $[x, y] = -[y, x]$. Then for a Lie algebra ideal, it would be equivalent to say that $[y, x] \in I$, whenever $x \in \mathcal{L}, y \in I$. So an ideal of a Lie algebra is analogous to a two-sided ideal of a ring.

In group theory, a normal subgroup arises as the kernel of some group homomorphism. In ring theory, an ideal arises as the kernel of some ring homomorphism, and every kernel is an ideal. The same is true for Lie algebras. For a Lie algebra $\mathcal{L}, I$ is an ideal of $\mathcal{L}$ if and only if I is the kernel of some Lie algebra homomorphism. One can construct a "factor Lie algebra" via the mapping $\mathcal{L} \to \mathcal{L}/I$, in the same way we form a factor ring $R \to R/I$, or a factor group $G \to G/N$. The usual homomorphism theorems apply to quotients of Lie algebras. Table 3.3.4 compares and contrasts substructure relationships, quotients, and classification results for finite groups, rings, finite dimensional vector spaces, and Lie algebras.

A group that has no nontrivial normal subgroups is called a simple group. This inspires the following definition for Lie algebras.

Definition 3.3.3 A Lie algebra $\mathcal{L}$ is *simple* if it has no nontrivial ideals.

Table 3.3.4 Comparisons and Contrasts Between Algebraic Structures

	Finite Groups	Rings	Finite dimensional vector spaces	Lie algebras
Substructure	subgroups normal subgroups	subrings two-sided ideal	subspace	subalgebras ideal
Quotients	G/N, N Normal	R/I	V/W	$\mathcal{L}/I$
Classification	Any finite abelian group is isomorphic to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$. All finite simple groups have been classified (included in the list are $\mathbb{Z}_p$ (p prime) and $A_n (n \geq 5)$).		For each positive integer n, there is a unique (up to isomorphism) vector space of dimension n.	The finite dimensional simple Lie algebras over $\mathbb{C}$ are of types: A_n, B_n, C_n, D_n (the classical Lie algebras) E_6, E_7, E_8, F_4, G_2 (the exceptional Lie algebras)

If $\mathcal{L}$ is simple and finite dimensional, then [Hum] Section 10 tells us how to associate an "irreducible root system" to $\mathcal{L}$. An irreducible root system is a finite spanning set of vectors that sit inside a Euclidean space. There is a certain symmetry to the arrangement of the vectors, and there are rigid restrictions on the possible angles the vectors can make. In [Hum] Chapter 3, Section 11, the irreducible root systems are classified, resulting in the following (irredundant) list: $A_n\ (n \geq 1), B_n\ (n \geq 3), C_n\ (n \geq 2), D_n\ (n \geq 4),\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$. There is a one-to-one correspondence between irreducible root systems and the "Cartan matrices" on page 59 of [Hum]. The number of rows in the Cartan matrix is called the *Lie rank* of the simple Lie algebra. This procedure of associating a matrix to a simple Lie algebra is due to Cartan and other late nineteenth century researchers in this field. Their results can be summarized as follows:

Theorem 3.3.5 (Cartan's Theorem)

(a) Cartan's procedure associates a Cartan matrix M to each simple Lie algebra.

(b) Distinct simple Lie algebras have distinct Cartan matrices.

(c) The Cartan matrices are classified on page 59 *of [Hum].*

So the question remains: Is there a simple Lie algebra associated to any given Cartan matrix on this list? A remarkable theorem due to Serre tells us how to start with a Cartan matrix and build a simple Lie algebra from it. Let M be an $n \times n$ Cartan matrix from the list on page 59 of [Hum]. Let $\{x_i, y_i, h_i\}_{i=1}^n$ be a set of generators, and impose the following relations:

$(S1)$ $[h_i h_j] = 0, (1 \leq i, j \leq n)$.

$(S2)$ $[x_i y_i] = h_i, [x_i y_j] = 0$ if $i \neq j$.

$(S3)$ $[h_i x_j] = m_{ji} x_j, [h_i y_j] = -m_{ji} y_j$.

(S_{ij}^{+}) $(ad x_i)^{-m_{ji}+1}(x_j) = 0, (i \neq j)$.

(S_{ij}^{-}) $(ad y_i)^{-m_{ji}+1}(y_j) = 0, (i \neq j)$.

The *ad* notation used in relations (S_{ij}^{+}) and (S_{ij}^{-}) is used to represent repeated bracket operations on the same object. For example, $(ad \;\; x_i)^3(x_j) = [x_i[x_i[x_i x_j]]]$. Note that it is known that m_{ji} is a non-positive integer for $i \neq j$.

Theorem 3.3.6 (Serre's Theorem) *Let M be a Cartan matrix and build a Lie algebra $\mathcal{L}$ out of it in the manner prescribed above. Then the resulting Lie algebra $\mathcal{L}$ will be (finite dimensional and) simple. Moreover, the Cartan matrix M' associated to this simple Lie algebra by Cartan's Theorem is the same as the matrix M here. Finally, the generators $\{x_i, y_i, h_i\}_{i=1}^{n}$ form a linearly independent (though not always spanning) set of vectors.*

3.4 Representations.

The next two definitions define the main algebraic objects we will be interested in. After we define these objects, we will consider some examples and then we will describe the connection to the combinatorial results of this thesis.

Definition 3.4.1 A *representation* of a Lie algebra $\mathcal{L}$ is a Lie algebra homomorphism $\phi : \mathcal{L} \to gl(V)$, whose target set is a general linear Lie algebra. If n=dim V and if we fix a basis for V, we get a homomorphism $\phi : \mathcal{L} \to gl(n, \mathbb{C})$, which we refer to as a *matrix representation*. We call n the *dimension* of the representation. For a matrix representation $\phi : \mathcal{L} \to gl(n, \mathbb{C})$, if there exists a change of basis in the representation space so that the representation matrices for the elements of $\mathcal{L}$ are all block diagonal and the corresponding blocks are the same size, then ϕ is a *reducible* matrix representation. (We assume there are at least two such blocks for each matrix, and that each is at least 1×1.) A matrix representation that is not reducible is said to be *irreducible*. (Notice, for example, that any one-dimensional representation is irreducible.) So, a matrix representation is a mapping that identifies each element of a Lie algebra with a matrix. The irreducible representations of a Lie algebra literally form the building blocks for all other reducible representations.

Example 3.4.2 We will look at an example of a representation using the Lie algebra A_1, which is associated to the 1×1 Cartan matrix [2]. Let A_1 denote the Lie algebra with generators x, y, h satisfying the relations $[xy] = h, [hx] = 2x$, and $[hy] = -2y$. Since these generators are independent (cf. Serre's Theorem), it suffices to say where the generators are mapped when describing a representation.

Define a map $\phi : A_1 \to sl(2, \mathbb{C})$ by the following: $x \mapsto X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y \mapsto Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h \mapsto H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and extend ϕ linearly. This linear map is well-defined since x, y and h are independent.

To see that ϕ is indeed a homomorphism of Lie algebras, it suffices now to check that $X, Y, H \in sl(2, \mathbb{C})$ mirror the relations of $x, y, h \in A_1$; that is, $[XY] = H, [HX] = 2X$, and $[HY] = -2Y$. It is easy to check by hand that these relations do indeed hold, which means that ϕ is a Lie algebra homomorphism.

Moreover, we would like to show that ϕ maps A_1 onto $sl(2, \mathbb{C})$. Recall that $sl(2, \mathbb{C})$ is the set of 2×2 matrices, with complex entries, having trace 0. Consider any such matrix, $\begin{bmatrix} c & a \\ b & -c \end{bmatrix}$. Note that $\begin{bmatrix} c & a \\ b & -c \end{bmatrix}$ can be written as the linear combination $aX + bY + cH$. So X, Y, H span $sl(2, \mathbb{C})$. Since ϕ takes x, y, h to X, Y, H, respectively, we also have that any matrix in $sl(2,\mathbb{C})$ can be written as the image of a linear combination of x, y, h. Thus, ϕ maps A_1 onto $sl(2,\mathbb{C})$.

By applying Serre's Theorem, we have that A_1 is simple. Since A_1 is a simple Lie algebra, it has no nontrivial ideals. So the only possible ideals of A_1 are A_1 and $\{0\}$. Then ker $\phi = A_1$ or ker $\phi = \{0\}$. If ker $\phi = A_1$, that would mean that every element in A_1 gets mapped to the zero element in $sl(2,\mathbb{C})$. We know that ϕ is onto and that $sl(2,\mathbb{C})$ is 3-dimensional, so every element in A_1 cannot be mapped to only one element in $sl(2,\mathbb{C})$. So ker $\phi \neq A_1$. Thus, ker $\phi = \{0\}$, which means ϕ is injective, making ϕ an isomorphism.

Note that the diagonal entries for X, Y are the respective eigenvalues for each matrix, since they are both triangular. Therefore, if there were a change of basis that would make X, Y into a block matrices, it would make them both into the zero matrix. We know that X, Y cannot be zero matrices since they do not annihilate all column vectors. Since there is not a change of basis that would make X, Y into block diagonal matrices, we conclude that $\phi : A_1 \to sl(2,\mathbb{C})$ is an irreducible representation. $\square$

Example 3.4.3 Let us now consider representations of A_1 in $gl(4,\mathbb{C})$. Consider the mapping $\theta : A_1 \to gl(4,\mathbb{C})$ defined by the following: $x \mapsto X = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $y \mapsto Y =$

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$, and $h \mapsto H = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$. For reasons similar to those given for $\phi : A_1 \to sl(2, \mathbb{C})$, this representation is irreducible. This is not the only representation of A_1 in $gl(4, \mathbb{C})$, however.

Define a mapping $\psi : A_1 \to gl(4, \mathbb{C})$ by $x \mapsto X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$, $y \mapsto Y = \begin{bmatrix} -2 & -1 & 3 & -\frac{3}{2} \\ \frac{5}{2} & 2 & -\frac{3}{2} & \frac{3}{2} \\ -1 & -1 & 0 & -\frac{1}{2} \\ -1 & -2 & -3 & 0 \end{bmatrix}$, and $h \mapsto H = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & -6 & 1 \end{bmatrix}$.

Is this representation also irreducible? If we apply the change of basis matrix $P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & -1 \\ -\frac{1}{2} & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ to each of X, Y, H, we have the following results: $X \to X_P =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, Y \to Y_P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{and } H \to H_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \text{ (Recall}$$

that the change of basis is done by multiplying the matrices in this manner: $P^{-1}XP = X_P$.) Clearly, X_P, Y_P, and H_P are block diagonal matrices. Therefore, the representation $\psi : A_1 \to gl(4, \mathbb{C})$ is reducible. □

3.5 Weights. Given a representation $\phi : \mathcal{L} \to gl(V)$, where $\mathcal{L}$ is a simple Lie algebra and V is a complex n-dimensional vector space, it is a nontrivial major result that there is a basis $\{v_1, v_2, \ldots, v_d\}$ for V consisting of eigenvectors for the $H_i's$; i.e., $H_i.v_t = m_i^{(t)} v_t$, where $m_i^{(t)}$ is a constant. This means that each H_i is a diagonal matrix with respect to the basis $\{v_1, v_2, \ldots, v_d\}$. Such a basis is called a *weight basis.* The *weight* of the basis vector v_t is $(m_1^{(t)}, m_2^{(t)}, \ldots, m_n^{(t)})$, which is an n-tuple of numbers, possibly complex. It can be shown that each of these eigenvalues $m_i^{(t)}$ is actually an integer.

Thus, each weight is an n-tuple of eigenvalues, and there is one eigenvalue for each operator H_i. To say that a vector v has weight $\mu = (m_1, m_2, \ldots, m_n)$, then, is to say that when H_i acts on the vector v, it simply multiplies it by the scalar m_i. So a vector of weight μ is really just an eigenvector common to all of the H_i's, and a weight could be thought of as a "generalized eigenvalue."

Given a representation $\phi : \mathcal{L} \to gl(V)$ of a simple Lie algebra $\mathcal{L}$, the set of all vectors in the representing space V with the same weight μ forms a vector subspace of V. The dimension of this subspace is called the *weight number* $d_V(\mu)$. Since the sum of the dimensions of all such subspaces must equal the (finite) dimension of V, there can only be a finite number of weights that have any associated nonzero weight vectors.

For a simple Lie algebra $\mathcal{L}$ of rank n, let $\lambda = (a_1, a_2, \ldots, a_n)$ be a shape for $\mathcal{L}$ (i.e. a_i is a non-negative integer, and λ has a_i columns of length i). There is an irreducible representation "corresponding to" λ, which we will call $\mathcal{L}(\lambda)$. That is, there is a map $\phi_\lambda : \mathcal{L} \rightarrow gl(\mathcal{L}(\lambda))$, but for brevity we will often just refer to this representation by referring to the representing space $\mathcal{L}(\lambda)$. The dimension numbers for $\mathcal{L}(\lambda)$ are denoted $d_\lambda(\mu)$. There is a beautiful formula that can be used to compute the dimension of $\mathcal{L}(\lambda)$, which is called the Weyl dimension formula (see [Hum] Chapter 24, Section 3). This formula is expressed as a quotient of products.

One remarkable thing is that the weights for an irreducible representation of a simple Lie algebra can all be determined without ever having a basis in hand, and the corresponding weight numbers can be determined by a formula (called the Freudenthal's multiplicity formula).

Chapter 4

Connection to Combinatorics

One of our goals is to find an appropriate combinatorial setting for constructing and understanding representations. One approach is as follows.

4.1 Supporting Graphs and Representations. Begin with a simple Lie algebra $\mathcal{L}$ and a representation $\phi : \mathcal{L} \to gl(V)$ with weight basis $\{v_{\mathbf{t}}\}_{\mathbf{t}\in P}$. (Here P is just an index set, e.g., the integers $\{1, 2, \ldots, \alpha\}$.) From this, we will build a picture, which will be an edge-colored directed graph.

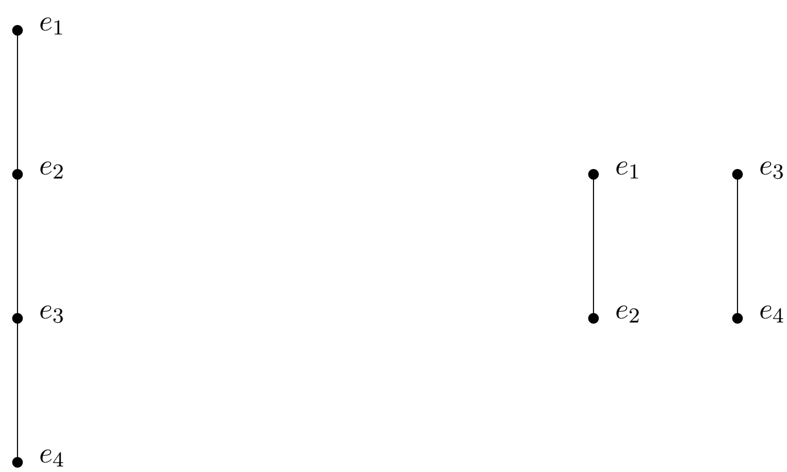
Each basis vector $v_{\mathbf{t}}$ will be a vertex in the picture associated with the representation. We will place colored, directed edges between these vertices as follows:

(1) for a basis vector $v_{\mathbf{s}}$, if $X_i.v_{\mathbf{s}} = u = \sum_{\mathbf{r}\in P} c_{\mathbf{r},\mathbf{s}} v_{\mathbf{r}}$ with $c_{\mathbf{t},\mathbf{s}} \neq 0$, place an edge color i from $v_{\mathbf{s}}$ to $v_{\mathbf{t}}$; i.e., $v_{\mathbf{s}} \xrightarrow{i} v_{\mathbf{t}}$.

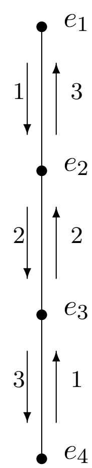
(2) for a basis vector $v_{\mathbf{t}}$, if $Y_i.v_{\mathbf{t}} = w = \sum_{\mathbf{r}\in P} c_{\mathbf{r},\mathbf{t}} v_{\mathbf{r}}$ with $c_{\mathbf{s},\mathbf{t}} \neq 0$, place an edge color i from $v_{\mathbf{t}}$ to $v_{\mathbf{s}}$; i.e., $v_{\mathbf{t}} \xrightarrow{i} v_{\mathbf{s}}$.

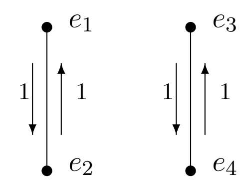
Now attach the two coefficients $c_{\mathbf{s},\mathbf{t}}$ and $c_{\mathbf{t},\mathbf{s}}$ to the edge $v_{\mathbf{s}} \xrightarrow{i} v_{\mathbf{t}}$. Note that the coefficient $c_{\mathbf{t},\mathbf{s}}$ (the "X-coefficient") is attached to the up direction and the coefficient $c_{\mathbf{s},\mathbf{t}}$ (the "Y-coefficient) is attached to the down direction of the edge between the vertices $v_{\mathbf{s}}$ and $v_{\mathbf{t}}$. The resulting picture (an edge colored directed graph with the coefficients attached to each edge) is called a *representation diagram* for the representation $\phi : \mathcal{L} \to gl(V)$ with weight basis $\{v_{\mathbf{t}}\}$. If we remove the edge coefficients, the underlying edge-colored directed graph is called a *supporting graph* for the representation $\phi : \mathcal{L} \to gl(V)$.

Example 4.1.1 We will now construct a representation diagram for each of the representations for A_1 presented in Example 3.4.3. The definition above suggests we should start by looking at the matrices for X and Y, and then use them as "instructions" for building the pictures. We will actually work in the reverse direction: we will start with some pictures and then attach coefficients in order to obtain matrices for X and Y. For both of the representations given, the weight basis is $\{e_1, e_2, e_3, e_4\}$. Using these basis vectors as vertices, we will look at two different supporting graphs:

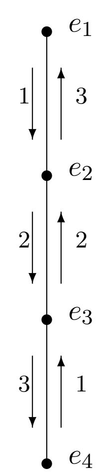


On chains, it is known that for the "X-coefficients", we can start at one on the bottom edge and number each edge in increments of one as we go up the chain. For the "Y-coefficients", we then number the top edge directed down as one and number each "down-directed" edge in increments of one as we move down the chain. So for our two sets of vertices and edges, we now have:

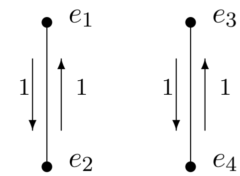




To determine matrices for X and Y from these representation diagrams, recall that $X.e_i$ is the ith column in the X-matrix (similarly, for Y). For the matrix H, we use the relation $H = XY - YX$. Doing this, we see that



is just a representation diagram for the representation $\theta : A_1 \to gl(4, \mathbb{C})$ described in Example 3.4.3. Similarly, one can see that



is a representation diagram for the representation $\psi : A_1 \to gl(4, \mathbb{C})$. □

How can we produce good candidates for representation diagrams for irreducible representations $\mathcal{L}(\lambda)$ of a simple Lie algebra $\mathcal{L}$? There are certain conditions that must be satisfied by any representation diagram, so these conditions limit the scope of our search somewhat. Let P be an edge-colored ranked poset with edge colors from $\{1, \ldots, n\}$. (Here, n is the *Lie* rank of the Lie algebra–not to be confused with *poset* rank–which corresponds to the number of x, y, and h generators for $\mathcal{L}$.) Attach two coefficients (denoted $c_{\mathbf{t},\mathbf{s}}$ and $c_{\mathbf{s},\mathbf{t}}$) to each $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in P. For any $\mathbf{t}$ in P, let $l_i(\mathbf{t})$ denote the length of the i-component containing $\mathbf{t}$, and let $\rho_i(\mathbf{t})$ be the rank of $\mathbf{t}$ within this i-component. For any $\mathbf{t} \in P$, let the weight of $\mathbf{t}$ be the n-tuple $wt_P(\mathbf{t}) = (2\rho_1(\mathbf{t}) - l_1(\mathbf{t}), \ldots, 2\rho_n(\mathbf{t}) - l_n(\mathbf{t}))$. We call this the "combinatorial weight rule" for P. Let $V[P]$ be the complex vector space with basis vectors $\{v_{\mathbf{t}}\}_{\mathbf{t}\in P}$, and for $1 \leq i \leq n$ define linear maps X_i and Y_i on $V[P]$ by $X_i.v_{\mathbf{s}} = \sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} c_{\mathbf{t},\mathbf{s}} v_{\mathbf{s}}$ and $Y_i.v_{\mathbf{t}} = \sum_{\mathbf{s}:\mathbf{s}\xrightarrow{i}\mathbf{t}} c_{\mathbf{t},\mathbf{s}} v_{\mathbf{t}}$ and set $H_i = X_iY_i - Y_iX_i$. The proof of the following proposition can be found in [Don].

Proposition 4.1.2 *Keeping the notation of the previous paragraph, suppose that*

(1) $X_iY_j = Y_jX_i$ *(equivalently,* $[X_iY_j] = 0$*) for* $i \neq j$*;*

(2) $H_i.v_{\mathbf{t}} = (2\rho_i(\mathbf{t}) - l_i(\mathbf{t}))v_{\mathbf{t}}$ *for* $1 \leq i \leq n$ *and for each* $\mathbf{t} \in P$ *(that is,* $[X_iY_i] = H_i$*);*

and

(3) for $1 \leq i \leq n$*, we have* $2\rho_i(s) - l_i(s) + m_{ji} = 2\rho_i(t) - l_i(t)$ *whenever* $s \xrightarrow{j} t$ *with* $i \neq j$*.*

Then the linear map $\phi : \mathcal{L} \rightarrow gl(V[P])$ *determined by* $x_i \mapsto X_i, y_i \mapsto Y_i$*, and* $h_i \mapsto H_i$ *is a representation of* $\mathcal{L}$*, and* P *is a representation diagram for* ϕ*.*

4.2 Identifying Supporting Graphs. Next we ask: How can we identify supporting graphs for an irreducible representation? For any irreducible representation, one can see that for $\mathcal{L}(\lambda)$, there is at least one and at most a finite number of supporting graphs. See [Don] for the proof of the following proposition.

Proposition 4.2.1 *Any supporting graph* P *for the irreducible representation* $\mathcal{L}(\lambda)$ *must satisfy the following three necessary conditions:*

(1) The number of vertices of P *is equal to the dimension of the representation* $\mathcal{L}(\lambda)$*.*

(2) The picture P *is a rank symmetric, rank unimodal, and strongly Sperner poset.*

(3) The combinatorial weight rule for P*, denoted* wt_P*, is well-defined; i.e., for any weight* μ*, we have that the weight number* $d_\lambda(\mu) = |\{t \in P : wt_P(t) = \mu\}|$*.*

It is important to remember that these conditions for a support P are necessary, but are not sufficient for proving that a graph is a support. In addition, it can be shown that P must have $a_1r_1 + \cdots + a_nr_n + 1$ ranks, where r_i is the length of any supporting graph for the fundamental representation $\mathcal{L}(\lambda_i)$ (where $\lambda_i = (0, \ldots, a_i, \ldots, 0)$ and $a_i = 1$). The numbers r_i are easily computed for each simple Lie algebra λ. In fact, it can be shown that r_i is twice the sum of the entries in the ith row of the inverse of the Cartan matrix for $\mathcal{L}$.

Chapter 5

The Lie Algebra A_2

The Lie algebra A_2 is the simple Lie algebra whose Cartan matrix is $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Since A_2 is a "rank two" Lie algebra (it has two x's, two y's, and two h's in its generating set), a shape for A_2 has the form $\lambda = (a, b)$, where a is the number of columns in λ that have only one box and b is the number of columns in λ with two boxes. It is known by the Weyl dimension formula that the dimension of the representing space for $A_2(\lambda)$ is $\frac{1}{2}(a+1)(b+1)(a+b+2)$ (see [Hum] p. 140). It follows that this is the number of vertices in any supporting graph for $A_2(\lambda)$.

There are two distinguished families of representation diagrams for the Lie algebra A_2. We will present combinatorial objects from which the pictures can be constructed, we will say how the objects are ordered and how the edges are colored. We will also investigate the combinatorics of the associated supporting graphs.

Actually, these supports are just as easy to describe for A_n, where $n \geq 1$, so we will be working in the general rank n case at the outset. Later in this section, and in all of our examples, we will specialize to the case $n = 2$. We call the supporting graphs underlying these representation diagrams *Gelfand-Tsetlin* supports because it can be proved that they are supporting graphs for the Gelfand-Tsetlin bases for the irreducible representations $A_n(\lambda)$ (see [Don]). The Gelfand-Tsetlin bases were first produced in 1950 [GT], but have been reproduced many times (e.g. [Mol2]). For examples of the Gelfand-Tsetlin supports for $A_2(\lambda)$ with $\lambda = (2,1)$, see Figure $B.1$ in Appendix B.

5.1 Family 1 of Gelfand-Tsetlin Supports for $A_2(\lambda)$. The vertices for one of these pictures will be the *semi-standard tableaux of shape* $\lambda = (a_1, a_2, \ldots, a_n)$, where a_i is the number of columns of length i boxes. A *tableau of shape* λ is a filling of the boxes of λ with entries from the set $\{1, 2, \ldots n+1\}$. A tableau $\mathbf{t}$ is semi-standard if the entries weakly increase left to right across rows of $\mathbf{t}$, and if the entries strictly increase down the columns of $\mathbf{t}$.

To build a picture out of these objects, we order them by "reverse component-wise comparison". Thus, for example, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ is less than $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ in this partial order on semi-standard tableaux of shape $\lambda = (1,1)$. Now let $\mathbf{s}$ and $\mathbf{t}$ be semi-standard tableaux of shape λ, and suppose $\mathbf{t}$ covers $\mathbf{s}$ in this partial order. We attach the color i to the edge $\mathbf{s} \to \mathbf{t}$, and we write $\mathbf{s} \xrightarrow{i} \mathbf{t}$, if an entry in $\mathbf{s}$ is changed from an $i+1$ to an i to form $\mathbf{t}$. We denote the resulting edge-colored directed graphs by $L_A^{GT-left}(n, \lambda)$.

Theorem 5.1.1 *$L_A^{GT-left}(n, \lambda)$ is (the order diagram for) a distributive lattice.*

Proof. For $S, T \in L_A^{GT-left}(n, \lambda)$, let $min(S,T)$ denote the component-wise max of S, T (since we order $L_A^{GT-left}(n, \lambda)$ by reverse component-wise comparison). That is, the (i,j)-entry of $min(S,T)$ is $s_{i,j} \vee t_{i,j}$ (where $s_{i,j}, t_{i,j}$ is the i,jth entry of S, T, respectively) for every i, j in shape λ. There are three things about $min(S,T)$ that need to be checked: (1)$min(S,T)$ is a semi-standard tableau, (2) that $min(S,T) \leq S$ and $min(S,T) \leq T$, and (3) if $R \leq S, R \leq T$, then $R \leq min(S,T)$. If so, we can write $min(S,T) = S \wedge T$.

(1) Let $Q = min(S, T)$. Consider the entry $q_{i,j} \in Q$. Then $q_{i,j} \geq s_{i,j}$ and $q_{i,j} \geq t_{i,j}$. Also, $q_{i,j+1} \geq s_{i,j+1}$ and $q_{i,j+1} \geq t_{i,j+1}$. Then, since S, T are semi-standard tableaux, $s_{i,j+1} \geq s_{i,j}$ and $t_{i,j+1} \geq t_{i,j}$. Note that $s_{i,j+1} \geq q_{i,j}$ and $t_{i,j+1} \geq q_{i,j}$. Thus, $q_{i,j+1} \geq s_{i,j+1} \geq q_{i,j}$ and $q_{i,j+1} \geq t_{i,j+1} \geq q_{i,j}$. Hence, $q_{i,j+1} \geq q_{i,j}$ for every i, j.

Now consider entry $q_{i+1,j}$. By definition of $Q, q_{i+1,j} \geq s_{i+1,j}$ and $q_{i+1,j} \geq t_{i+1,j}$. Note that $s_{i+1,j} > s_{i,j}$ and $t_{i+1,j} > t_{i,j}$. Then note that $s_{i+1,j} > q_{i,j}$ and $t_{i+1,j} > q_{i,j}$. So $q_{i+1,j} \geq s_{i+1,j} > q_{i,j}$ and $q_{i+1,j} \geq t_{i+1,j} > t_{i,j}$. Thus, $q_{i+1,j} > q_{i,j}$ for all i, j. Therefore, $Q = min(S, T)$ is a semi-standard tableau. By the way Q is defined (in terms of component-wise maximums of S, T), we have that $Q \in L_A^{GT-left}(n, \lambda)$.

(2) Since $Q = min(S, T)$ and $q_{i,j} = \max(s_{i,j}, t_{i,j})$ for every i, jth entry of λ, $q_{i,j} \geq s_{i,j}$ and $q_{i,j} \geq t_{i,j}$ for every i, j. Then in the reverse component-wise comparison ordering on $L_A^{GT-left}(n, \lambda), Q \leq S$ and $Q \leq T$.

(3) Suppose $R \leq S$ and $R \leq T$. Then $r_{i,j} \geq s_{i,j}$ and $r_{i,j} \geq t_{i,j}$ since this is ordered by reverse component-wise comparison. So $r_{i,j} \geq \max(s_{i,j}, t_{i,j}) = q_{i,j}$. Thus, $R \leq min(S, T)$.

A similar argument for $S \vee T$ can be done for the three analogous criteria. Therefore, $L_A^{GT-left}(n, \lambda)$ is a distributive lattice. □

Corollary 5.1.2 $L_A^{GT-left}(2, \lambda)$ *is a distributive lattice.* □

For a proof of the following theorem and more on the history of these lattices, see [Don] and the references therein.

Theorem 5.1.3 $L_A^{GT-left}(n, \lambda)$ *is a supporting graph for the irreducible representation* $A_n(\lambda)$.

Corollary 5.1.4 *$L_A^{GT-left}(2,\lambda)$ is a supporting graph for the irreducible representation of $A_2(\lambda)$.*

5.2 Family 2 of Gelfand-Tsetlin Supports for $A_2(\lambda)$. For the second family of lattices in the Lie algebra A_n, we have two versions. The first version is constructed directly from $L_A^{GT-left}(n,\lambda)$. The second version is built independently of $L_A^{GT-left}(n,\lambda)$, but it is related to that family of lattices and the method of building them is similar to the one for the first family of Gelfand-Tsetlin lattices.

Version 1: One way to construct the second family of lattices for A_n is to take the poset dual of $L_A^{GT-left}(n,\lambda)$ and then recolor the edges by switching the colors on all edges (i.e., change all edges of color 1 to color n, color 2 to color $n-1$, etc.). We will call this family of lattices $L_A^{GT-right}(n,\lambda)$.

Alternatively, we can use the same objects (the semi-standard tableaux of shape λ with entries from the set $\{1,2,\ldots n+1\}$) that were used in the first family of lattices, order them by component-wise comparison and color the edges according to the following rule: if a vertex $\mathbf{t}$ is obtained from a vertex $\mathbf{s}$ by changing an $(i+1)$-entry to an i, then place an edge of color $(n+1-i)$ between them.

For a particular λ, it is not hard to see that each of these methods will produce the same lattice $L_A^{GT-right}(n,\lambda)$.

Version 2: For the second version of the second family of Gelfand-Tsetlin supports for the Lie algebra A_n, the objects are semi-standard tableaux of shape $\lambda^{sym} = (a_n, a_{n-1}, \ldots, a_1)$. The tableaux λ^{sym} are filled with entries from the set $\{1, 2, \ldots, n+1\}$, so that they satisfy the semi-standard condition. To construct the lattice from these tableaux, we order them by reverse component-wise comparison and color the edges according to the following rule: if $\mathbf{t}$ is obtained from $\mathbf{s}$ by changing an $i+1$ to an i, then place an edge of color $(n+1-i)$ between them. We will call this version $K_A^{GT-right}(n,\lambda)$.

Theorem 5.2.1 *For the second family of Gelfand-Tsetlin supports in the A_n case, version one and version two produce the same (edge-colored) distributive lattice.*

Proof. For a shape λ, take the dual of the $L_A^{GT-left}(n,\lambda)$ lattice. Since $L_A^{GT-left}(n,\lambda)$ is distributive, the dual $L_A^{GT-right}(n,\lambda)$ is also distributive, by Proposition 2.2.2 . (We disregard edge colors here.) Now we need to establish a bijection between the two versions.

For $\mathbf{t}$ in $L_A^{GT-right}(n,\lambda)$, extend the shape down so that there are $n+1$ rows in each column that is in λ. Fill the empty boxes of each column in the new shape with numbers from $\{1,2,\ldots,n+1\} \setminus \{m|m$ is in the filling of that column in $\mathbf{t}\}$. Then each column in the "extended shape" has each of $1,2,\ldots,n+1$, and the extended part of the new shape strictly decreases down the columns and weakly decreases across the rows from left to right. Remove $\mathbf{t}$ from the new shape and rotate the newly filled part 180°. Call this $\mathbf{t}'$. (See Example 5.2.2 below.) The mapping $\phi : L_A^{GT-right}(n,\lambda) \to K_A^{GT-right}(n,\lambda)$, which takes each vertex $\mathbf{t}$ to the vertex $\mathbf{t}'$ in the dual, is a clear bijection. Thus, for any shape λ, version one and version two produce the same distributive lattice in the second family of the A_2 lattices. □

Example 5.2.2 Consider $L_A^{GT-right}(2,\lambda)$ (version one), where $\lambda=(2,1)$. Let $\mathbf{t}=$

1	2	3
3		

. Then the mapping ϕ in the proof of Theorem 5.2.1 acts as follows:

$$\mathbf{t} = \begin{array}{ccc} 1 & 2 & 3 \\ 3 & & \end{array} \rightarrow \begin{array}{ccc} 1 & 2 & 3 \\ 3 & \; & \; \\ \; & \; & \; \end{array} \rightarrow \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 3 & 2 \\ 2 & 1 & 1 \end{array} \rightarrow \begin{array}{ccc} & 3 & 2 \\ 2 & 1 & 1 \end{array} \rightarrow \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array} = \mathbf{t}'.$$

Note that $\mathbf{t}$ has shape $\lambda = (2,1)$ and $\mathbf{t}'$ has shape $\lambda^{sym} = (1,2)$, and that $\mathbf{t}'$ is an element of the set of $K_A^{GT-right}(2,\lambda)$. $\square$

Chapter 6

The Lie Algebra G_2

We would like to find supporting graphs for irreducible representations of G_2 that are analogous to the supports we presented for A_2. Ideally, we would like to find distributive lattice supports that can be built easily from tableaux. We have some partial results in this direction. We have found one family of distributive lattices (cf. Theorem 6.2.2 below) analogous to $L_A^{GT-left}(2,\lambda)$ that are good candidates for supporting graphs for the irreducible representations of G_2 (cf. Conjecture 6.2.3 below). It remains an open problem to find a second family of distributive lattices for G_2 that will be analogous to the lattices $L_A^{GT-right}(2,\lambda)$. (However, recent work of Donnelly, Lewis, and Pervine can be used to show that certain lattices defined by Reiner and Stanton in [RS] actually become supporting graphs for the irreducible representations $G_2(\lambda)$ where $\lambda = (a, 0)$. These supporting graphs are distributive lattices distinct from those presented here.)

The Lie algebra G_2 has Cartan matrix $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$. It is a rank 2 Lie algebra, so a shape for G_2 has the form $\lambda = (a, b)$, where a is the number of columns of λ with one box and b is the number of columns of λ with two boxes. If a picture P is a support for the representation $G_2(\lambda)$, where $\lambda = (a, b)$, we know from the Weyl dimension formula (see [Hum], p. 140) that P will have

$$\tfrac{1}{5!}(a+1)(b+1)(a+b+2)(a+3b+4)(2a+3b+5)$$

vertices.

6.1 Lattices from Littelmann's $6 \times \lambda$ tableaux. Littelmann produced a set of objects with the right number of elements, and he produced a weight rule for these objects. To form one of Littelmann's objects, each column in the shape λ should be thought of as a "block" of six columns. These "expanded" blocks are then filled with numbers from the set $\{1, 2, 3, 4, 5, 6\}$, so that the semi-standard condition (weakly increasing across rows, strictly increasing down columns) is satisfied. Also, with his construction, only certain fillings of these six-column blocks are "admissible". The set of admissible blocks can be found in [Lit], page 348-9 (reflected through the line $y = -x$) and are reproduced here in Figure $B.2$ in Appendix B. (An example of an admissible tableau for shape $\lambda = (1, 1)$ is $\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&2&3&3&3&3&3&3&4&4&4\\\hline 3&3&4&5&5&5\\\cline{1-6}\end{array}$.) We will call the set of admissible blocks for a shape λ, "Littelmann's $6 \times \lambda$ tableaux." We order Littelmann's $6 \times \lambda$ tableaux by reverse component-wise comparison. We will call the poset of Littelmann's tableaux $K_G^{Lit}(2, 6 \times \lambda)$. See Figure $B.2$ in Appendix B for the order diagram for $K_G^{Lit}(2, 6\times \square)$ and Figure $B.3$ for the order diagram for $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\\square\end{smallmatrix})$.

Observation 6.1.1 *Suppose that* $\mathbf{r}, \mathbf{s}$*, and* $\mathbf{t}$ *are "expanded" blocks from* $K_G^{Lit}(2, 6\times \square)$ *or* $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\\square\end{smallmatrix})$ *. Suppose that semi-standardness allows* $\mathbf{s}$ *to follow* $\mathbf{r}$ *(so, of course, if* $\mathbf{r}$ *appears in* $K_G^{Lit}(2, 6\times \square)$ *we cannot have* $\mathbf{s}$ *appearing in* $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\\square\end{smallmatrix})$ *). Suppose further that* $\mathbf{s} \leq \mathbf{t}$ *(in particular, we assume* $\mathbf{s}$ *and* $\mathbf{t}$ *appear in the same picture* $K_G^{Lit}(2, 6\times \square)$ *or* $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\\square\end{smallmatrix})$ *). Then semi-standardness allows* $\mathbf{t}$ *to follow* $\mathbf{r}$*.* $\square$

For his set of tableaux, Littelmann formulated a "well-defined" weight rule wt_{Lit} based on the entries of the blocks. That is, the number of Littelmann's $6 \times \lambda$ tableaux with weight μ is equal to the weight number $d_\lambda(\mu)$. Define $c_i(t)$ to be the number of entries of i in one of these extended blocks. Now define Littelmann's weight rule for a block $\mathbf{t}$ in his set of tableaux to be

$$wt_{Lit}(\mathbf{t}) = (\tfrac{1}{6}[c_1(\mathbf{t}) - c_2(\mathbf{t}) + 2c_3(\mathbf{t}) - 2c_4(\mathbf{t}) + c_5(\mathbf{t}) - c_6(\mathbf{t})], \tfrac{1}{6}[c_2(\mathbf{t}) - c_3(\mathbf{t}) + c_4(\mathbf{t}) - c_5(\mathbf{t})]).$$

We have computed the weights for the tableaux in $K_G^{Lit}(2, 6\times \square)$ and $K_G^{Lit}(2, 6\times \boxminus)$ in Figure $B.2$ and Figure $B.3$. For any shape λ for G_2 and any $\mathbf{t} \in K_G^{Lit}(2, 6 \times \lambda)$, we may write $\mathbf{t} = (t_1, \ldots, t_k)$, where each t_i corresponds to one of the "expanded" blocks of 6 columns which form the tableau $\mathbf{t}$. (Here, k is the number of such blocks, or equivalently, the number of columns in the shape λ.)

Lemma 6.1.2 *Littelmann's weight rule is "additive". That is, in the notation of the preceding paragraph, if $\mathbf{t} = (t_1, \ldots, t_k) \in K_G^{Lit}(2, 6 \times \lambda)$ for some shape λ for G_2, then $wt_{Lit}(\mathbf{t}) = wt_{Lit}(t_1) + \cdots + wt_{Lit}(t_k)$.*

Proof. Let $c : K_G^{Lit}(2, 6 \times \lambda) \to \mathbb{Z}^6$ be the function defined by $c(\mathbf{t}) = (c_1(\mathbf{t}), \ldots, c_6(\mathbf{t}))$. Clearly, the function c is additive, so $c(\mathbf{t}) = c(t_1) + \cdots + c(t_k)$. Now let $\phi : \mathbb{Z}^6 \to \mathbb{Z}^2$ be the function given by

$$\phi(a_1, \ldots, a_6) = (\tfrac{1}{6}[a_1 - a_2 + 2a_3 - 2a_4 + a_5 - a_6], \tfrac{1}{6}[a_2 - a_3 + a_4 - a_5]).$$

Observe that ϕ is $\mathbb{Z}$-linear. Then we see that

$$\begin{aligned} wt_{Lit}(\mathbf{t}) &= \phi(c(\mathbf{t})) = \phi(c(t_1) + \cdots + c(t_k)) \\ &= \phi(c(t_1)) + \cdots + \phi(c(t_k)) \\ &= wt_{Lit}(t_1) + \cdots + wt_{Lit}(t_k). \end{aligned}$$

□

Remark 6.1.3 We know that wt_{Lit} is integer-valued (despite the appearance of the fraction $\frac{1}{6}$) because weights are vectors with integer coordinates, and Littelmann's weight rule is well-defined. This fact can also be deduced from the previous result together with the observation that wt_{Lit} is integer-valued when $\mathbf{t} \in K_G^{Lit}(2, 6\times \square)$ or $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\ \square\end{smallmatrix})$. $\square$

Now let $\mathbf{s} = (s_1, \ldots, s_k)$ and $\mathbf{t} = (t_1, \ldots, t_k) \in K_G^{Lit}(2, 6 \times \lambda)$, where s_l and t_l are in $K_G^{Lit}(2, 6\times \square)$ or $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\ \square\end{smallmatrix})$. If $s_l = t_l$ for $l \neq j$ with $s_j \to t_j$, then one can easily see that $\mathbf{s} \to \mathbf{t}$ in $K_G^{Lit}(2, 6 \times \lambda)$. Conversely, suppose that $\mathbf{s} \to \mathbf{t}$ in $K_G^{Lit}(2, 6 \times \lambda)$. Now clearly $s_l \leq t_l$ for $1 \leq l \leq k$. Let j be the smallest index for which $s_j < t_j$. Suppose now that there is another $l_1 > j$ such that $s_{l_1} < t_{l_1}$. Construct a tableau $\mathbf{u}$ by setting $u_l = s_l$ for $l < l_1$ and $u_l = t_l$ for $l \geq l_1$. Does semi-standardness allow t_{l_1} to follow $s_{l_1 - 1}$? Yes, by Observation 6.1.1. Thus, $\mathbf{u} \in K_G^{Lit}(2, 6 \times \lambda)$ and $\mathbf{s} < \mathbf{u} < \mathbf{t}$. This contradicts the fact that $\mathbf{s} \to \mathbf{t}$, so it must be the case that $s_{j+1} = t_{j+1}, \ldots, s_k = t_k$. Next, we claim that $s_j \to t_j$. If not, let $s_j < u_j < t_j$. Then construct a tableau $\mathbf{u}$ by $u_l = s_l = t_l$ for $l \neq j$. Does semi-standardness allow u_j to follow s_{j-1}, and t_{j+1} to follow u_j? Yes, again by Observation 6.1.1. Then we see that $\mathbf{s} < \mathbf{u} < \mathbf{t}$ in $K_G^{Lit}(2, 6 \times \lambda)$, which contradicts our assumption that $\mathbf{s} \to \mathbf{t}$. We summarize this in the following lemma:

Lemma 6.1.4 *Let $\mathbf{s} = (s_1, \ldots, s_k)$ and $\mathbf{t} = (t_1, \ldots, t_k)$ be in $K_G^{Lit}(2, 6 \times \lambda)$, where each s_l and t_l is in $K_G^{Lit}(2, 6\times \square)$ or $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\ \square\end{smallmatrix})$. Then $\mathbf{s} \to \mathbf{t}$ in $K_G^{Lit}(2, 6 \times \lambda)$ if and only if $s_l = t_l$ for $l \neq j$ with $s_j \to t_j$.*

One can check by hand in the pictures $K_G^{Lit}(2, 6\times \square)$ and $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\ \square\end{smallmatrix})$, if $\mathbf{s} \to \mathbf{t}$, then $wt_{Lit}(\mathbf{s}) + \alpha_i = wt_{Lit}(\mathbf{t})$, where α_i is the ith row of the Cartan matrix $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ for G_2 (so $i = 1$ or $i = 2$). We have depicted this in Figures $B.2$ and $B.3$ by placing the "colors"

1 and 2 on the edges of $K_G^{Lit}(2, 6\times \square)$ and $K_G^{Lit}(2, 6\times \begin{smallmatrix}\square\\\square\end{smallmatrix})$. In light of Lemma 6.1.4 above, we get that if $\mathbf{s} \rightarrow \mathbf{t}$ in $K_G^{Lit}(2, 6 \times \lambda)$, then $wt_{Lit} + \alpha_i = wt_{Lit}(\mathbf{t})$ for $i = 1$ or $i = 2$. So we make the following definition.

Definition 6.1.5 Let $\mathbf{s}$ and $\mathbf{t}$ be in $K_G^{Lit}(2, 6 \times \lambda)$, and suppose $\mathbf{s} \rightarrow \mathbf{t}$. Then we give this edge color $i(i = 1 \textit{ or } i = 2)$ if $wt_{Lit}(\mathbf{s}) + \alpha_i = wt_{Lit}(\mathbf{t})$, where α_i is the ith row of the Cartan matrix for G_2, $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

6.2 Lattices from Littelmann's G_2 tableaux of shape λ. Littelmann's expanded blocks are a bit cumbersome to work with on bigger shapes. We would like to translate his objects into tableaux analogous to the A_2 case, where λ is the actual shape of the tableaux. Figure $B.4$ shows the translation of Littelmann's $6 \times \lambda$ tableaux into what we will call "Littelmann's G_2 tableaux of shape λ" when the shape λ is a single column.

By inspection, one can check (by comparing Figures $B.2$ and $B.3$ to Figure $B.4$) that the semi-standard condition which determines when an "expanded" block $\mathbf{s}$ can be followed by another "expanded" block $\mathbf{t}$ results in the restrictions on our "translated" tableaux recorded in Table $B.5$. Then, given a shape λ for G_2, a *Littelmann G_2 tableau* $\mathbf{t}$ *of shape* λ is a semi-standard filling of the boxes of λ with entries from $\{1, 2, 3, 4, 5, 6, 7\}$ so that the columns of $\mathbf{t}$ come from Figure $B.4$ and successive columns in $\mathbf{t}$ satisfy the restrictions of Table $B.5$. So our example tableau for $\lambda = (1, 1)$ from page 38 becomes $\begin{smallmatrix} 1 & 4 \\ 6 & \end{smallmatrix}$.

To construct the order diagram of these tableaux, we order them by reverse component-wise comparison. We will call the poset constructed from these tableaux $L_G^{Lit}(2, \lambda)$.

An edge in $L_G^{Lit}(2, \lambda)$ gets the same color as the corresponding edge in $K_G^{Lit}(2, 6 \times \lambda)$. But now if we take advantage of Lemma 6.1.4, we see that the rule for coloring the edges of $L_G^{Lit}(2, \lambda)$ is greatly simplified. For $\mathbf{s}, \mathbf{t} \in L_G^{Lit}(2, \lambda)$ and $\mathbf{s} \to \mathbf{t}$:

(1) If an entry of $\mathbf{s}$ changes from a 3 to a 2 or from a 6 to a 5 to form $\mathbf{t}$, we give the edge $\mathbf{s} \to \mathbf{t}$ color 2,

(2) For any other change in an entry of $\mathbf{s}$ to form $\mathbf{t}$, we give the edge $\mathbf{s} \to \mathbf{t}$ color 1.

Lemma 6.2.1 *$L_G^{Lit}(2, \ydiagram{2,1})$, $L_G^{Lit}(2, \ydiagram{2,2})$, and $L_G^{Lit}(2, \ydiagram{2})$ are distributive lattices. (Here, the order is reverse component-wise comparison.)*

Proof. The lemma was verified using Stembridge's *poset package* for *Maple*. (See [Stem].) It is possible to confirm this by hand by applying the Fundamental Theorem of Finite Distributive Lattices (see [Sta]). Begin by identifying the "poset of join irreducibles" in each case, and then checking that the distributive lattice of order ideals for each poset of join irreducibles gives you back the picture with which you started. □

Theorem 6.2.2 *$L_G^{Lit}(2, \lambda)$ is a distributive lattice for any shape λ.*

Proof. Let $\lambda = (a, b)$ be a shape with a columns of length one box and b columns of length two boxes. Let $P_1 = P_2 = \cdots = P_b = L_G^{Lit}(2, \ydiagram{1,1})$. Let $P_{b+1} = P_{b+2} = \cdots = P_{b+a} = L_G^{Lit}(2, \ydiagram{1})$. Then $L_G^{Lit}(2, \lambda) \subseteq P_1 \times P_2 \times \cdots \times P_{a+b}$. Then for $\mathbf{t} \in L_G^{Lit}(2, \lambda)$, let $\mathbf{t} = (t_1, \ldots, t_{a+b})$, where t_i is the ith column of $\mathbf{t}$. Let $\mathbf{s}$ be another tableau in $L_G^{Lit}(2, \lambda)$, and write $\mathbf{s} = (s_1, \ldots, s_{a+b})$. In particular, (s_i, s_{i+1}) and (t_i, t_{i+1}) (for $1 \leq i \leq a + b - 1$) are in $L_G^{Lit}(2, \mu)$, where $\mu = \ydiagram{2,1}$, $\ydiagram{2}$, or $\ydiagram{2,2}$.

Then, by the previous lemma, we have that $(s_i \vee t_i, s_{i+1} \vee t_{i+1}) \in L_G^{Lit}(2, \mu)$ and $(s_i \wedge t_i, s_{i+1} \wedge t_{i+1}) \in L_G^{Lit}(2, \mu)$, for $1 \leq i \leq a + b - 1$ and $\mu = \ydiagram{2,1}$, $\ydiagram{2}$, or $\ydiagram{2,2}$. Therefore,

$(s_1 \vee t_1, s_2 \vee t_2, \ldots, s_{a+b} \vee t_{a+b}) \in L_G^{Lit}(2, \lambda)$ and $(s_1 \wedge t_1, s_2 \wedge t_2, \ldots, s_{a+b} \wedge t_{a+b}) \in L_G^{Lit}(2, \lambda)$. So $L_G^{Lit}(2, \lambda)$ is closed under the component-wise max and component-wise min operations. By Lemma 2.2.3, $L_G^{Lit}(2, \lambda)$ is a distributive lattice for any shape λ. □

Conjecture 6.2.3 *$L_G^{Lit}(2, \lambda)$ is a support for $G_2(\lambda)$.*

We will look at some theorems, their implications, and other evidence that will provide some support for this conjecture. Since the Littelmann G_2 tableaux of shape λ are in one to one correspondence with Littelmann's $6 \times \lambda$ tableaux, from [Lit] we get:

Theorem 6.2.4 (Littelmann) *The number of vertices in $L_G^{Lit}(2, \lambda)$ is equal to the dimension of $G_2(\lambda)$.*

We say a tableau $\mathbf{t}$ in $L_G^{Lit}(2, \lambda)$ is *i-maximal* (respectively, *i-minimal*) if it is a maximal (respectively, minimal) element of the i-component that contains it. The next proposition allows us to easily locate the i-maximal and i-minimal elements of $L_G^{Lit}(2, \lambda)$.

Proposition 6.2.5 *Let $\mathbf{t}$ be i-maximal in $L_G^{Lit}(2, \lambda)$. Then each column of $\mathbf{t}$ is i-maximal in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \substack{\square\\\square})$. Similarly, when $\mathbf{s}$ is i-minimal in $L_G^{Lit}(2, \lambda)$, then each column of $\mathbf{s}$ is i-minimal in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \substack{\square\\\square})$.*

Proof. Let $\mathbf{t}$ be any tableau in $L_G^{Lit}(2, \lambda)$, where $\lambda = (a, b)$. Write $\mathbf{t} = (t_1, \ldots, t_{a+b})$. We will first consider the maximal case. Suppose that $t_1, \ldots, t_{j-1}$ are all i-maximal in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \substack{\square\\\square})$ (that is, they are all at the top of their respective i-components). If the column t_j is not i-maximal, we want to show that it can be made i-maximal.

First, we need to consider the possible fillings of the column t_{j-1}. By supposition, t_{j-1} is i-maximal, so the set of possibilities for t_{j-1} are the i-maximal elements in $L_G^{Lit}(2, \square)$ and $L_G^{Lit}(2, \substack{\square\\\square})$ that are allowed to precede t_j by the semi-standardness of $\mathbf{t}$ and by the filling restrictions (which can be found in Table $B.5$ in Appendix B). The set of possible fillings for column t_{j+1} are just the fillings allowed to follow t_j by the semi-standard condition and by the filling restrictions .

Now, if $t_j \xrightarrow{i} u$ in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \begin{smallmatrix}\square\\ \square\end{smallmatrix})$, change the jth column of $\mathbf{t}$ to u. If u is i-maximal, the jth column of $\mathbf{t}$ is now i-maximal. If u is not i-maximal, then there exists a v such that $u \xrightarrow{i} v$ in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \begin{smallmatrix}\square\\ \square\end{smallmatrix})$. Now change the jth column of $\mathbf{t}$ from the filling of u to that of v. Continue in this process until the jth column of $\mathbf{t}$ is i-maximal.

As we change the filling of t_j, the numerical value of the entry in each box weakly decreases, so note that its values will always be less than the values in the column t_{j+1}. We have verified by hand for every possibility for t_{j-1} and t_j that the process described in the previous paragraph never violates the semi-standard condition between the $(j-1)$st and jth columns of $\mathbf{t}$. Thus, changing t_j to maximal does not violate the semi-standard condition of $\mathbf{t}$. It can also be verified by hand (by considering each filling of t_j together with the set of possibilities for the fillings of t_{j-1} and t_{j+1}), that making t_j i-maximal does not violate any of the filling restrictions. We show how we went about verifying these claims by hand in Example 6.2.6 below. One observation that should be noted is that as we change the filling of t_j to i-maximal ("move up" an i-component), the new filling will not have any restrictions that the previous filling did not have, but could possibly have fewer (this can be seen by analyzing the covering relations in any i-component of t_j in Figure $B.4$ along with the filling restrictions listed in Table $B.4$).

Now we will consider the minimal case. Let $\mathbf{s}$ be any tableau in $L_G^{Lit}(2, \lambda)$, where $\lambda = (a, b)$. Write $\mathbf{s} = (s_1, \ldots, s_{a+b})$. Suppose that $s_{j+1}, \ldots, s_{a+b-1}, s_{a+b}$ are all i-minimal.

If the column s_j is not i-minimal, it can be made i-minimal with a process similar to that for the maximal case.

If $r \xrightarrow{i} s_j$ in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array})$, change the filling of column s_j to that of r. Continue this process of "walking down" the i-component until s_j is i-minimal. $\square$

We will now consider an example that will illustrate how to make a column in a tableau i-maximal and i-minimal.

Example 6.2.6 In the notation of the proof above, suppose $t_j = \begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array}$. To change the filling of the jth column of $\mathbf{t}$ to make it 1-maximal, we need to consider the possible fillings of t_{j-1} and t_{j+1}. Since t_{j-1} is assumed to be 1-maximal, we must have $t_{j-1} \in \{ \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} , \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} , \begin{array}{|c|}\hline 1\\ \hline 6\\ \hline\end{array} , \begin{array}{|c|}\hline 3\\ \hline 6\\ \hline\end{array} \}$, since those are the 1-maximal elements allowed to precede t_j by the semi-standard condition and the filling restrictions. Similarly, $t_{j+1} \in \{ \begin{array}{|c|}\hline 5\\ \hline 7\\ \hline\end{array} , \begin{array}{|c|}\hline 6\\ \hline 7\\ \hline\end{array} , \boxed{5} , \boxed{6} , \boxed{7} \}$. Now, since $\begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array} \xrightarrow{1} \begin{array}{|c|}\hline 3\\ \hline 7\\ \hline\end{array}$, change t_j to $\begin{array}{|c|}\hline 3\\ \hline 7\\ \hline\end{array}$. Observe that $\begin{array}{|c|}\hline 3\\ \hline 7\\ \hline\end{array}$ will not produce any violations on either side of t_j . Also, note that $\begin{array}{|c|}\hline 3\\ \hline 7\\ \hline\end{array}$ is not 1-maximal. But since $\begin{array}{|c|}\hline 3\\ \hline 7\\ \hline\end{array} \xrightarrow{1} \begin{array}{|c|}\hline 3\\ \hline 6\\ \hline\end{array}$, change t_j to $\begin{array}{|c|}\hline 3\\ \hline 6\\ \hline\end{array}$. Note that $\begin{array}{|c|}\hline 3\\ \hline 6\\ \hline\end{array}$ is 1-maximal and that it does not produce any violations. So $\begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array}$ can be made 1-maximal by changing $\begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array}$ to $\begin{array}{|c|}\hline 3\\ \hline 7\\ \hline\end{array}$ to $\begin{array}{|c|}\hline 3\\ \hline 6\\ \hline\end{array}$.

Now let $s_j = t_j = \begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array}$. To force s_j to be 1-minimal, note that by supposition, s_{j+1} is 1-minimal, so $s_{j+1} \in \{ \begin{array}{|c|}\hline 5\\ \hline 7\\ \hline\end{array} , \begin{array}{|c|}\hline 6\\ \hline 7\\ \hline\end{array} , \boxed{5} , \boxed{7} \}$. Also, $s_{j-1} \in \{ \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} , \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} , \begin{array}{|c|}\hline 1\\ \hline 4\\ \hline\end{array} , \begin{array}{|c|}\hline 1\\ \hline 5\\ \hline\end{array} , \begin{array}{|c|}\hline 1\\ \hline 6\\ \hline\end{array} , \begin{array}{|c|}\hline 2\\ \hline 5\\ \hline\end{array} , \begin{array}{|c|}\hline 2\\ \hline 6\\ \hline\end{array} , \begin{array}{|c|}\hline 3\\ \hline 6\\ \hline\end{array} \}$. We can "walk $\begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array}$ down" its 1-component to $\begin{array}{|c|}\hline 5\\ \hline 7\\ \hline\end{array}$, which is 1-minimal. So we can change s_j from $\begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array}$ to $\begin{array}{|c|}\hline 5\\ \hline 7\\ \hline\end{array}$ and not cause any filling violation.

Note that $\begin{array}{|c|}\hline 4\\ \hline 7\\ \hline\end{array}$ is its own 2-component, so it is already 2-maximal and 2-minimal. $\square$

Corollary 6.2.7 *Let* $\mathbf{t} \in L_G^{Lit}(2, \lambda)$, *where* $\lambda = (a, b)$. *Write* $\mathbf{t} = (t_1, \ldots, t_{a+b})$. *Then* $l_i(\mathbf{t}) = l_i(t_1) + \cdots + l_i(t_{a+b})$ *and* $\rho_i(\mathbf{t}) = \rho_i(t_1) + \cdots + \rho_i(t_{a+b})$.

Proof. The previous proposition tells us how to form the max element $\mathbf{t}_{max}$ in the i-component of $\mathbf{t}$ and the min element $\mathbf{t}_{min}$ in the i-component of $\mathbf{t}$. To form $\mathbf{t}_{max}$, we take the max element in the i-component of each t_l $(1 \leq l \leq a+b)$. Then, we put these columns together (in the same order as their respective columns in $\mathbf{t}$) into a tableau, forming $\mathbf{t}_{max}$. Similarly, form $\mathbf{t}_{min}$.

Now we ask: How many steps are there from $\mathbf{t}_{min}$ to $\mathbf{t}_{max}$? In the proof of the previous proposition, we saw that we could take each column $(\mathbf{t}_{min})_l$ (which is i-minimal in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \begin{smallmatrix}\square\\\square\end{smallmatrix})$) of $\mathbf{t}_{min}$ and in successive steps change it to $(\mathbf{t}_{max})_l$. We do this for each column of $\mathbf{t}_{min}$, starting with column $l = 1$ and working our way left to right across the tableau, completing the process for column $l = a+b$. (Similarly, we could take each column $(\mathbf{t}_{max})_l$ in $\mathbf{t}_{max}$, starting with column $l = a+b$ and then successively change each column working left to column $l = 1$ ending up with t_{min}.) Thus, there are $l_i(t_1)+l_i(t_2)+\cdots+l_i(t_{a+b})$ steps between $\mathbf{t}_{min}$ and $\mathbf{t}_{max}$. Therefore, $l_i(\mathbf{t}) = l_i(t_1) + l_i(t_2) + \cdots + l_i(t_{a+b})$.

Next we ask: How many steps are there from $\mathbf{t}_{min}$ to $\mathbf{t}$? In the first column, there are $\rho_i(t_1)$ steps, in the second column $\rho_i(t_2)$, etc. So there are $\rho_i(t_1) + \rho_i(t_2) + \cdots + \rho_i(t_{a+b})$ steps between $\mathbf{t}_{min}$ and $\mathbf{t}$. Hence, $\rho_i(\mathbf{t}) = \rho_i(t_1) + \rho_i(t_2) + \cdots + \rho_i(t_{a+b})$. $\square$

The next result is the main result of this thesis. It says, in effect, that the lattices $L_G^{Lit}(2, \lambda)$ provide a good combinatorial setting for studying the irreducible representations of G_2.

Theorem 6.2.8 *Let* $\mathbf{t} \in L_G^{Lit}(2, \lambda)$, *where* $\lambda = (a, b)$. *Then,* $wt_P(\mathbf{t}) = wt_{Lit}(\mathbf{t})$.

Proof. Let $\mathbf{t} \in L_G^{Lit}(2,\lambda)$, where $\lambda = (a,b)$. Since $L_G^{Lit}(2,\lambda)$ is a distributive lattice (cf. Corollary 6.2.2), $L_G^{Lit}(2,\lambda)$ has a unique maximal element. (In fact, any lattice has a unique maximal element.)

Note that for a shape $\lambda = (a,b)$, $\mathbf{m} \in L_G^{Lit}(2,\lambda)$ is the (unique) maximal element if all the entries in the first row are 1 and all the entries in the second row (if $b \neq 0$) are 2. (The element $\mathbf{m}$ is maximal since there is not an entry that we can decrease without violating the semi-standard condition.) That is, $\mathbf{m}$ has b columns of $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ and a columns of $\boxed{1}$.

Since $\lambda = (a,b)$, then it can be observed that the 1-component of the maximal element $\mathbf{m} \in L_G^{Lit}(2,\lambda)$ is a chain of length a, and the 2-component of $\mathbf{m}$ is a chain of length b. Thus, $wt_P(\mathbf{m}) = (a,b)$.

To compute the weight of $\mathbf{m}$ using Littelmann's weight rule, we will need to consider Littelmann's $6 \times \lambda$ tableaux. That is, $\boxed{1} = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\\hline\end{array} \in K_G^{Lit}(2, 6\times \square\,)$ and $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\\hline 2&2&2&2&2&2\\\hline\end{array} \in K_G^{Lit}(2, 6\times \begin{array}{|c|}\hline \\\hline \\\hline\end{array}\,)$. Thus,

$$wt_{Lit}(\,\boxed{1}\,) = (\tfrac{1}{6}[6], \tfrac{1}{6}[0]) = (1,0) \text{ and}$$
$$wt_{Lit}(\,\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,) = (\tfrac{1}{6}[6-6], \tfrac{1}{6}[6]) = (0,1).$$

Now, we will use the fact that Littelmann's weight rule is additive (cf. Lemma 6.1.2). That is, for $\mathbf{t} = (t_1, \ldots, t_{a+b}) \in L_G^{Lit}(2,\lambda)$, $wt_{Lit}(\mathbf{t}) = wt_{Lit}(t_1) + \cdots + wt_{Lit}(t_{a+b})$. Since $\mathbf{m}$ has b columns of $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ and a columns of $\boxed{1}$, the corresponding Littelmann $6 \times \lambda$ tableau would have $6b$ columns of $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ and $6a$ columns of $\boxed{1}$. Thus, for Littelmann's weight rule, $c_1(\mathbf{m}) = 6a + 6b = 6(a+b)$ and $c_2(\mathbf{m}) = 6b$. Hence,

$$\begin{aligned} wt_{Lit}(\mathbf{m}) &= (\tfrac{1}{6}[6(a+b) - 6b], \tfrac{1}{6}[6b]) \\ &= (\tfrac{1}{6}[6a + 6b - 6b], b) \end{aligned}$$

$$= (a, b).$$

Therefore, for the maximal element $\mathbf{m} \in L_G^{Lit}(2, \lambda)$, $wt_P(\mathbf{m}) = wt_{Lit}(\mathbf{m})$.

Now suppose $\mathbf{s} \xrightarrow{i} \mathbf{t}$ is any edge in $L_G^{Lit}(2, \lambda)$. We will show that $wt_P(\mathbf{s}) + \alpha_i = wt_P(\mathbf{t})$, where α_i is the ith row of the Cartan matrix for G_2, $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

Since $\mathbf{s}, \mathbf{t}$ are in the same i-component and $\rho_i(\mathbf{s}) + 1 = \rho_i(\mathbf{t})$, we have $2\rho_i(\mathbf{s}) - l_i(\mathbf{s}) + 2 = 2\rho_i(\mathbf{t}) - l_i(\mathbf{t})$. If $i = 1$, then $\mathbf{s}, \mathbf{t}$ are in the same 1-component and we can add 2 to the first coordinate of $wt_P(\mathbf{s})$ to obtain the first coordinate of $wt_P(\mathbf{t})$. Similarly, if $i = 2$, we can add 2 to the second coordinate of $wt_P(\mathbf{s})$ to obtain the second coordinate of $wt_P(\mathbf{t})$.

Now we will analyze the second coordinate of $wt_P(\mathbf{s})$ and $wt_P(\mathbf{t})$ for $i = 1$. Let $\mathbf{s} = (s_1, \ldots, s_j, \ldots, s_{a+b})$ and $\mathbf{t} = (t_1, \ldots, t_j, \ldots, t_{a+b})$, where $s_l = t_l$ $(l \neq j)$ and $s_j \xrightarrow{1} t_j$ in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \substack{\square\\\square})$. The previous corollary tells us that

$$l_2(\mathbf{s}) = l_2(s_1) + \cdots + l_2(s_j) + \cdots + l_2(s_{a+b}),$$

$$l_2(\mathbf{t}) = l_2(t_1) + \cdots + l_2(t_j) + \cdots + l_2(s_{t+b}),$$

$$\rho_2(\mathbf{s}) = \rho_2(s_1) + \cdots + \rho_2(s_j) + \cdots + \rho_2(s_{a+b}), \text{ and}$$

$$\rho_2(\mathbf{t}) = \rho_2(t_1) + \cdots + \rho_2(t_j) + \cdots + \rho_2(t_{a+b}).$$

But since $s_l = t_l$ $(l \neq j)$,

$$2\rho_2(\mathbf{t}) - l_2(\mathbf{t}) - (2\rho_2(\mathbf{s}) - l_2(\mathbf{s})) = 2\rho_2(t_j) - l_2(t_j) - (2\rho_2(s_j) - l_2(s_j)).$$

(Recall that s_j, t_j are in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \substack{\square\\\square})$.)

Below we will consider the possibilities for $s_j \xrightarrow{1} t_j$ in $L_G^{Lit}(2, \square)$ and $L_G^{Lit}(2, \substack{\square\\\square})$ and show that for each possibility the difference $2\rho_2(t_j) - l_2(t_j) - (2\rho_2(s_j) - l_2(s_j))$ is -1.

$s_j \xrightarrow{1} t_j$	$\rho_2(t_j) - l_2(t_j)$	$2\rho_2(s_j) - l_2(s_j)$	*Difference*
$\boxed{2} \xrightarrow{1} \boxed{1}$	0	1	-1
$\boxed{4} \xrightarrow{1} \boxed{3}$	-1	0	-1
$\boxed{5} \xrightarrow{1} \boxed{4}$	0	1	-1
$\boxed{7} \xrightarrow{1} \boxed{6}$	-1	0	-1
$\begin{array}{\|c\|}\hline 1\\\hline 4\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array}$	-1	0	-1
$\begin{array}{\|c\|}\hline 1\\\hline 5\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 1\\\hline 4\\\hline\end{array}$	0	1	-1
$\begin{array}{\|c\|}\hline 2\\\hline 5\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 1\\\hline 5\\\hline\end{array}$	1	2	-1
$\begin{array}{\|c\|}\hline 2\\\hline 6\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array}$	-1	0	-1
$\begin{array}{\|c\|}\hline 2\\\hline 7\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 2\\\hline 6\\\hline\end{array}$	0	1	-1
$\begin{array}{\|c\|}\hline 1\\\hline 7\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array}$	-1	0	-1
$\begin{array}{\|c\|}\hline 2\\\hline 7\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 1\\\hline 7\\\hline\end{array}$	0	1	-1
$\begin{array}{\|c\|}\hline 3\\\hline 7\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 3\\\hline 6\\\hline\end{array}$	-2	-1	-1
$\begin{array}{\|c\|}\hline 4\\\hline 7\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 3\\\hline 7\\\hline\end{array}$	-1	0	-1
$\begin{array}{\|c\|}\hline 5\\\hline 7\\\hline\end{array} \xrightarrow{1} \begin{array}{\|c\|}\hline 4\\\hline 7\\\hline\end{array}$	0	1	-1

Thus, for $i = 1$, we can add -1 to the second coordinate of $wt_P(\mathbf{s})$ to obtain the second coordinate of $wt_P(\mathbf{t})$. Therefore, if $\mathbf{s} \xrightarrow{1} \mathbf{t}$, $wt_P(\mathbf{s}) + \alpha_1 = wt_P(\mathbf{t})$.

Now for $i = 2$, we still need to analyze the first coordinate of $wt_P(\mathbf{s})$ and $wt_P(\mathbf{t})$. Now, $\mathbf{s} = (s_1, \ldots, s_j, \ldots, s_{a+b})$ and $\mathbf{t} = (t_1, \ldots, t_j, \ldots, t_{a+b})$, where $s_l = t_l (l \neq j)$ and $s_j \xrightarrow{2} t_j$ in $L_G^{Lit}(2, \square)$ or $L_G^{Lit}(2, \begin{array}{|c|}\hline \\\hline \\\hline\end{array})$. By the previous corollary, we have

$$2\rho_1(\mathbf{t}) - l_1(\mathbf{t}) - (2\rho_1(\mathbf{s}) - l_1(\mathbf{s})) = 2\rho_1(t_j) - l_1(t_j) - (2\rho_1(s_j) - l_1(s_j)).$$

Below we will consider the possibilities for $s_j \xrightarrow{2} t_j$ in $L_G^{Lit}(2, \square)$ and $L_G^{Lit}(2, \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array})$ and show that for each, the difference $2\rho_1(t_j) - l_1(t_j) - (2\rho_1(s_j) - l_1(s_j))$ is -3.

$s_j \xrightarrow{2} t_j$	$\rho_1(t_j) - l_1(t_j)$	$2\rho_1(s_j) - l_1(s_j)$	*Difference*
$\boxed{3} \xrightarrow{2} \boxed{2}$	-1	2	-3
$\boxed{6} \xrightarrow{2} \boxed{5}$	-2	1	-3
$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array} \xrightarrow{2} \begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$	0	3	-3
$\begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array} \xrightarrow{2} \begin{array}{\|c\|}\hline 1\\\hline 5\\\hline\end{array}$	-1	2	-3
$\begin{array}{\|c\|}\hline 2\\\hline 6\\\hline\end{array} \xrightarrow{2} \begin{array}{\|c\|}\hline 2\\\hline 5\\\hline\end{array}$	-3	0	-3
$\begin{array}{\|c\|}\hline 3\\\hline 6\\\hline\end{array} \xrightarrow{2} \begin{array}{\|c\|}\hline 2\\\hline 6\\\hline\end{array}$	0	3	-3
$\begin{array}{\|c\|}\hline 3\\\hline 7\\\hline\end{array} \xrightarrow{2} \begin{array}{\|c\|}\hline 2\\\hline 7\\\hline\end{array}$	-2	1	-3
$\begin{array}{\|c\|}\hline 6\\\hline 7\\\hline\end{array} \xrightarrow{2} \begin{array}{\|c\|}\hline 5\\\hline 7\\\hline\end{array}$	-3	0	-3

Thus, if $\mathbf{s} \xrightarrow{2} \mathbf{t}$, $wt_P(\mathbf{s}) + \alpha_2 = wt_P(\mathbf{t})$.

Therefore, if $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in $L_G^{Lit}(2, \lambda)$, we have $wt_P(\mathbf{s}) + \alpha_i = wt_P(\mathbf{t})$, where α_i is the ith row of the Cartan matrix $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

Thus, since $wt_P(\mathbf{m}) = wt_{Lit}(\mathbf{m})$, where $\mathbf{m}$ is the maximal element of $L_G^{Lit}(2, \lambda)$, if $\mathbf{r} \xrightarrow{i} \mathbf{m}$, we can subtract α_i from each of $wt_P(\mathbf{m})$ and $wt_{Lit}(\mathbf{m})$, and then we'll have that $wt_P(\mathbf{r}) = wt_{Lit}(\mathbf{r})$. We can continue this process of subtracting α_i as we "work down" the lattice. Thus, for any $\mathbf{t} \in L_G^{Lit}(2, \lambda)$, $wt_P(\mathbf{t}) = wt_{Lit}(\mathbf{t})$. □

6.3 Conclusions. We should now make some remarks about Conjecture 6.2.3. If $L_G^{Lit}(2,\lambda)$ is to be a support for $G_2(\lambda)$, $L_G^{Lit}(2,\lambda)$ must satisfy the conditions listed in Proposition 4.2.1. We know from Theorem 6.2.4 that the number of vertices in $L_G^{Lit}(2,\lambda)$ equals the dimension of $G_2(\lambda)$, so the first necessary condition is satisfied. Although we will not prove it in this thesis, it actually follows as a consequence of Theorem 6.2.8 that $L_G^{Lit}(2,\lambda)$ is a rank symmetric and rank unimodal poset. It can also be seen from $L_G^{Lit}(2, \square\ )$ and $L_G^{Lit}(2, \begin{smallmatrix}\square\\\square\end{smallmatrix}\ )$ in Figure $B.4$ (the fundamental representations of $G_2(\lambda)$) that $r_1 = 6$ and $r_2 = 10$. In light of Theorem 6.2.2 and the description of covering relations on page 42, one can easily see that for a shape $\lambda = (a, b)$, $L_G^{Lit}(2,\lambda)$ has $6a + 10b + 1$ ranks. So we have established (most of) necessary condition (2) of Proposition 4.2.1 for $L_G^{Lit}(2,\lambda)$. By Theorem 6.2.8, for $\mathbf{t} \in L_G^{Lit}(2,\lambda)$, $wt_P(\mathbf{t}) = wt_{Lit}(\mathbf{t})$, so since Littelmann's weight rule, wt_{Lit} is well-defined, the combinatorial weight rule, wt_P, is also well-defined, and thus, $L_G^{Lit}(2,\lambda)$ satisfies condition (3). Therefore, we have good evidence to support the conjecture, and we can say with some confidence that $L_G^{Lit}(2,\lambda)$ is a good candidate for being a supporting graph for the representation $G_2(\lambda)$.

We did not find a second family of lattices to analogize the supporting graphs $L_A^{GT-right}(2,\lambda)$ for A_2. In [Mol1] and [Mol2], Molev constructs the irreducible representations of the simple rank two Lie algebra C_2 in two ways. It is possible that the supporting graphs for these constructions are the "right" analogs of the Gelfand-Tsetlin supports, and their existence may shed some light on the possibility of finding a second family of supports for representations of G_2.

Appendix A

Basis Matrices for the Lie Algebras C_2 and B_2

Type C_2: The basis matrices for $so(2n+1,\mathbb{C})$ in the case $n=2$ are shown here.

For $sp(4,\mathbb{C})$, $B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$. The matrices $X = \begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix}$ with $N = N^T$ are $\begin{bmatrix} 0 & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \\ 0 & 0 \end{bmatrix}$. The matrices $X = \begin{bmatrix} 0 & 0 \\ P & 0 \end{bmatrix}$ with $P = P^T$ are $\begin{bmatrix} 0 & 0 \\ \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \end{bmatrix}$. The matrices $X = \begin{bmatrix} M & 0 \\ 0 & Q \end{bmatrix}$ with $M^T = -Q$ ($Q = -M^T$) are $\begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} -1 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}$, $\begin{bmatrix} \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 \\ 0 & -1 \end{matrix} \end{bmatrix}$, $\begin{bmatrix} \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix} \end{bmatrix}$, and $\begin{bmatrix} \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} \end{bmatrix}$. Note that the set of these matrices is a basis and spans $sp(4,\mathbb{C})$. Thus, $sp(4,\mathbb{C})$ is a 10-dimensional Lie subalgebra of $gl(4,\mathbb{C})$.

Type B_2: The basis matrices for $sp(2n, \mathbb{C})$ for $n = 2$ are worked out here as an example.

For $so(5, \mathbb{C}),\ B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. For $n = 2$, the matrix $X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & N \\ 0 & 0 & 0 \end{bmatrix}$ with

$-N = N^T$ is $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \text{\Large 0} & & 0 & 1 \\ 0 & & & -1 & 0 \\ 0 & \text{\Large 0} & & \text{\Large 0} & \\ 0 & & & & \end{bmatrix}$. The matrix $X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & P & 0 \end{bmatrix}$ with $-P = P^T$ is

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \text{\Large 0} & & \text{\Large 0} & \\ 0 & & & & \\ 0 & 0 & -1 & \text{\Large 0} & \\ 0 & 1 & 0 & & \end{bmatrix}$. The matrices $X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & Q \end{bmatrix}$ with $Q = -M^T$ are

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \text{\Large 0} & \\ 0 & 0 & 0 & & \\ 0 & \text{\Large 0} & & 0 & 0 \\ 0 & & & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{\Large 0} & \\ 0 & 1 & 0 & & \\ 0 & \text{\Large 0} & & 0 & -1 \\ 0 & & & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \text{\Large 0} & \\ 0 & 0 & 0 & & \\ 0 & \text{\Large 0} & & -1 & 0 \\ 0 & & & 0 & 0 \end{bmatrix}$, and

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & 1 & & \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & 0 & 0 \\ 0 & & & 0 & -1 \end{bmatrix}$. The matrices $X = \begin{bmatrix} 0 & 0 & S_2 \\ T_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where $-S_2 = {T_1}^T$ are

$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & & & & \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & & & & \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & & & & \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & & & & \end{bmatrix}$. The matrices $X = \begin{bmatrix} 0 & S_1 & 0 \\ 0 & 0 & 0 \\ T_2 & 0 & 0 \end{bmatrix}$

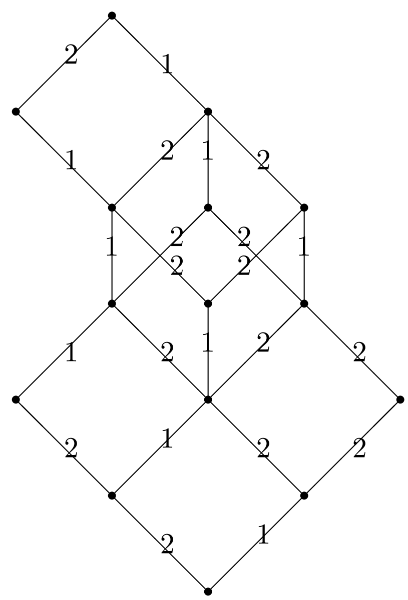
where $-S_1 = {T_2}^T$ are $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & & & & \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & & & & \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & & & & \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & & & & \end{bmatrix}$. This set of matrices forms a basis and spans $so(5, \mathbb{C})$. Thus, we have that $so(5, \mathbb{C})$ is a 10-dimensional Lie subalgebra of $gl(5, \mathbb{C})$.

Appendix B

Lattices for $A_2(\lambda)$ and $G_2(\lambda)$

Figure B.1 Gelfand-Tsetlin supports for $A_2(\lambda)$ with $\lambda = (2, 1)$.

$L_A^{GT-left}(2, \lambda)$



$L_A^{GT-right}(2, \lambda)$

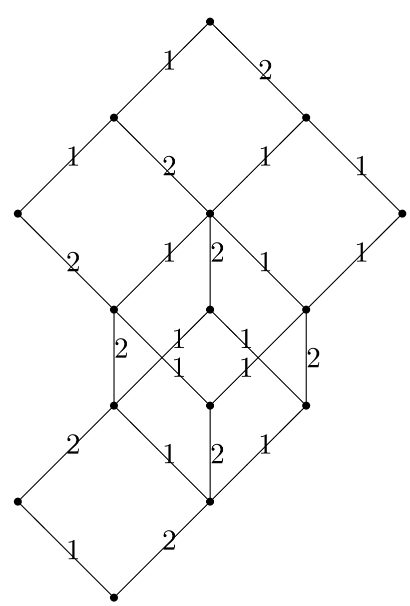


Figure B.2 Lattice of Littelmann's $6 \times \lambda$ tableaux for $\lambda = (1, 0)$.

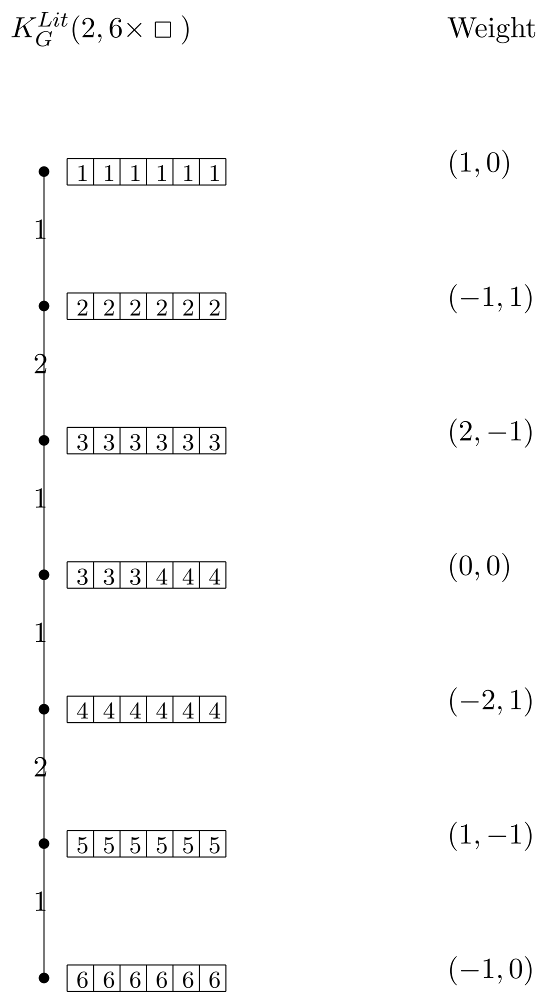
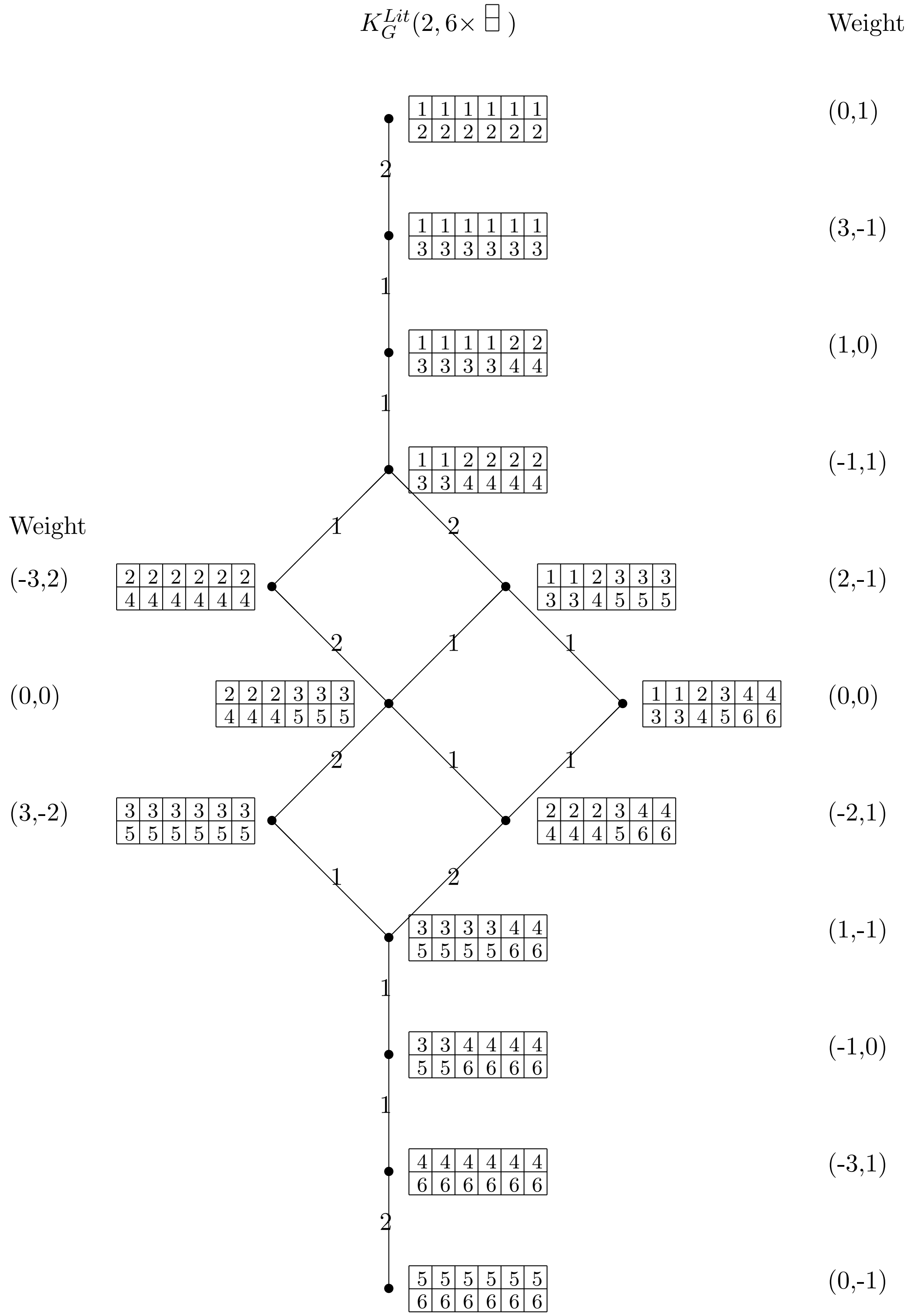
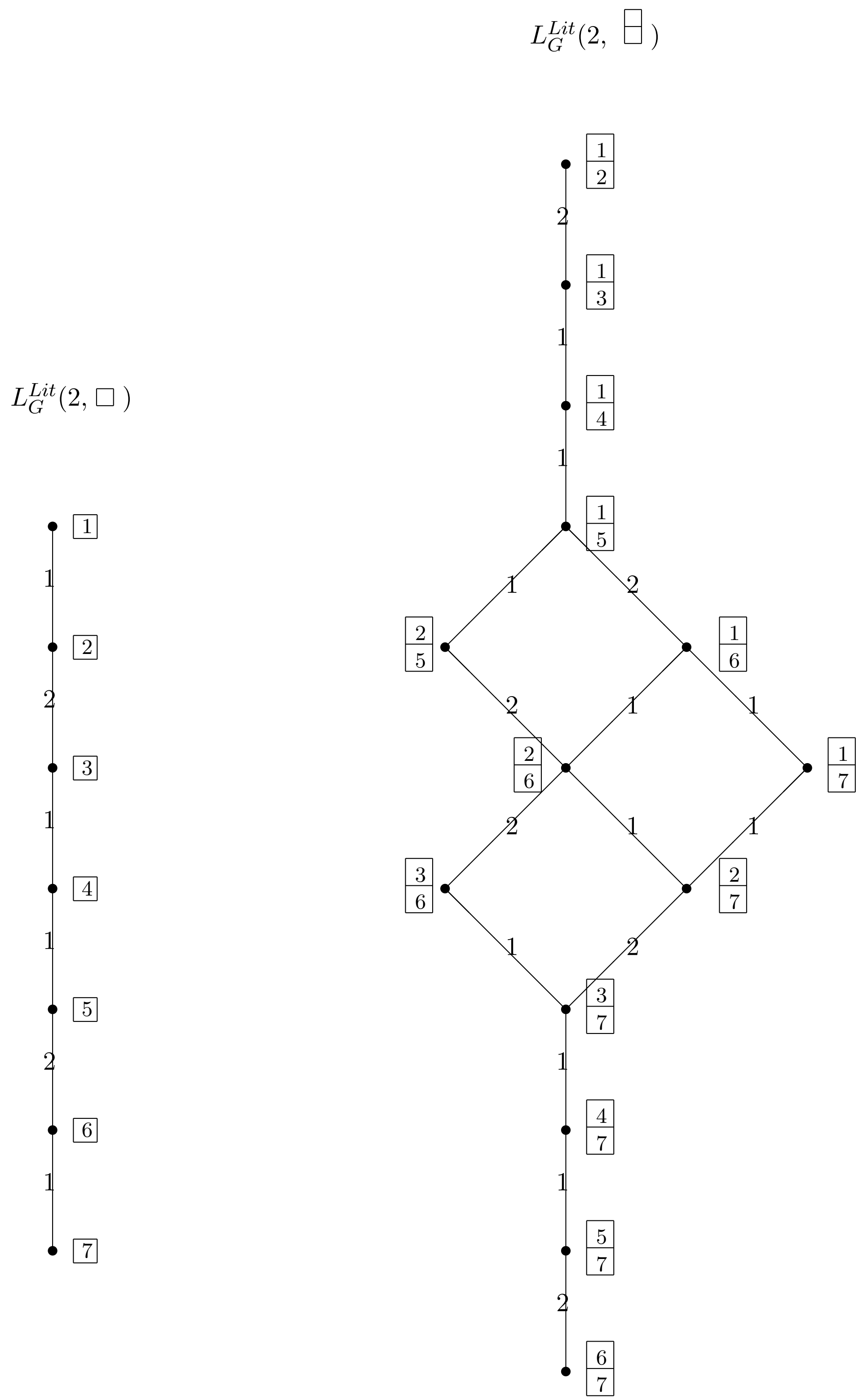


Figure B.3 Lattice of Littelmann's $6 \times \lambda$ tableaux for $\lambda = (0,1)$.



□

Figure B.4 Lattices of Littelmann's G_2 tableaux of shapes $\lambda = (1, 0)$ and $\lambda = (0, 1)$.



□

Table B.5 Filling Restrictions for Littelmann's G_2 tableaux

Filling	Cannot be followed by a column with a
$\boxed{4}$	$\boxed{4}$
$\begin{array}{\|c\|}\hline 1 \\ \hline 4 \\ \hline\end{array}$	$\boxed{1}$
$\begin{array}{\|c\|}\hline 1 \\ \hline 5 \\ \hline\end{array}$	$\boxed{1}$
$\begin{array}{\|c\|}\hline 1 \\ \hline 6 \\ \hline\end{array}$	$\boxed{1}$ or $\boxed{2}$
$\begin{array}{\|c\|}\hline 1 \\ \hline 7 \\ \hline\end{array}$	$\boxed{1}$, $\boxed{2}$, $\boxed{3}$, or $\boxed{4}$
$\begin{array}{\|c\|}\hline 2 \\ \hline 6 \\ \hline\end{array}$	$\boxed{2}$
$\begin{array}{\|c\|}\hline 2 \\ \hline 7 \\ \hline\end{array}$	$\boxed{2}$, $\boxed{3}$, or $\boxed{4}$
$\begin{array}{\|c\|}\hline 3 \\ \hline 7 \\ \hline\end{array}$	$\boxed{3}$ or $\boxed{4}$
$\begin{array}{\|c\|}\hline 4 \\ \hline 7 \\ \hline\end{array}$	$\boxed{4}$

Figure B.6 Lattice of Littelmann's G_2 tableaux for shape $\lambda = (2, 0)$.

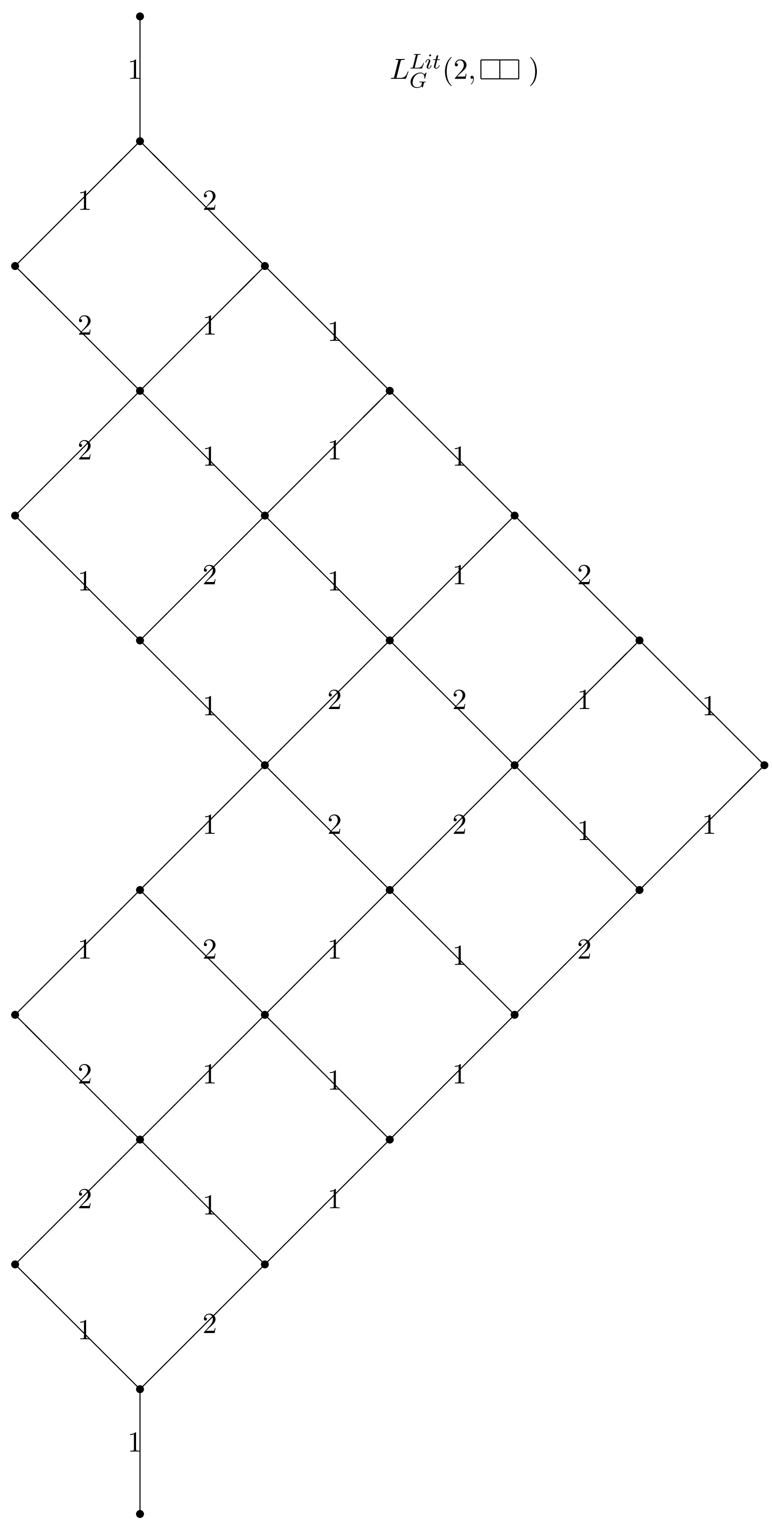
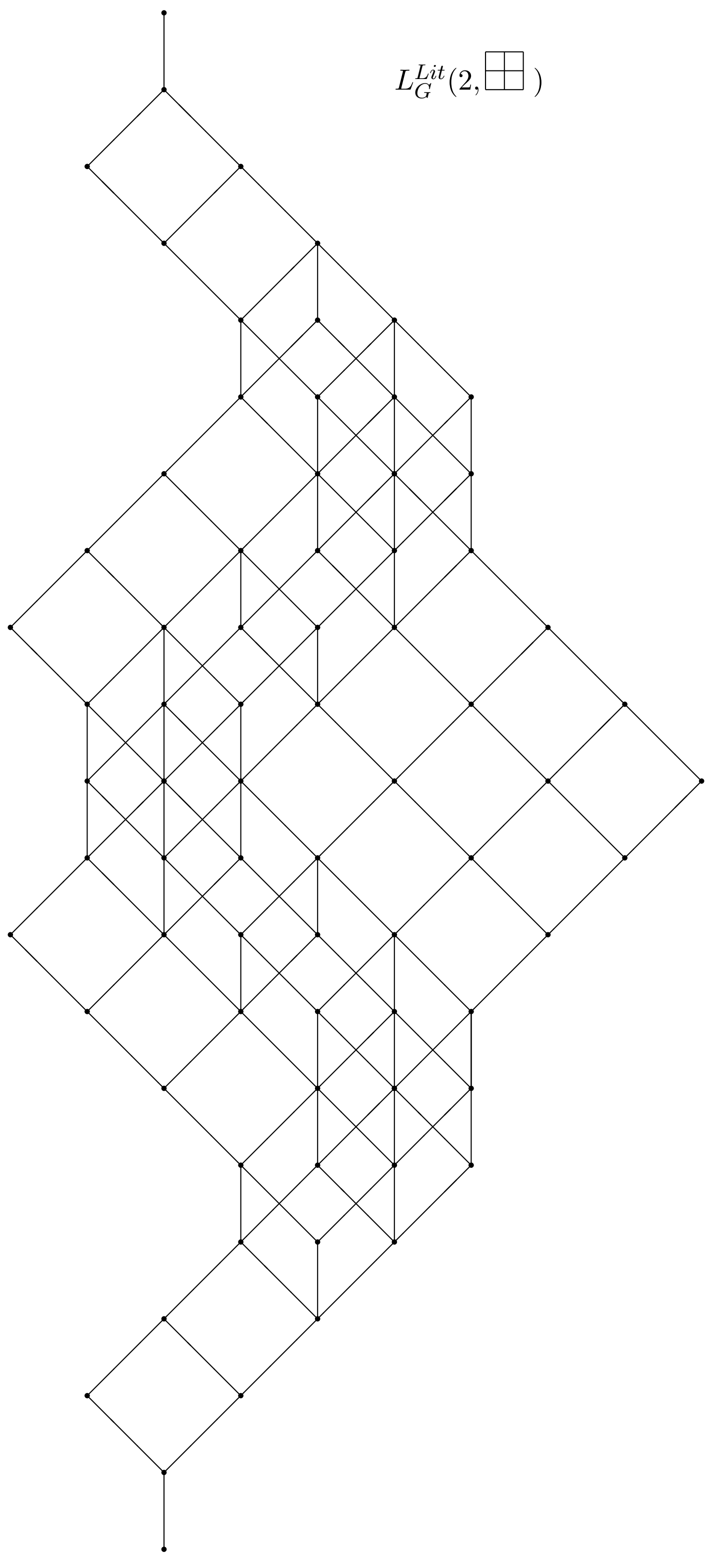


Figure B.7 Lattice of Littelmann's G_2 tableaux for $\lambda = (0, 2)$.



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