DISTRIBUTIVE LATTICES AND WEYL CHARACTERS OF EXOTIC TYPE F_4

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DISTRIBUTIVE LATTICES AND WEYL CHARACTERS OF EXOTIC TYPE F_4

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Matthew Ross Gilliland

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ABSTRACT

Posets and distributive lattices that model Weyl characters of the F_4 type are investigated. For the two 'smallest' characters, distributive lattice models and their posets of irreducibles obtained by Donnelly are presented. The existence of distributive lattice models for other 'small' characters is explored here using known methods and some new approaches. One of the main contributions of this thesis is a demonstration that for certain small characters, distributive lattice models do not exist. Another contribution is the discovery of distributive lattice models for certain other small characters; these models were found using posets of irreducibles. Obtaining these new existence/nonexistence results was aided by a new concept presented here, the so-called 'distributive core'.

TABLE OF CONTENTS

LIST OF FIGURES AND TABLES vii

Chapter 1 OVERVIEW 1

Chapter 2 COMBINATORIAL AND ALGEBRAIC SETTING FOR OUR MAIN RESULTS 7

Part 1: Some combinatorial preliminaries 7

Part 2: Weyl groups and Weyl characters 24

Chapter 3 THE DISTRIBUTIVE CORE 60

Chapter 4 F_4 SPECIFICS 69

Chapter 5 SPLITTING DISTRIBUTIVE LATTICES FOR F_4-CHARACTERS 77

Chapter 6 REMARKS AND SPECULATIONS 101

BIBLIOGRAPHY 104

LIST OF FIGURES AND TABLES

Figure 2.1: A vertex-colored poset P and an edge-colored distributive lattice L 9

Figure 2.2: A product of chains 10

Figure 2.3: L^* and $(L^*)^\sigma$ for the lattice L from Figure 2.1 11

Figure 2.4: The disjoint sum of the β-components of the lattice L from Figure 2.1 12

Figure 2.5: The lattice L from Figure 2.1 is rank symmetric and rank unimodal 13

Figure 2.6: The lattice L from Figure 2.1 has a symmetric chain decomposition 14

Figure 2.7: The lattice L from Figure 2.1 recognized as $\mathrm{J}_{color}(P)$ 17

Figure 2.8: An illustration of principles from Proposition 2.4 19

Figure 2.9: Weights of elements of the lattice L from Figure 2.1 23

Figure 2.10: Connected Dynkin diagrams of finite type 26

Figure 2.11: Action of the permutation σ_0 when σ_0 is not the identity 36

Figure 2.12: $L^\triangle$ for the edge-colored lattice L from Figure 2.1 37

Figure 2.13: The numbers game for the Dynkin diagram C_2 46

Figure 2.14: A game sequence played on $\mathsf{G}_2^{\mathsf{T}}$ 50

Figure 2.15: A game sequence played on G_2 53

Figure 2.16: $\mathcal{M}(\omega_1) = \Pi(\omega_1)$ is an SDL for the fundamental G_2-character χ_{ω_1} 54

Figure 2.17: An SDL for the fundamental G_2-character χ_{ω_2} 56

Figure 2.18: Another SDL for the fundamental G_2-character χ_{ω_2} 57

Figure 2.19: A poset P for which $\mathrm{J}_{color}(P)$ is an SDL for an irreducible G_2-character . . . 58

Figure 2.20: A poset Q for which $\mathrm{J}_{color}(Q)$ is an SDL for an irreducible G_2-character . . . 59

Figure 5.1: A maximal and minimal splitting poset for χ_{ω_4} . . . 78

Figure 5.2: Posets of irreducibles for the two SDL's for χ_{ω_4} . . . 80

Figure 5.3: The two SDL's for χ_{ω_4} . . . 81

Figure 5.4: Poset of irreducibles for the SDL for $\chi_{2\omega_4}$ . . . 84

Figure 5.5: Minimal splitting poset for χ_{ω_1} . . . 86

Figure 5.6: Maximal splitting poset for χ_{ω_1} . . . 87

Figure 5.7: Posets of irreducibles for the two SDL's for χ_{ω_1} . . . 88

Figure 5.8: An SDL for χ_{ω_1} . . . 89

Figure 5.9: The other SDL for χ_{ω_1} . . . 90

Figure 5.10: Poset of irreducibles for the SDL for $\chi_{2\omega_1}$ . . . 92

Table 5.1: Weights of vertices of Figure 5.11 . . . 96

Figure 5.11: Necessary subgraph of any SDL for χ_{ω_3} . . . 97

Figure 5.12: Necessary subgraph of any SDL for χ_{ω_2} . . . 99

Table 5.2: Weights of vertices of Figure 5.12 . . . 100

CHAPTER 1: OVERVIEW

This thesis is part of a program whose goal is to produce interesting combinatorial models for Weyl characters. Weyl characters can be thought of as multivariate Laurent polynomials with nonnegative integer coefficients and which are symmetric with respect to the action of certain distinctive finite symmetry groups. These polynomials have many combinatorial and representation theoretic manifestations (as the basis of Schur functions for the ring of symmetric polynomials, as invariants for Lie algebra representations, in harmonic analysis on a compact Lie group G as objects related to orthonormal bases for the Hilbert space of class functions on G), and so it is desirable to find simple models for these fundamental objects. In pursuing such models for the 'F_4' class of Weyl characters, this thesis connects to work of Proctor ([Pro1], [Pro2], [Pro3], [Pro4]), Donnelly ([Don2], [Don3], [Don4], [Don5]), and other researchers ([Alv], [ADLP], [ADLMPPW], [DLP1], [DLP2], [DW], [KN], [LS], [LP], [Lit], [Mc], [Stem3], [Wil1], [Wil2], [Wil3], etc). For discussion of background and motivation, this thesis borrows extensively from the monograph [Don9].

The models we are most interested in are finite partially ordered sets called 'splitting posets'. Distributive lattices are partially ordered sets with certain nice properties, so for this reason a distributive lattice model for a Weyl character will sometimes be referred to by the special name 'splitting distributive lattice', or SDL. There are combinatorial and algebraic motivations for seeking such models.

First, splitting posets and distributive lattices have many salient combinatorial features. Such posets and distributive lattices have nice quotient-of-products expressions for their rank generating functions, cf. Theorem 2.30 below. Many classical enumerations, such as the binomial theorem, q-binomial coefficients, and q-Catalan numbers arise in this way. Splitting posets have been used to obtain solutions to purely combinatorial problems, see for example [Pro2]. Furthermore, the problem of finding splitting posets and distributive lattices is a refinement of a noted problem posed by Stanley (Problem 3 from [Sta1]), which we rephrase as: Which distributive lattices (or posets) naturally yield Weyl characters as 'weight generating functions'? Distributive lattice answers are of particular interest because of the compression of information afforded by their 'posets of irreducibles'.

Second, it is sometimes possible to use certain splitting posets to obtain explicit realizations of representations of semisimple Lie algebras. Obtaining such explicit constructions is a fundamental problem in representation theory. Further, splitting posets that realize semisimple Lie algebra representations (henceforth, 'supporting graphs') have additional combinatorial structure. In particular, connected supporting graphs possess the 'Sperner property', an extremal property of some combinatorial interest, see for example [Pro2], [Eng], [Don2], [DLP1]. Sometimes the combinatorics of a supporting graph is sufficient to uniquely specify or otherwise characterize a particular representation construction (cf. the 'solitary' and 'edge-minimal' properties introduced in [Don4]). Besides helping identify naturally occurring representation constructions, such combinatorial properties can have advantages for relating different constructions, see for example [Don4], [DLP1].

Of particular interest are the 'irreducible' Weyl characters. All other Weyl characters are nonnegative integer linear combinations of irreducible Weyl characters. The groups with respect to which these characters exhibit special symmetry are the finite Weyl groups. To relate the Weyl group and the polynomials requires a certain geometric representation of the group. For the 'irreducible' finite Weyl groups – those which cannot be expressed as the Cartesian product of two smaller Weyl groups – such geometric representations are classified by Dynkin diagrams into seven families (type A, B, C, D, E, F, and G), see Figure 2.10. (A subscript in that figure indicates the 'rank' of the group, a quantity which coincides with the number of generators for the group and the dimension of the representing space for the relevant geometric representation. For example, F_4 has rank four.)

Kashiwara and Stembridge have studied certain special splitting posets for Weyl characters. For the discussion of this paragraph, we consider an arbitrarily fixed irreducible Weyl character. For this Weyl character, Kashiwara's 'crystal graph', introduced in [Kash1] and [Kash2], is one particular connected splitting poset. An 'admissible system' for this Weyl character is a connected splitting poset characterized axiomatically in [Stem3]. Kashiwara's crystal graph satisfies these axioms and hence is an admissible system. Donnelly has observed that Stembridge's admissible systems are minimal in the following sense: no other splitting poset for the same Weyl character has fewer edges. In the other direction, among all splitting posets corresponding to this Weyl character, there is one that has the maximum number of edges possible. Call this the 'maximal splitting poset'. It is connected. (In fact, any such maximal splitting poset is a supporting graph, but admissible systems are rarely supporting graphs, cf. [Don4].) So for a given irreducible Weyl character, Stembridge's admissible systems and the maximal splitting poset can be viewed as extremal answers to the existence question: Does an irreducible Weyl character possess a connected splitting poset?

Less is known about splitting distributive lattices. In this paragraph we survey the known SDL's for irreducible Weyl characters with symmetry from irreducible Weyl groups. The Gel'fand-Tsetlin lattices are SDL's for the irreducible A_n-characters, cf. [GT], [Pro4], [Don4]. The symplectic lattices of [Don2], [Don3] are SDL's for the 'fundamental' C_n-characters. SDL's for the fundamental B_n-characters are described in [Don1]. SDL's for the 'one-rowed' irreducible B_n- and G_2-characters are produced in [DLP1]. SDL's for the one-rowed irreducible C_n-characters are easily obtained, see e.g. [ADLP]. A consequence of [Don5] is that there are exactly n splitting 'modular' lattices (see §2.4 of Chapter 2 below for definitions of lattice properties) for the 'adjoint' character for a rank n irreducible Weyl group. Of these splitting modular lattices in the A_n, B_n, C_n, F_4, and G_2 cases, precisely two are distributive. All of the SDL's mentioned so far in this paragraph are supporting graphs. SDL's for all irreducible characters for the rank two cases (A_2, C_2, and G_2) are uniformly constructed in [ADLMPPW]. The main objective of [DW] is to produce SDL's for all irreducible Weyl characters corresponding to nonnegative integer linear combinations of 'adjacency-free fundamental weights'.* This encompasses the rank two work of [ADLMPPW]. Not all of the SDL's of [ADLMPPW] are supporting graphs, as shown in [ADLP]. Some irreducible Weyl characters are known to have no SDL's, e.g. the adjoint character for D_4. No E_8-character is known to have a splitting distributive lattice.

With this context in mind, we turn our attention to Weyl characters of type F_4. We view the Weyl group for F_4 as a finite group generated by reflections in a four-dimensional Euclidean space. We note in passing that it is the 1152 element symmetry group of the famous

*The adjacency-free fundamental weights are defined in [Don6] and include the 'minuscule' fundamental weights. SDL's for the corresponding minuscule Weyl characters were produced in [Pro3].

24-cell, one of the six regular four-dimensional convex polytopes, although this realization will not play any role in the development that follows. As will be seen in Chapter 4 the two smallest irreducible F_4-characters are Laurent polynomials in four variables and have 26 and 52 terms (counting multiplicities). We refer to the numbers 26 and 52 as the 'dimensions' of the corresponding F_4-characters since the corresponding irreducible representations of the simple Lie algebra of type F_4 have these dimensions. Two SDL's (in fact, supporting graphs) were identified in [Don5] for each of these characters. Prior to this thesis, these were the only known SDL's for irreducible F_4-characters. Our goal is to find SDL's (or say why none exist) for the next smallest F_4-characters. Our focus will be on F_4-characters with dimensions 273, 324, 1053, and 1274.

We have both positive and negative results to report. In the 324 and one of the 1053 dimensional cases we produce new splitting distributive lattices. At this time these are the only known splitting distributive lattices for these F_4-Weyl characters. Finding these SDL's was a taxing needle-in-a-haystack search. Our proof that these distributive lattices do indeed model the appropriate Weyl characters uses iterative algorithms implemented in a computer algebra system to perform the necessary computations. While it is possible for these computations to be performed by hand, such labor would require many human-hours to complete. It is not clear at this time whether a shorter proof is possible. In Chapter 3 we introduce an algorithm which produces a certain 'edge-colored' distributive lattice. We will refer to this object as the 'distributive core'. While more general statements concerning the nature of the distributive core are not available at this time, computational evidence suggests that the distributive core might play a fundamental role in the study of SDL's.

In particular, we are able to apply this concept in the special case of the 273 and 1274 dimensional fundamental F_4-characters to show that these have no SDL's.

The thesis is organized as follows. In Chapter 2, we excerpt from the monograph [Don9] to provide the necessary combinatorial and algebraic background for our work here. In Chapter 3 we introduce the distributive core. Beginning in Chapter 4 we consider the case of F_4, offering a case-by-case analysis of the small dimension irreducible F_4-characters and presenting our new results for these characters.

CHAPTER 2: COMBINATORIAL AND ALGEBRAIC SETTING FOR OUR MAIN RESULTS

While the main focus of this thesis is combinatorial/algebraic structures of type F_4, our results take place within a broader context. The purpose of this chapter is provide a reasonably self-contained description of that context, giving the relevant definitions and key results. The reader might choose to browse this chapter at the outset, and then consult as necessary from Chapter 3. The reader might also find the examples and figures of this chapter to be useful illustrations of many of the main background concepts. In the combinatorics and Weyl groups discussions below, the reader should be alert to some important finiteness assumptions made early on in each part. These two parts are excerpts from the monograph [Don9]. Proofs are omitted in order to streamline the presentation.

Part 1: Some combinatorial preliminaries

Some of the definitions, notational conventions, and results of this part borrow from [Don4], [DLP1], [DLP2], [ADLP], [ADLMPPW], and [Sta2]. We use "R" (and when necessary, "Q") as a generic name for most of the combinatorial objects we define here ("edge-colored directed graph," "vertex-colored directed graph," "ranked poset"). The letter "P" is reserved for posets (and "vertex-colored" posets) that will be viewed as posets of irreducibles for distributive lattices; we reserve use of the letter "L" for reference to lattices and "edge-colored" lattices.

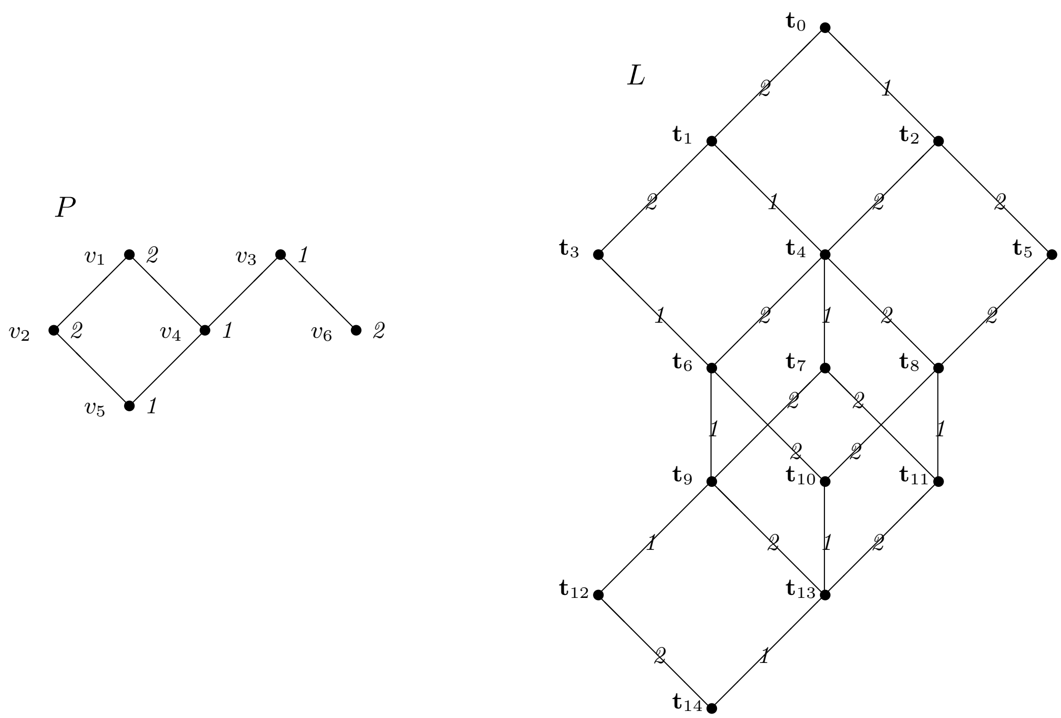
§2.1 Vertex- and edge-colored directed graphs. Let I be any set. An *edge-colored directed graph with edges colored by the set* I is a directed graph R with vertex set $\mathcal{V}(R)$ and directed-edge set $\mathcal{E}(R)$ together with a function $\mathbf{edgecolor}_R : \mathcal{E}(R) \longrightarrow I$ assigning to each edge of R an element ("color") from the set I. If an edge $\mathbf{s} \rightarrow \mathbf{t}$ in R is assigned color $i \in I$, we write $\mathbf{s} \xrightarrow{i} \mathbf{t}$. For $i \in I$, we let $\mathcal{E}_i(R)$ denote the set of edges in R of color i, so $\mathcal{E}_i(R) = \mathbf{edgecolor}_R^{-1}(i)$. If J is a subset of I, remove all edges from R whose colors are not in J; connected components of the resulting edge-colored directed graph are called *J-components* of R. For any $\mathbf{t}$ in R and any $J \subset I$, we let $\mathbf{comp}_J(\mathbf{t})$ denote the J-component of R containing $\mathbf{t}$. The *dual* R^* is the edge-colored directed graph whose vertex set $\mathcal{V}(R^*)$ is the set of symbols $\{\mathbf{t}^*\}_{\mathbf{t} \in R}$ together with colored edges $\mathcal{E}_i(R^*) := \{\mathbf{t}^* \xrightarrow{i} \mathbf{s}^* \,|\, \mathbf{s} \xrightarrow{i} \mathbf{t} \in \mathcal{E}_i(R)\}$ for each $i \in I$. Let Q be another edge-colored directed graph with edge colors from I. If R and Q have disjoint vertex sets, then the *disjoint sum* $R \oplus Q$ is the edge-colored directed graph whose vertex set is the disjoint union $\mathcal{V}(R) \cup \mathcal{V}(Q)$ and whose colored edges are $\mathcal{E}_i(R) \cup \mathcal{E}_i(Q)$ for each $i \in I$. If $\mathcal{V}(Q) \subseteq \mathcal{V}(R)$ and $\mathcal{E}_i(Q) \subseteq \mathcal{E}_i(R)$ for each $i \in I$, then Q is an *edge-colored subgraph* of R. Let $R \times Q$ denote the edge-colored directed graph whose vertex set is the Cartesian product $\{(\mathbf{s}, \mathbf{t}) | \mathbf{s} \in R, \mathbf{t} \in Q\}$ and with colored edges $(\mathbf{s}_1, \mathbf{t}_1) \xrightarrow{i} (\mathbf{s}_2, \mathbf{t}_2)$ if and only if $\mathbf{s}_1 = \mathbf{s}_2$ in R with $\mathbf{t}_1 \xrightarrow{i} \mathbf{t}_2$ in Q or $\mathbf{s}_1 \xrightarrow{i} \mathbf{s}_2$ in R with $\mathbf{t}_1 = \mathbf{t}_2$ in Q. Two edge-colored directed graphs are *isomorphic* if there is a bijection between their vertex sets that preserves edges and edge colors. If R is an edge-colored directed graph with edges colored by the set I, and if $\sigma : I \longrightarrow I'$ is a mapping of sets, then we let R^σ be the edge-colored directed graph with edge color function $\mathbf{edgecolor}_{R^\sigma} := \sigma \circ \mathbf{edgecolor}_R$. We call R^σ a *recoloring* of R. Observe that $(R^*)^\sigma \cong (R^\sigma)^*$. We similarly define a *vertex-colored directed graph* with a function $\mathbf{vertexcolor}_R : \mathcal{V}(R) \longrightarrow I$ that assigns colors to the vertices of R.

Figure 2.1: A vertex-colored poset P and an edge-colored distributive lattice L.

(The set of vertex colors for P and the set of edge colors for L are $\{\mathit{1},\mathit{2}\}$.

Elements of P are denoted v_i and elements of L are denoted $\mathbf{t}_i$.

Edges in P and L are directed "up".)

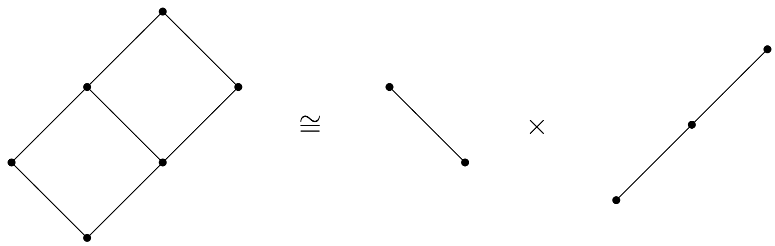


In this context, we speak of the *dual vertex-colored directed graph* R^*, the *disjoint sum* of two vertex-colored directed graphs with disjoint vertex sets, *isomorphism* of vertex-colored directed graphs, *recoloring*, etc. See Figures 2.1, 2.2, 2.3, and 2.4 for examples.

§2.2 Finiteness hypothesis. In this thesis, all directed graphs, including all partially ordered sets (discussed in the next subsection) will be assumed to be finite.

§2.3 Posets. A *partially ordered set* ('poset') is a set R together with a relation $\leq_R$ that is reflexive ($\mathbf{s} \leq_R \mathbf{s}$ for all $\mathbf{s} \in R$), transitive ($\mathbf{r} \leq_R \mathbf{s}$ and $\mathbf{s} \leq_R \mathbf{t} \Rightarrow \mathbf{r} \leq_R \mathbf{t}$ for all $\mathbf{r}, \mathbf{s}, \mathbf{t} \in R$), and antisymmetric ($\mathbf{s} \leq_R \mathbf{t}$ and $\mathbf{t} \leq_R \mathbf{s} \Rightarrow \mathbf{s} = \mathbf{t}$ for all $\mathbf{s}, \mathbf{t} \in R$). In this thesis, we identify a poset $(R, \leq_R)$ with its Hasse diagram ([Sta2] p. 98): For elements $\mathbf{s}$ and $\mathbf{t}$ of a

Figure 2.2: A product of chains.

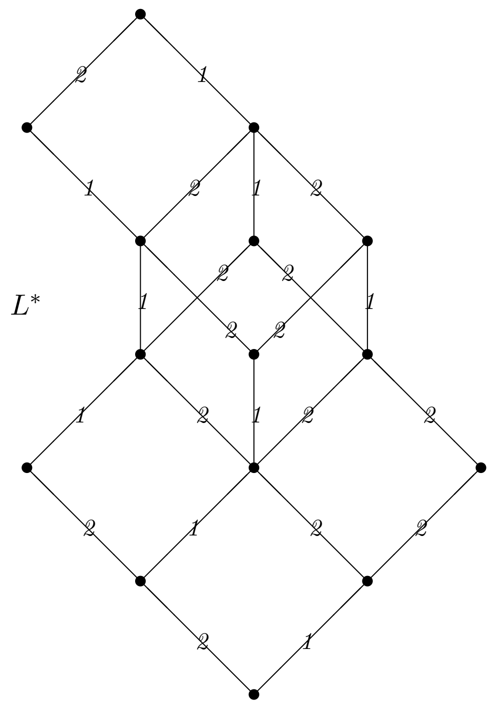


poset R, there is a directed edge $\mathbf{s} \to \mathbf{t}$ in the Hasse diagram if and only if $\mathbf{s} < \mathbf{t}$ and there is no $\mathbf{x}$ in R such that $\mathbf{s} < \mathbf{x} < \mathbf{t}$, i.e. $\mathbf{t}$ *covers* $\mathbf{s}$. Thus, terminology that applies to directed graphs (*connected*, *edge-colored*, *dual*, *vertex-colored*, etc) will also apply to posets. When we depict the Hasse diagram for a poset, its edges are directed 'up'. In an edge-colored poset R, we say the vertex $\mathbf{s}$ and the edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ are *below* $\mathbf{t}$, and the vertex $\mathbf{t}$ and the edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ are *above* $\mathbf{s}$. The vertex $\mathbf{s}$ is a *descendant* of $\mathbf{t}$, and $\mathbf{t}$ is an *ancestor* of $\mathbf{s}$. The edge-colored and vertex-colored directed graphs studied in this thesis will turn out to be posets. Given a subset Q of the elements of a poset R, let Q inherit the partial ordering of R; call Q a *subposet in the induced order*. Suppose $Q \subseteq R$ for another poset $(Q, \leq_Q)$, and suppose that $\mathbf{s} \leq_Q \mathbf{t} \Rightarrow \mathbf{s} \leq_R \mathbf{t}$ for all $\mathbf{s}, \mathbf{t} \in Q$. Then Q is a *weak subposet* of R. The terminology "weak subposet" applies in the case that Q and R are vertex-colored (resp. edge-colored) if the colors of vertices (resp. edges) from Q are the same as their colors when viewed as vertices (resp. edges) of R. An *antichain* in R is a subset whose elements are pairwise incomparable with respect to the partial order. A *chain* in R is a subset whose elements are pairwise comparable.

For a directed graph R, a *rank function* is a surjective function $\rho : R \longrightarrow \{0, \ldots, l\}$ (where $l \geq 0$) with the property that if $\mathbf{s} \to \mathbf{t}$ in R, then $\rho(\mathbf{s}) + 1 = \rho(\mathbf{t})$; if such a rank

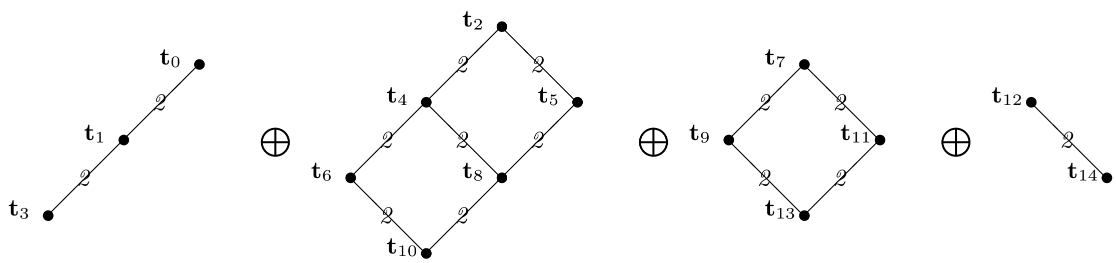
Figure 2.3: L^* and $(L^*)^\sigma$ for the lattice L from Figure 2.1.

(Here $\sigma(1) = \alpha$ and $\sigma(2) = \beta$.)



function exists then R is the Hasse diagram for a poset — a *ranked* poset. We call l the *length* of R with respect to ρ, and the set $\rho^{-1}(i)$ is the *ith rank* of R. The *rank generating function* $RGF(R, q)$ for such a ranked poset R is the polynomial $\sum_{i=0}^{l} |\rho^{-1}(i)| q^i$ in the variable q. Given another ranked poset Q, a simple counting argument can be used to show that $RGF(Q \times R, q) = RGF(Q, q) \cdot RGF(R, q)$. A ranked poset that is connected has a unique rank function. A ranked poset R with rank function ρ and length l is *rank symmetric* if $|\rho^{-1}(i)| = |\rho^{-1}(l - i)|$ for $0 \leq i \leq l$. It is *rank unimodal* if there is an m such that $|\rho^{-1}(0)| \leq |\rho^{-1}(1)| \leq \cdots \leq |\rho^{-1}(m)| \geq |\rho^{-1}(m + 1)| \geq \cdots \geq |\rho^{-1}(l)|$. It is *strongly Sperner* if for every $k \geq 1$, the largest union of k antichains is no larger than the largest union of k ranks. It has a *symmetric chain decomposition* if there exist chains $R_1, \ldots, R_k$ in R such

Figure 2.4: The disjoint sum of the *2*-components
of the edge-colored lattice L from Figure 2.1.

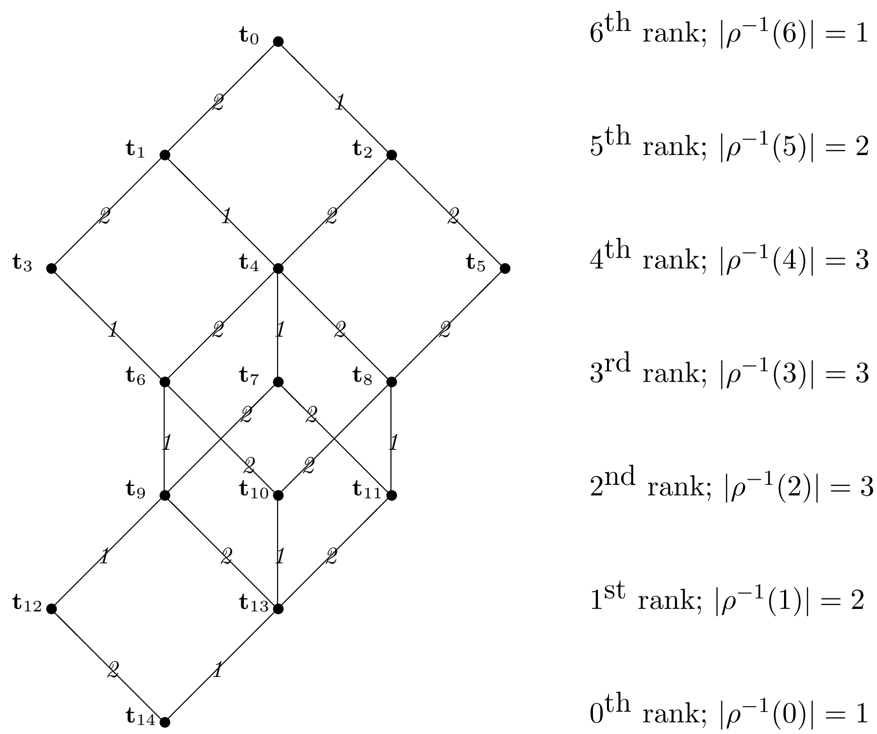


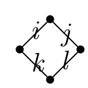
that (1) as a set $R = R_1 \cup \cdots \cup R_k$ (disjoint union), and (2) for $1 \leq i \leq k$, $\rho(\mathbf{y}_i) + \rho(\mathbf{x}_i) = l$ and $\rho(\mathbf{y}_i) - \rho(\mathbf{x}_i) = l_i$, where $\mathbf{x}_i$ and $\mathbf{y}_i$ are respectively the minimal and maximal elements of the chain R_i, and l_i is the length of the chain R_i. See Figures 2.5 and 2.6. If R has a symmetric chain decomposition, then one can see that R is rank symmetric, rank unimodal, and strongly Sperner; however, the converse does not hold. In an edge-colored ranked poset R, $\mathbf{comp}_i(\mathbf{t})$ will be a ranked poset for each $\mathbf{t} \in R$ and $i \in I$. We let $l_i(\mathbf{t})$ denote the length of $\mathbf{comp}_i(\mathbf{t})$, and we let $\rho_i(\mathbf{t})$ denote the rank of $\mathbf{t}$ within this component. We define the *depth* of $\mathbf{t}$ in its i-component to be $\delta_i(\mathbf{t}) := l_i(\mathbf{t}) - \rho_i(\mathbf{t})$.

A *path from* $\mathbf{s}$ *to* $\mathbf{t}$ in a poset R is a sequence $(\mathbf{s}_0 = \mathbf{s}, \mathbf{s}_1, \ldots, \mathbf{s}_r = \mathbf{t})$ such that for $1 \leq p \leq r$ it is the case that either $\mathbf{s}_{p-1} \rightarrow \mathbf{s}_p$ or $\mathbf{s}_p \rightarrow \mathbf{s}_{p-1}$. We say this path has *length* r. In notating paths, we sometimes include the directed edges between sequence elements. The distance $\mathbf{dist}(\mathbf{s}, \mathbf{t})$ between $\mathbf{s}$ and $\mathbf{t}$ in a connected poset R is the minimum length achieved when all paths from $\mathbf{s}$ to $\mathbf{t}$ in R are considered. (For example, the distance from $\mathbf{t}_3$ to $\mathbf{t}_5$ in the lattice L from Figure 2.5 is $\mathbf{dist}(\mathbf{t}_3, \mathbf{t}_5) = 4$.) If R is a ranked poset and if $\mathbf{s} \leq \mathbf{t}$ in R, then $\mathbf{dist}(\mathbf{s}, \mathbf{t}) = \rho(\mathbf{t}) - \rho(\mathbf{s})$. We say a poset R *has no open vees* if (1) whenever $\mathbf{r} \rightarrow \mathbf{s}$ and $\mathbf{r} \rightarrow \mathbf{t}$ in R, then there exists a unique $\mathbf{u}$ in R such that $\mathbf{s} \rightarrow \mathbf{u}$ and $\mathbf{t} \rightarrow \mathbf{u}$, and (2) whenever $\mathbf{s} \rightarrow \mathbf{u}$ and $\mathbf{t} \rightarrow \mathbf{u}$ in R, then there exists a unique $\mathbf{r}$ in R such that $\mathbf{r} \rightarrow \mathbf{s}$ and

Figure 2.5: The lattice L from Figure 2.1 is rank symmetric and rank unimodal.

$$RGF(L,q) = 1 + 2q + 3q^2 + 3q^3 + 3q^4 + 2q^5 + q^6$$

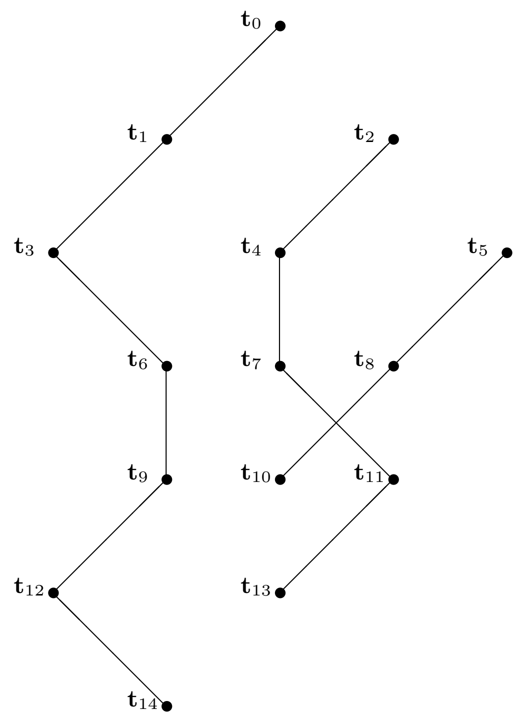


$\mathbf{r} \to \mathbf{t}$. An edge-colored poset R has the *diamond coloring property* if whenever 

is an edge-colored subgraph of the Hasse diagram for R, then $i = l$ and $j = k$.

Let R be an edge-colored ranked poset. For this paragraph, the elements of R will be denoted by $v_1, \ldots, v_n$, so $n = |R|$. For an integer $k \geq 0$, let $\bigwedge^k(R)$ denote the set of all k-element subsets of the vertex set of R. If $k > n$, then $\bigwedge^k(R) = \emptyset$. If $k = 0$ or $k = n$ then $\bigwedge^k(R)$ is a set with one element. For $\mathbf{s}, \mathbf{t} \in \bigwedge^k(R)$, write $\mathbf{s} \xrightarrow{i} \mathbf{t}$ if and only if $\mathbf{s}$ and $\mathbf{t}$ differ by exactly one element in the sense that $(\mathbf{s} - \mathbf{t}, \mathbf{t} - \mathbf{s}) = (\{v_p\}, \{v_q\})$ and $v_p \xrightarrow{i} v_q$ in R. Use the notation $\bigwedge^k(R)$ to refer to this edge-colored directed graph, which we call the kth *exterior power* of R. Similarly let $\mathbb{S}^k(R)$ denote the set of all k-element multisubsets of the vertex set of R and define colored, directed edges $\mathbf{s} \xrightarrow{i} \mathbf{t}$ between elements of $\mathbb{S}^k(R)$. Call

Figure 2.6: The lattice L from Figure 2.1 has a symmetric chain decomposition.



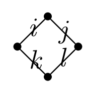
$\mathbb{S}^k(R)$ the kth *symmetric power* of R. It can be shown that $\bigwedge^k(R)$ and $\mathbb{S}^k(R)$ are ranked posets whose covering relations are the colored, directed edges prescribed in this paragraph.

§2.4 Lattices, modular lattices, and distributive lattices. A *lattice* is a poset for which any two elements $\mathbf{s}$ and $\mathbf{t}$ have a unique least upper bound $\mathbf{s} \vee \mathbf{t}$ (the *join* of $\mathbf{s}$ and $\mathbf{t}$) and a unique greatest lower bound $\mathbf{s} \wedge \mathbf{t}$ (the *meet* of $\mathbf{s}$ and $\mathbf{t}$). That is, whenever $\mathbf{s} \leq \mathbf{x}$ and $\mathbf{t} \leq \mathbf{x}$ then $(\mathbf{s} \vee \mathbf{t}) \leq \mathbf{x}$, and whenever $\mathbf{x} \leq \mathbf{s}$ and $\mathbf{x} \leq \mathbf{t}$ then $\mathbf{x} \leq (\mathbf{s} \wedge \mathbf{t})$. A lattice L is necessarily connected, and finiteness implies that there is a unique maximal element $\mathbf{max}(L)$ and a unique minimal element $\mathbf{min}(L)$. For any $\mathbf{r}, \mathbf{s}, \mathbf{t} \in L$, the facts that $\mathbf{r} \wedge (\mathbf{s} \wedge \mathbf{t}) = (\mathbf{r} \wedge \mathbf{s}) \wedge \mathbf{t}$ and $\mathbf{r} \vee (\mathbf{s} \vee \mathbf{t}) = (\mathbf{r} \vee \mathbf{s}) \vee \mathbf{t}$ follow easily from transitivity and antisymmetry of the partial order on L. That is, the meet and join operations are associative. Thus for a nonempty subset S of L, the meet $\wedge_{\mathbf{s} \in S}(\mathbf{s})$ and the join $\vee_{\mathbf{s} \in S}(\mathbf{s})$ are well-defined. We take $\wedge_{\mathbf{s} \in S}(\mathbf{s}) = \mathbf{min}(L)$ and $\vee_{\mathbf{s} \in S}(\mathbf{s}) = \mathbf{max}(L)$ if S is empty.

A lattice L is *modular* if it is ranked and $\rho(\mathbf{s}) + \rho(\mathbf{t}) = \rho(\mathbf{s} \vee \mathbf{t}) + \rho(\mathbf{s} \wedge \mathbf{t})$ for all $\mathbf{s}, \mathbf{t} \in L$. One can easily check that a modular lattice L is a ranked lattice with no open vees. If L is a lattice with no open vees, then one can see that L is ranked and for any $\mathbf{s}$ and $\mathbf{t}$, $\mathbf{dist}(\mathbf{s}, \mathbf{t}) = 2\rho(\mathbf{s} \vee \mathbf{t}) - \rho(\mathbf{s}) - \rho(\mathbf{t}) = \rho(\mathbf{s}) + \rho(\mathbf{t}) - 2\rho(\mathbf{s} \wedge \mathbf{t})$; hence L is a modular lattice (see [Sta2] Proposition 3.3.2). A lattice L is *distributive* if for any $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ in L it is the case that $\mathbf{r} \vee (\mathbf{s} \wedge \mathbf{t}) = (\mathbf{r} \vee \mathbf{s}) \wedge (\mathbf{r} \vee \mathbf{t})$ and $\mathbf{r} \wedge (\mathbf{s} \vee \mathbf{t}) = (\mathbf{r} \wedge \mathbf{s}) \vee (\mathbf{r} \wedge \mathbf{t})$. One can see that this distributive lattice L is a ranked lattice with no open vees. It follows that L is also a modular lattice. The following lemma shows how the modular lattice and diamond-coloring properties can interact.

Lemma 2.1 *Let L be a diamond-colored modular lattice. Suppose $\mathbf{s} \leq \mathbf{t}$. Suppose $\mathbf{s} = \mathbf{r}_0 \stackrel{i_1}{\to} \mathbf{r}_1 \stackrel{i_2}{\to} \mathbf{r}_2 \stackrel{i_3}{\to} \cdots \stackrel{i_{p-1}}{\to} \mathbf{r}_{p-1} \stackrel{i_p}{\to} \mathbf{r}_p = \mathbf{t}$ and $\mathbf{s} = \mathbf{r}'_0 \stackrel{j_1}{\to} \mathbf{r}'_1 \stackrel{j_2}{\to} \mathbf{r}'_2 \stackrel{j_3}{\to} \cdots \stackrel{j_{p-1}}{\to} \mathbf{r}'_{p-1} \stackrel{j_p}{\to} \mathbf{r}'_p = \mathbf{t}$ are two paths from $\mathbf{s}$ up to $\mathbf{t}$. Moreover, if $\mathbf{r}_1$ and $\mathbf{r}'_{p-1}$ are incomparable, then $i_1 = j_p$.*

The following discussion of edge-colored distributive lattices and certain related vertex-colored posets encompasses the classical uncolored situation (for example as in Ch. 3 of [Sta2]). These concepts have antecedents in work of Proctor and Stembridge (see e.g. [Pro3], [Pro4], [Stem2], [Stem1]), but there seems to be no standard treatment of these ideas. The main idea is that for a certain kind of edge-colored distributive lattice, all the information about the lattice can be compressed into a much smaller vertex-colored poset in such a way that the information can be fully recovered.

Edge-colored distributive lattices can be constructed as follows: Let P be a poset with vertices colored by a set I. An *order ideal* $\mathbf{x}$ from P is a vertex subset of P with the property that $u \in \mathbf{x}$ whenever $v \in \mathbf{x}$ and $u \leq v$ in P. For order ideals $\mathbf{x}$ and $\mathbf{y}$ from P, write $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x} \subseteq \mathbf{y}$ (subset containment). This is a partial ordering on the set L of order ideals from P. With respect to this partial ordering, L is a distributive lattice: $\mathbf{x} \vee \mathbf{y} = \mathbf{x} \cup \mathbf{y}$ (set union) and $\mathbf{x} \wedge \mathbf{y} = \mathbf{x} \cap \mathbf{y}$ (set intersection) for all $\mathbf{x}, \mathbf{y} \in L$. One can easily see that $\mathbf{x} \rightarrow \mathbf{y}$ in L if and only if $\mathbf{x} \subset \mathbf{y}$ (proper containment) and $\mathbf{y} \setminus \mathbf{x} = \{v\}$ for some maximal element v of $\mathbf{y}$ (thought of as a subposet of P in the induced order). In this case, we declare that $\mathbf{edgecolor}_L(\mathbf{x} \rightarrow \mathbf{y}) := \mathbf{vertexcolor}_P(v)$, making L an edge-colored distributive lattice. One can easily check that whenever 

is an edge-colored subgraph of the Hasse diagram for L, then $i = l$ and $j = k$. Therefore L has the diamond-coloring property. The diamond-colored distributive lattice just constructed is given special notation: we write $L := \mathrm{J}_{color}(P)$. See Figure 2.7. Note that if $P \cong Q$ as vertex-colored posets, then $\mathrm{J}_{color}(P) \cong \mathrm{J}_{color}(Q)$ as edge-colored posets. Moreover, L is ranked with rank function given by $\rho(\mathbf{t}) = |\mathbf{t}|$, the number of elements in the subset $\mathbf{t}$ from P. In particular, the length of L is $|P|$.

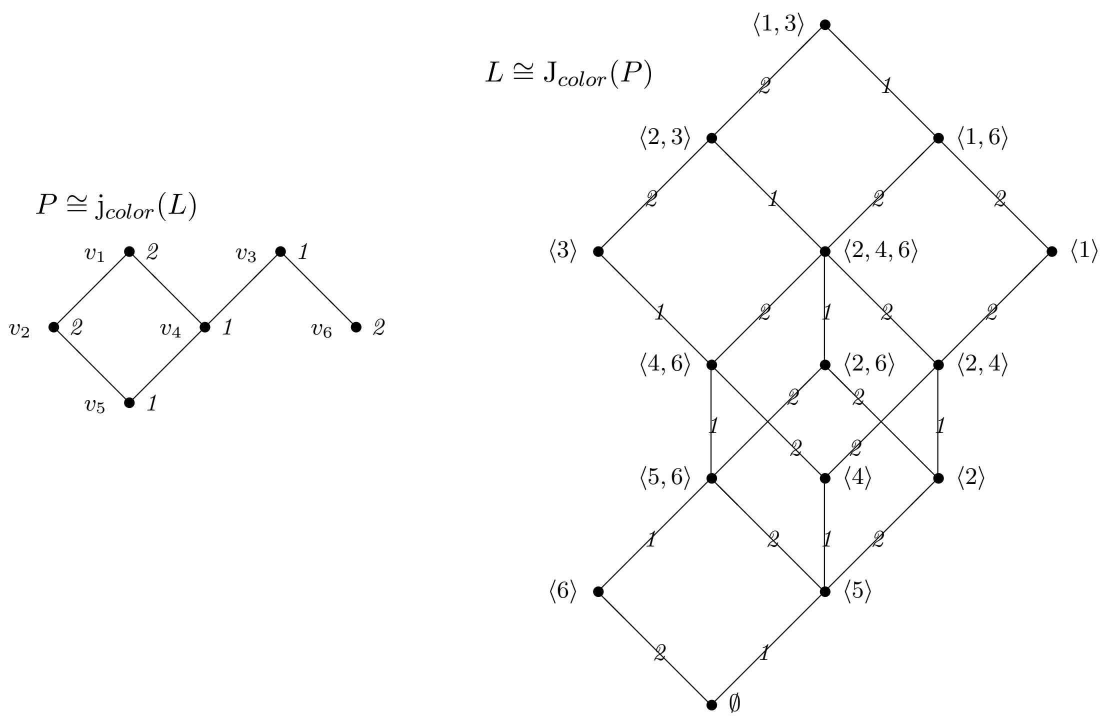
The process described in the previous paragraph can be reversed. Given a diamond-colored distributive lattice L, an element $\mathbf{x}$ is *join irreducible* if $\mathbf{x} \neq \mathbf{min}(L)$ and whenever $\mathbf{x} = \mathbf{y} \vee \mathbf{z}$ then $\mathbf{x} = \mathbf{y}$ or $\mathbf{x} = \mathbf{z}$. One can see that $\mathbf{x}$ is join irreducible if and only if $\mathbf{x}$ has precisely one descendant $\mathbf{x}'$ in L, i.e. $|\{\mathbf{x}' \,|\, \mathbf{x}' \rightarrow \mathbf{x}\}| = 1$. Let P be the set of all join irreducible elements of L with the induced partial ordering. Color the vertices of P by the rule: $\mathbf{vertexcolor}_P(\mathbf{x}) := \mathbf{edgecolor}_L(\mathbf{x}' \rightarrow \mathbf{x})$. We call P the *vertex-colored poset of join*

Figure 2.7: The lattice L from Figure 2.1 recognized as $\mathrm{J}_{color}(P)$.

(In this figure, each order ideal from P is identified by the indices of its maximal vertices.

For example, $\langle 2, 3\rangle$ in L denotes the order ideal $\{v_2, v_3, v_4, v_5, v_6\}$ in P.

A join irreducible in L is an order ideal $\langle k\rangle$ whose only maximal element is v_k.)



irreducibles and denote it by $P := \mathrm{j}_{color}(L)$. If $K \cong L$ is an isomorphism of diamond-colored lattices, then $\mathrm{j}_{color}(K) \cong \mathrm{j}_{color}(L)$ is an isomorphism of vertex-colored posets.

Example 2.2 Let P be an antichain whose elements all have the same color. Then the elements of $L := \mathrm{J}_{color}(P)$ are just the subsets of P. In particular, $|L| = 2^{|P|}$. Moreover, the rank $\rho_L(\mathbf{t})$ of a subset $\mathbf{t}$ from P is just $|\mathbf{t}|$. The join irreducible elements of L are just the singleton subsets of P. Covering relations in L are easy to describe: $\mathbf{s} \to \mathbf{t}$ if and only if $\mathbf{t}$ is formed from $\mathbf{s}$ by adding to $\mathbf{s}$ exactly one element from $P \setminus \mathbf{s}$. Any such lattice L is called a *Boolean lattice*. $\square$

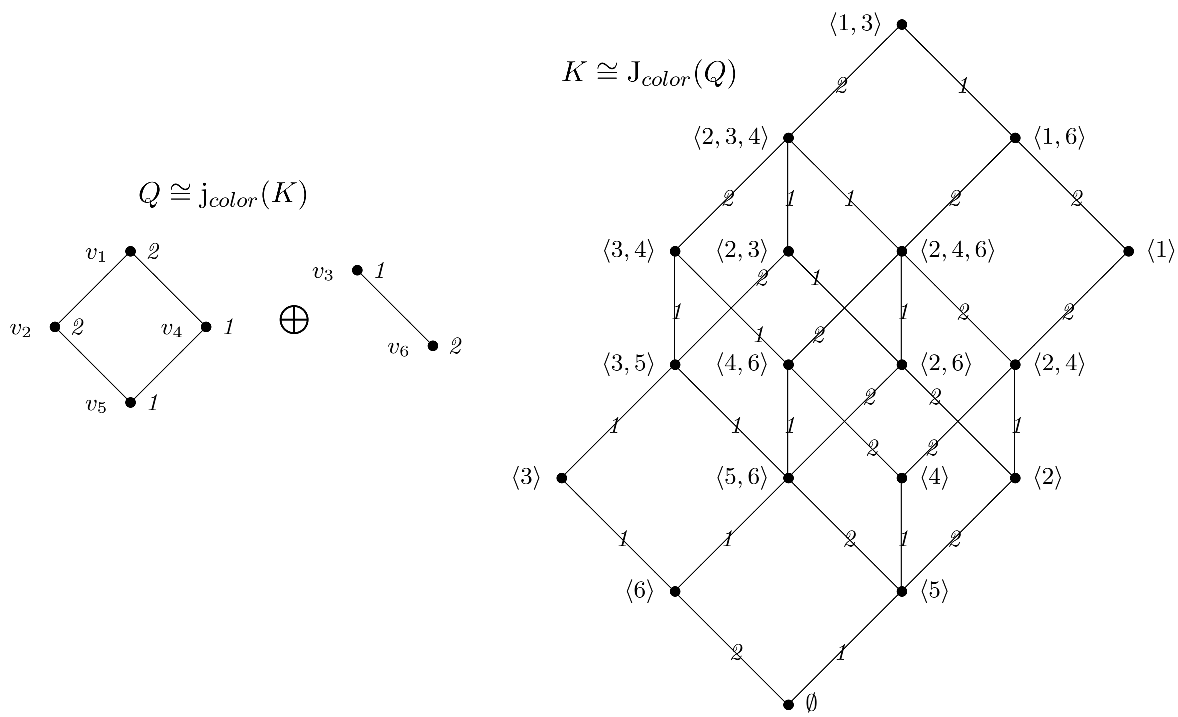
What follows is a dual to the above constructions of edge-colored distributive lattices. A *filter* from a vertex-colored poset P is a subset $\mathbf{x}$ with the property that if $u \in \mathbf{x}$ and $u \leq v$ in P then $v \in \mathbf{x}$. Note that for $\mathbf{x} \subseteq P$, $\mathbf{x}$ is a filter if and only if the set complement $P \setminus \mathbf{x}$ is an order ideal. Now partially order all filters from P by reverse containment: $\mathbf{x} \leq \mathbf{y}$ if and only if $\mathbf{x} \supseteq \mathbf{y}$ for filters $\mathbf{x}, \mathbf{y}$ from P. The resulting partially ordered set L is a distributive lattice. We color the edges of L as we did in the case of order ideals. The result is a diamond-colored distributive lattice which we denote by $L = \mathrm{M}_{color}(P)$. In the other direction, given a diamond-colored distributive lattice L, we say $\mathbf{x} \in L$ is *meet irreducible* if and only if $\mathbf{x} \neq \mathbf{max}(L)$ and whenever $\mathbf{x} = \mathbf{y} \wedge \mathbf{z}$ then $\mathbf{x} = \mathbf{y}$ or $\mathbf{x} = \mathbf{z}$. One can see that $\mathbf{x}$ is meet irreducible if and only if $\mathbf{x}$ has exactly one ancestor. Now consider the set P of meet irreducible elements in L with the order induced from L. Color the vertices of P in the same way we colored the vertices of the poset of join irreducibles. The vertex-colored poset P is the *poset of meet irreducibles* for L. Write $P = \mathrm{m}_{color}(L)$. We have $\mathrm{m}_{color}(P) \cong \mathrm{m}_{color}(Q)$ if $P \cong Q$ (an isomorphism of vertex-colored posets). We also have $\mathrm{M}_{color}(L) \cong \mathrm{M}_{color}(K)$ if $L \cong K$ (an isomorphism of diamond-colored distributive lattices).

The next result shows that the operations J_{color} (respectively, M_{color}) and j_{color} (respectively, m_{color}) are inverses in a certain sense. This is a straightforward generalization of the classical Fundamental Theorem of Finite Distributive Lattices (cf. Theorem 3.4.1 of [Sta2]). The latter result is formulated for uncolored posets and distributive lattices.

Theorem 2.3 (The Fundamental Theorem of Finite Diamond-colored Distributive Lattices) *(1) Let L be any diamond-colored distributive lattice. Then it is the case that $L \cong \mathrm{J}_{color}(\mathrm{j}_{color}(L)) \cong \mathrm{M}_{color}(\mathrm{m}_{color}(L))$. (2) Let P be any vertex-colored poset. Then $P \cong \mathrm{j}_{color}(\mathrm{J}_{color}(P)) \cong \mathrm{m}_{color}(\mathrm{M}_{color}(P))$.*

Figure 2.8: An illustration of the principles that $\mathrm{J}_{color}(P_1 \oplus P_2) \cong \mathrm{J}_{color}(P_1) \times \mathrm{J}_{color}(P_2)$ and $\mathrm{j}_{color}(L_1 \times L_2) \cong \mathrm{j}_{color}(L_1) \oplus \mathrm{j}_{color}(L_2)$, cf. Proposition 2.4.

(As in Figure 2.7, here each order ideal from Q is identified by the indices of its maximal vertices. A join irreducible in K is an order ideal $\langle k \rangle$ whose only maximal element is v_k.)



As a consequence, we note that a necessary and sufficient condition for an edge-colored distributive lattice L to be isomorphic (as an edge-colored poset) to $\mathrm{J}_{color}(P)$ or $\mathrm{M}_{color}(P)$ for some vertex-colored poset P is for L to have the diamond coloring property. We will often refer to P simply as the *poset of irreducibles.*

The details justifying the next result are routine.

Proposition 2.4 *Let P and Q be posets with vertices colored by a set I, and let K and L be diamond-colored distributive lattices with edges colored by I. In what follows, $*$, σ, $\oplus$, $\times$, and $\cong$ account for colors on vertices or edges as appropriate. (1) Then*

$\mathrm{J}_{color}(P^*) \cong (\mathrm{J}_{color}(P))^*$, $\mathrm{J}_{color}(P^\sigma) \cong (\mathrm{J}_{color}(P))^\sigma$ *(recoloring), and* $\mathrm{J}_{color}(P \oplus Q) \cong \mathrm{J}_{color}(P) \times \mathrm{J}_{color}(Q)$. *(2) Also,* $\mathrm{j}_{color}(L^*) \cong (\mathrm{j}_{color}(L))^*$, $\mathrm{j}_{color}(L^\sigma) \cong (\mathrm{j}_{color}(L))^\sigma$, *and* $\mathrm{j}_{color}(L \times K) \cong \mathrm{j}_{color}(L) \oplus \mathrm{j}_{color}(K)$. *(3) Further,* $\mathrm{M}_{color}(P^*) \cong (\mathrm{M}_{color}(P))^*$, $\mathrm{M}_{color}(P^\sigma) \cong (\mathrm{M}_{color}(P))^\sigma$, *and* $\mathrm{M}_{color}(P \oplus Q) \cong \mathrm{M}_{color}(P) \times \mathrm{M}_{color}(Q)$. *(4) In addition it is the case that* $\mathrm{m}_{color}(L^*) \cong (\mathrm{m}_{color}(L))^*$, $\mathrm{m}_{color}(L^\sigma) \cong (\mathrm{m}_{color}(L))^\sigma$, *and* $\mathrm{m}_{color}(L \times K) \cong \mathrm{m}_{color}(L) \oplus \mathrm{m}_{color}(K)$. *(5) If* $K \cong L$, *then* $\mathrm{j}_{color}(K) \cong \mathrm{m}_{color}(L)$. *If* $P \cong Q$, *then* $\mathrm{J}_{color}(P) \cong \mathrm{M}_{color}(Q)$.

§2.5 Sublattices. Let L be a lattice with partial ordering $\leq_L$ and meet and join operations $\wedge_L$ and $\vee_L$ respectively. Let K be a vertex subset of L. Suppose that K has a lattice partial ordering $\leq_K$ of its own with meet and join operations $\wedge_K$ and $\vee_K$ respectively. We say K is a *sublattice* of L if for all $\mathbf{x}$ and $\mathbf{y}$ in K we have $\mathbf{x} \wedge_K \mathbf{y} = \mathbf{x} \wedge_L \mathbf{y}$ and $\mathbf{x} \vee_K \mathbf{y} = \mathbf{x} \vee_L \mathbf{y}$. It is easy to see that if K is a sublattice of L then for all $\mathbf{x}$ and $\mathbf{y}$ in K we have $\mathbf{x} \leq_K \mathbf{y}$ if and only if $\mathbf{x} \leq_L \mathbf{y}$. That is, K is a weak subposet of L and a subposet in the induced order.

Lemma 2.5 *Suppose K is a sublattice of L. Suppose K and L are ranked with rank functions $\rho^{(K)}$ and $\rho^{(L)}$ respectively. Suppose K and L have the same length. Then $\rho^{(K)}(\mathbf{x}) = \rho^{(L)}(\mathbf{x})$ for all $\mathbf{x}$ in K, and moreover for all $\mathbf{x}$ and $\mathbf{y}$ in K we have $\mathbf{x} \to \mathbf{y}$ in K if and only if $\mathbf{x} \to \mathbf{y}$ in L.*

When K satisfies the hypotheses of Lemma 2.5, we say K is a *full length* sublattice of L. Suppose L is an edge-colored lattice. Suppose K is a sublattice of L such that $\mathbf{x} \to \mathbf{y}$ in L whenever $\mathbf{x} \to \mathbf{y}$ in K. If K is also edge-colored and if the colors on edges of K match the colors when we view these as edges in L, then we say K is an *edge-colored sublattice* of L. The previous lemma gives us one way to know whether the edges of a sublattice are also edges of the 'parent' lattice. We now turn our attention to the special case of diamond-colored distributive lattices.

Theorem 2.6 *(1) Let P and Q be vertex-colored posets with vertices colored by a set I. Suppose that for each $i \in I$, the vertices of color i in P coincide with the vertices of color i in Q (so in particular $P = Q$ as vertex sets). Further suppose that P is a weak subposet of Q. Let $K := \mathrm{J}_{color}(Q)$ and $L := \mathrm{J}_{color}(P)$. Then K is a full-length edge-colored sublattice of L. (2) Conversely, suppose L is a diamond-colored distributive lattice with edges colored by a set I. Suppose K is a full-length edge-colored sublattice of L (so K is necessarily a diamond-colored distributive lattice). Let $P := \mathrm{j}_{color}(L)$ and $Q := \mathrm{j}_{color}(K)$. Then $P \cong P'$ (an isomorphism of vertex-colored posets) where P' is weak-subposet of Q, $P' = Q$ as vertex sets, and the color of a vertex in P' is the same as its color when viewed as a vertex in Q.*

To set up our next result we require some further notation. For elements $\mathbf{s}, \mathbf{t}$ in any poset R, the *interval* $[\mathbf{s}, \mathbf{t}]$ is the set $\{\mathbf{x} \in R \,|\, \mathbf{s} \leq_R \mathbf{x} \leq_R \mathbf{t}\}$ with partial order induced by R. One can check that the Hasse diagram for $[\mathbf{s}, \mathbf{t}]$ is just the induced subgraph of R on the vertices of $[\mathbf{s}, \mathbf{t}]$. Then we can regard $[\mathbf{s}, \mathbf{t}]$ as an edge-colored subposet of R in the induced order, if R is edge-colored. In a diamond-colored modular lattice L, it is not hard to see that any interval $[\mathbf{s}, \mathbf{t}]$ is naturally an edge-colored sublattice of L. Our next result concerns the distributive lattice structure of certain intervals in diamond-colored distributive lattices.

Proposition 2.7 *Let L be a diamond-colored distributive lattice. Let $\mathbf{t} \in L$. Let D be a subset of the descendants of $\mathbf{t}$. For any $\mathbf{s} \in D$, let $\mathbf{vertexcolor}_D(\mathbf{s}) := \mathbf{edgecolor}_L(\mathbf{s} \rightarrow \mathbf{t})$. Let $\mathbf{r} := \wedge_{\mathbf{s}\,\in\,D}(\mathbf{s})$. Then $[\mathbf{x}, \mathbf{t}] \cong \mathrm{M}_{color}(D)$ and $D \subseteq [\mathbf{x}, \mathbf{t}]$ if and only if $\mathbf{x} = \mathbf{r}$. Similarly let*

A be a subset of the ancestors of $\mathbf{t}$. For any $\mathbf{s} \in A$, let $\mathbf{vertexcolor}_A(\mathbf{s}) := \mathbf{edgecolor}_L(\mathbf{t} \rightarrow \mathbf{s})$*. Let $\mathbf{u} := \vee_{\mathbf{s} \in A}(\mathbf{s})$. Then $[\mathbf{t}, \mathbf{x}] \cong \mathrm{J}_{color}(A)$ and $A \subseteq [\mathbf{t}, \mathbf{x}]$ if and only if $\mathbf{x} = \mathbf{u}$.*

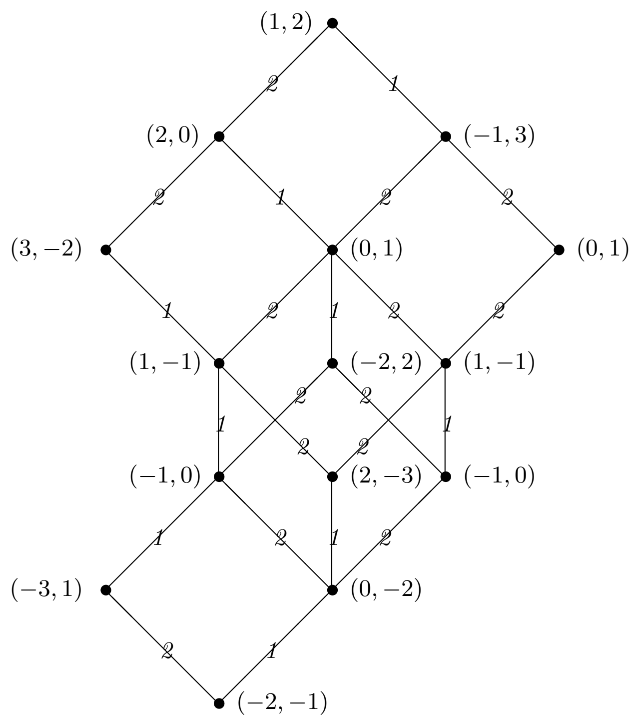
Note that any two descendants (respectively ancestors) of a given element of a poset are incomparable. It follows then that the intervals $[\mathbf{r}, \mathbf{t}]$ and $[\mathbf{t}, \mathbf{u}]$ of Proposition 2.7 are Boolean lattices, cf. Example 2.2.

The next result concerns the structure of J-components of a diamond-colored modular lattice.

Proposition 2.8 *Let L be a diamond-colored modular lattice with edge colors from a set I. If $\mathbf{t} \in L$ and $J \subseteq I$, then $\mathbf{comp}_J(\mathbf{t})$ is the Hasse diagram for a diamond-colored modular lattice. Moreover, $\mathbf{comp}_J(\mathbf{t})$ is a sublattice of L, and a covering relation in $\mathbf{comp}_J(\mathbf{t})$ is also a covering relation in L. If L is a distributive lattice, then so is $\mathbf{comp}_J(\mathbf{t})$.*

§2.6 A first look at the M-structure property. Let R be a ranked poset whose Hasse diagram edges are colored with colors taken from a totally ordered set I_n of cardinality n. For $i \in I_n$ and $\mathbf{s}$ in R, set $m_i(\mathbf{s}) := \rho_i(\mathbf{s}) - \delta_i(\mathbf{s}) = 2\rho_i(\mathbf{s}) - l_i(\mathbf{s})$, where ρ_i, δ_i, and l_i are defined as in §2.3. Fix an n-dimensional real vector space V with basis $\{\omega_i\}_{i \in I_n}$. Define a mapping $wt_R : R \rightarrow V$ by the rule $wt_R(\mathbf{s}) := \sum_{i \in I_n} m_i(\mathbf{s})\omega_i$, and call this vector the *weight* of $\mathbf{s}$. Given a matrix $M = (M_{pq})_{p,q \in I_n}$, then for fixed $i \in I_n$ let $M^{(i)}$ be the "ith row" vector $\sum_{j \in I_n} M_{ij}\omega_j$. We say R has the *M-structure property* if $wt_R(\mathbf{s}) + M^{(i)} = wt_R(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{i} \mathbf{t}$ for some $i \in I_n$, that is, for all $j \in I_n$ we have $m_j(\mathbf{s}) + M_{ij} = m_j(\mathbf{t})$ if $\mathbf{s} \xrightarrow{i} \mathbf{t}$. We also say R is an *M-structured* poset. It can be easily shown that if the edge color function $\mathbf{edgecolor}_R : \mathcal{E}(R) \longrightarrow I_n$ is surjective, then the all of the M_{ij}'s are uniquely determined integers and that $M_{ii} = 2$ for all $i \in I_n$. One can check by hand that the

Figure 2.9: For each element $\mathbf{t}$ of the lattice L from Figure 2.1, we compute $wt_L(\mathbf{t}) = (m_1(\mathbf{t}), m_2(\mathbf{t}))$.



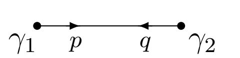
edge-colored distributive lattice of Figure 2.9 has the M-structure property for the matrix $M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The following result shows how the M-structure property interacts with some of our usual poset operations.

Proposition 2.9 *Let Q and R be ranked posets with edges colored by a set I_n. Let $M = (M_{ij})_{i,j \in I_n}$ be a real matrix. Suppose Q and R have the M-structure property. (1) Then so do $Q \oplus R$, $Q \times R$, and R^*. Let $J \subseteq I_n$, and let M' be the submatrix $(M_{ij})_{i,j \in J}$ of M. Then for each $\mathbf{t} \in R$, the J-component $\mathbf{comp}_J(\mathbf{t})$ is a ranked poset with edges colored by J and with the M'-structure property. (2) Suppose now that M is nonsingular. Then for any nonnegative integer k, $\bigwedge^k(R)$ and $\mathbb{S}^k(R)$ have the M-structure property. Moreover, if R is connected and $wt_R(\mathbf{s}) = wt_R(\mathbf{t})$, then $\rho(\mathbf{s}) = \rho(\mathbf{t})$.*

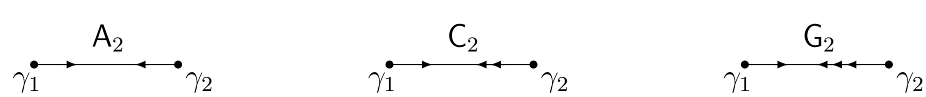
Part 2: Weyl groups and Weyl characters

Much of the discussion of Weyl groups and Weyl characters in the following subsections is borrowed from [Don6], [Don7], and [ADLMPPW] as well as standard treatments like [Hum1], [Hum2], [Bour], and [BB].

§2.7 GCM graphs and Dynkin diagrams. Following [Don7] we take as our starting point some given simple graph Γ on n nodes. In particular, Γ has no loops and no multiple edges. Nodes $\{\gamma_i\}_{i\in I_n}$ for Γ are indexed by elements of some fixed totally ordered set I_n of size n (usually $I_n = \{1 < 2 < \cdots < n\}$). For each pair of adjacent nodes γ_i and γ_j in Γ, choose two negative integers M_{ij} and M_{ji}. Extend this to an $n \times n$ matrix $M = (M_{ij})_{i,j\in I_n}$ where, in addition to the negative integers M_{ij} and M_{ji} on edges of Γ, we have $M_{ii} := 2$ for all $i \in I_n$ and $M_{ij} := 0$ if there is no edge in Γ between nodes γ_i and γ_j. We call the pair (Γ, M) a *GCM graph*, since M is a 'generalized Cartan matrix' as in [Kac] and [Kum]. Such matrices are the starting point for the study of Kac–Moody algebras. More importantly for us, these matrices also encode information about certain geometric representations of Weyl groups. Such representations provide a suitable environment for studying Weyl characters, which can be thought of as special multivariate Laurent polynomials which exhibit symmetry under the actions of the Weyl groups.

We say a GCM graph (Γ, M) is *connected* if Γ is. We depict a generic connected two-node GCM graph as 

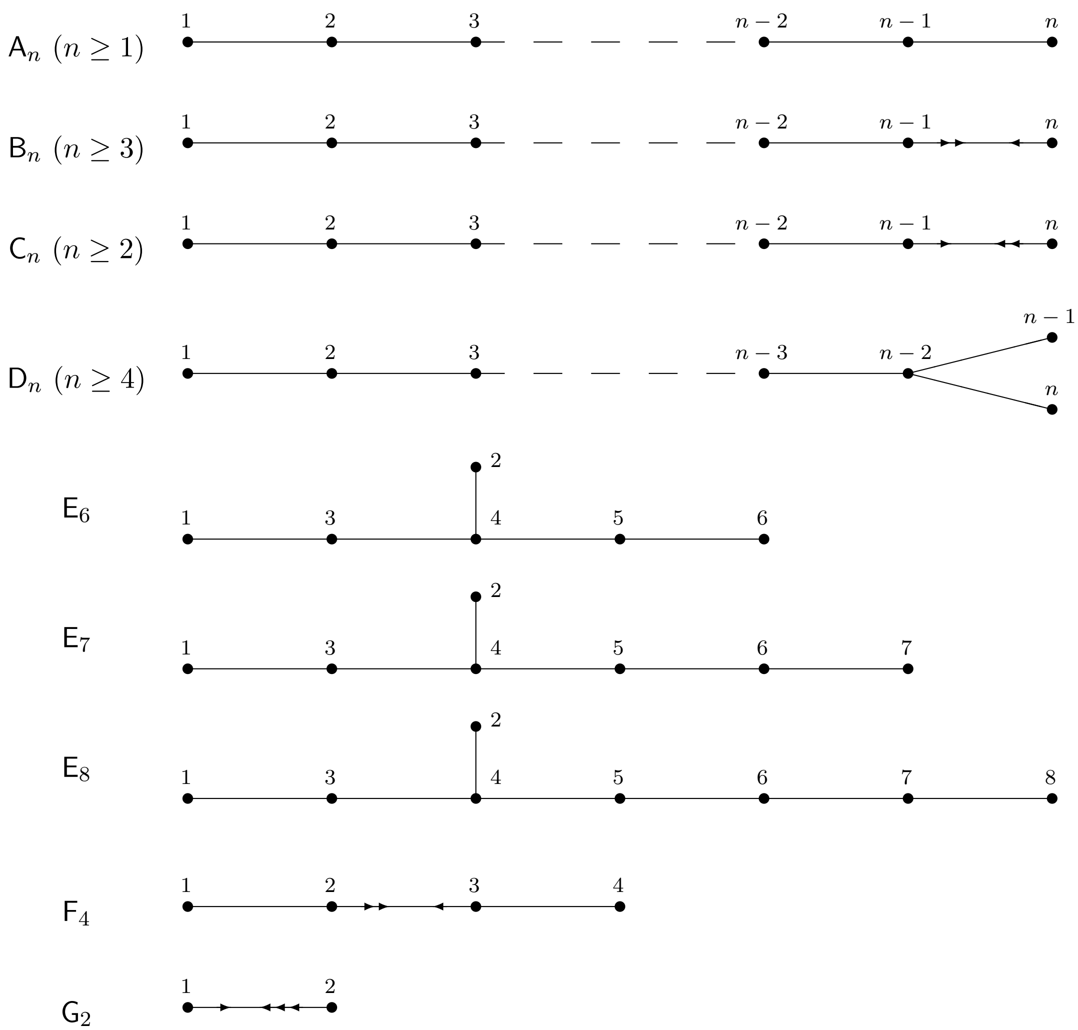
, where $p = -M_{12}$ and $q = -M_{21}$. We use special names and notation to refer to two-node GCM graphs which have $p = 1$ and $q = 1$, 2, or 3 respectively:



When $p = 1$ and $q = 1$ it is convenient to use the graph $\gamma_1 \bullet\!\!\frac{}{}\!\!\bullet \gamma_2$ to represent the GCM graph A_2. A GCM graph (Γ, M) is a *Dynkin diagram of finite type* (or *Dynkin diagram*, for short) if each 'connected component' of (Γ, M) (in the obvious sense, defined below) is one of the graphs of Figure 2.10; in this case the matrix M is called a *Cartan matrix*. We number the nodes of connected Dynkin diagrams of finite type as in §11.4 of [Hum1]. The special two-node GCM graphs A_2, C_2, and G_2 above are Dynkin diagrams with Cartan matrices $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, and $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

The following language concerning GCM graphs is sometimes useful. Given two GCM graphs $\mathfrak{g}_1 = (\Gamma_1, (A_{ij})_{i,j \in I_n})$ and $\mathfrak{g}_2 = (\Gamma_2, (B_{ij})_{i,j \in J_m})$, the *disjoint sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the GCM graph (Γ, M) with graph $\Gamma = \Gamma_1 \oplus \Gamma_2$ (a disjoint sum of undirected graphs in the obvious way, analogous to §2.1, and with nodes indexed by the disjoint union $I_n \dot{\cup} J_m$) and generalized Cartan matrix $M = \begin{pmatrix} P & O \\ O' & Q \end{pmatrix}$ (a block diagonal matrix in the obvious sense, where O and O' are a zero matrices of appropriate size). These GCM graphs are *isomorphic* if there is a bijection $\sigma : I_n \to J_m$ with respect to which $A_{ij} = B_{\sigma(i),\sigma(j)}$ for all $i, j \in I_n$. If I'_m is a subset of the index set I_n of a GCM graph (Γ, M), then let Γ' be the subgraph of Γ with nodes indexed I'_m and the induced set of edges, and let M' be the corresponding submatrix of the generalized Cartan matrix M; we call (Γ', M') a *GCM subgraph* of (Γ, M). (For example, in Figure 2.10 one can see that C_3 is a GCM subgraph of F_4.) The GCM subgraph (Γ', M') is a *connected component* if Γ' is a connected component of Γ. Given a one-to-one function $\sigma : I_n \to J_n$, obtain a graph Γ^σ by recoloring the nodes of the undirected graph Γ as in §2.1. Then the GCM graph $\mathfrak{g}^\sigma = (\Gamma^\sigma, M^\sigma)$ is the *re-coloring*

Figure 2.10: Connected Dynkin diagrams of finite type.



of the GCM graph $\mathfrak{g}$, where $(M^\sigma)_{\sigma(i),\sigma(j)} := M_{i,j}$ for all $i, j \in I_n$. We let $\mathfrak{g}^\mathsf{T} := (\Gamma, M^\mathsf{T})$, so that $(\mathfrak{g}^\mathsf{T})^\mathsf{T} = \mathfrak{g}$.

§2.8 Weyl groups and geometric representations. For the remainder of this chapter, let $\mathfrak{g} := (\Gamma, M)$ be a fixed GCM graph with index set I_n. The development in this subsection basically follows [BB] and [Don7]. For $i \neq j$ in I_n, declare

$$m_{ij} = \begin{cases} k_{ij} & \text{if } M_{ij}M_{ji} = 4\cos^2(\pi/k_{ij}) \text{ for some integer } k_{ij} \geq 2 \\ \infty & \text{if } M_{ij}M_{ji} \geq 4 \end{cases}$$

We have $m_{ij} = 2$ (respectively $3, 4, 6$) if $M_{ij}M_{ji} = 0$ (resp. $1, 2, 3$). Let $\mathcal{W} := \mathcal{W}_{\mathfrak{g}}$ be the group generated by $\{s_i\}_{i \in I_n}$ subject to relations $s_i^2 = \varepsilon$ for all $i \in I_n$ and $(s_i s_j)^{m_{ij}} = \varepsilon$ for all $i \neq j$ in I_n. (Conventionally, $m_{ij} = \infty$ means there is no relation between generators s_i and s_j.) Then $\mathcal{W}$ is called a *Weyl group*, and it is a special kind of Coxeter group.

Let V be a real vector space freely generated by vectors $\{\alpha_i\}_{i \in I_n}$. The α_i's are called *simple roots*. For each $i \in I_n$, define a linear transformation $S_i : V \to V$ by setting $S_i(\alpha_j) = \alpha_j - M_{ji}\alpha_i$ for each $j \in I_n$ and extending linearly.* The next result follows from Proposition 3.13 of [Kac] or Proposition 1.3.21 of [Kum] (see also §2 of [Don7]). Here $GL(V)$ is the group of invertible linear transformations on V and Id denotes the identity transformation on V.

Lemma 2.10 *For each $i \in I_n$, $S_i^2 = \mathrm{Id}$. In particular, $S_i \in GL(V)$. Now take $i \neq j$ in I_n. If m_{ij} is finite, then $(S_iS_j)^{m_{ij}} = \mathrm{Id}$. If $m_{ij} = \infty$, then the subgroup of $GL(V)$ generated by $\{S_i, S_j\}$ is infinite.*

The above lemma guarantees that the mapping $s_i \mapsto S_i$ extends uniquely to a group homomorphism $\phi : \mathcal{W} \to GL(V)$. Our next result, which is Theorem 4.2.7 of [BB], says that this mapping is injective. In the language of group representations we state this as:

Theorem 2.11 *The representation ϕ of $\mathcal{W}$ in the previous paragraph is faithful.*

§2.9 Finiteness hypothesis. Of interest to us are GCM graphs whose corresponding Weyl groups are finite. These have the following well-known classification (see e.g. [Hum1] or [Hum2]):

---

*This 'transpose' of the usual definition ($S_i(\alpha_j) = \alpha_j - M_{ij}\alpha_i$) facilitates connections with certain results such as the root system and weights results of Chapter III of [Hum1].

Theorem 2.12 *The Weyl group $\mathcal{W}$ is finite if and only if the connected components of $\mathfrak{g}$ are Dynkin diagrams of finite-type from Figure 2.10.*

Two of the most famous Dynkin diagram classification results come from Lie theory: the Dynkin diagrams of Figure 2.10 are in one-to-one correspondence with the finite-dimensional complex simple Lie algebras and the finite-dimensional irreducible Kac-Moody algebras. For examples of other Dynkin diagram classifications, see [HHSV], [Pro5], and [Pro6]. From here on, we restrict our attention to the finite cases unless stated otherwise. For connected Dynkin diagrams of finite type, we have the following important observation: one can verify case-by-case that the associated Cartan matrices are invertible.

§2.10 A Euclidean representation of the Weyl group. We would like to realize each transformation S_i as a reflection 'with respect to' α_i. Such a geometric realization of the Weyl group $\mathcal{W}$ will require an inner product $\langle\cdot,\cdot\rangle$ on V. The derivation of the inner product in this subsection is an interpretation of standard material. Assuming for the moment that such an inner product exists, we investigate in this paragraph its interactions with the Cartan matrix M. Relative to this inner product, the reflection $S : V \to V$ in the hyperplane orthogonal to some fixed nonzero α will act on vectors v in V by the rule $S(v) = v - 2\frac{\langle v,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$. Applied to the transformations S_i acting on vectors α_j, we determine that $M_{ji} = 2\frac{\langle \alpha_j,\alpha_i\rangle}{\langle \alpha_i,\alpha_i\rangle}$. Symmetry of the inner product now gives

$$M_{ji}\langle \alpha_i, \alpha_i\rangle = M_{ij}\langle \alpha_j, \alpha_j\rangle. \tag{1}$$

If $\mathfrak{g}$ is connected, fix the length of one of the end node simple roots. Then using the preceding relation, the remaining simple root lengths can be computed in terms of the fixed simple root length and entries from the Cartan matrix M. For A–D–E graphs, only one simple root length is possible. Inspection of the other connected Dynkin diagrams of finite type ($\mathsf{B}_n, \mathsf{C}_n, \mathsf{F}_4, \mathsf{G}_2$) shows that each has two root lengths. In the B–C–F cases, 'long' simple roots have squared length twice that of 'short' roots. For G_2, the long simple root α_2 has squared length three times that of the short simple root α_1. If $\mathfrak{g}$ is not connected, then we must choose a squared length for short simple roots in each connected component of $\mathfrak{g}$. With such a fixed choice of short simple root lengths for $\mathfrak{g}$, one can now determine that

$$\langle \alpha_j, \alpha_i \rangle = \frac{1}{2} \langle \alpha_i, \alpha_i \rangle M_{ji} \tag{2}$$

for all $i, j \in I_n$. So our hypothetical inner product is determined by the preceding relations (1) and (2) together with the choices for short simple root lengths for connected components of $\mathfrak{g}$. With this discussion in mind, now define a bilinear form B on V so that for each $i \in I_n$, $B(\alpha_i, \alpha_i)$ coincides with the choices for squared lengths of simple roots indicated above, and where $B(\alpha_i, \alpha_j) := \frac{1}{2} B(\alpha_j, \alpha_j) M_{ij}$ for all $i, j \in I_n$.

Theorem 2.13 *The bilinear form B defined above is symmetric and nondegenerate. Moreover, the Weyl group $\mathcal{W}$ preserves B in the sense that $B(w.v_1, w.v_2) = B(v_1, v_2)$ for all $w \in \mathcal{W}$ and $v_1, v_2 \in V$. Finally, relative to the form B each S_i is a reflection with respect to α_i: $S_i(v) = v - 2\frac{B(v,\alpha_i)}{B(\alpha_i,\alpha_i)}\alpha_i$ for all $v \in V$.*

It suffices to prove Theorem 2.13 for connected Dynkin diagrams. This can be done case by case. From here on, we use $\langle \cdot, \cdot \rangle$ to denote the inner product B of the preceding paragraph and theorem. Given $\langle \cdot, \cdot \rangle$, we call $\phi : \mathcal{W} \to GL(V, \langle \cdot, \cdot \rangle)$ a *Euclidean representation* of $\mathcal{W}$. Let $\mathcal{O}(V, \langle \cdot, \cdot \rangle)$ be the orthogonal group for the Euclidean space $(V, \langle \cdot, \cdot \rangle)$. A consequence of the preceding theorem is that $\phi(\mathcal{W}) \cong \mathcal{W}$ is actually a subgroup of $\mathcal{O}(V, \langle \cdot, \cdot \rangle)$. From here on, we consider ϕ to be a Euclidean representation for $\mathcal{W}_{\mathfrak{g}}$ with respect to some fixed choice of inner product.

Suppose $\mathfrak{g} = (\Gamma_1, A = (A_{ij})_{i,j \in I_n})$ and $\mathfrak{h} = (\Gamma_2, B = (B_{ij})_{i,j \in J_m})$ are connected Dynkin diagrams with corresponding Weyl groups $\mathcal{W}_{\mathfrak{g}} = \langle s_i \rangle_{i \in I_n}$ and $\mathcal{W}_{\mathfrak{h}} = \langle t_j \rangle_{j \in J_m}$. Let $\phi : \mathcal{W}_{\mathfrak{g}} \to GL(V_1, \langle \cdot, \cdot \rangle_1)$ and $\psi : \mathcal{W}_{\mathfrak{h}} \to GL(V_2, \langle \cdot, \cdot \rangle_2)$ be Euclidean representations of $\mathcal{W}_{\mathfrak{g}}$ and $\mathcal{W}_{\mathfrak{h}}$ respectively, with $V_1 := \mathrm{span}_{\mathbb{R}}(\{\alpha_i\}_{i \in I_n})$ and $V_2 := \mathrm{span}_{\mathbb{R}}(\{\beta_j\}_{j \in J_m})$ for simple roots $\{\alpha_i\}_{i \in I_n}$ and $\{\beta_j\}_{j \in J_m}$ respectively. We say ϕ and ψ are *isomorphic* if there is a bijection $\sigma : I_n \to J_m$ such that the mapping $s_i \mapsto t_{\sigma(i)}$ extends to a group isomorphism from $\mathcal{W}_{\mathfrak{g}}$ to $\mathcal{W}_{\mathfrak{h}}$ and such that the linear transformation $T : V_1 \to V_2$ induced by the set mapping $\alpha_i \mapsto \beta_{\sigma(i)}$ is 'angle-preserving', i.e. for some fixed (necessarily positive) real scalar κ we have $\langle T(u), T(v) \rangle_2 = \kappa \langle u, v \rangle_1$ for all $u, v \in V_1$. To emphasize the role of the bijection σ we say that ϕ and ψ are *isomorphic via* σ. In particular, it follows that for any two choices of inner products on V_1 from Theorem 2.13, the corresponding Euclidean representations of $\mathcal{W}_{\mathfrak{g}}$ are isomorphic. Some other results concerning isomorphic Euclidean representations are explored in Lemma 2.15. The Euclidean representations corresponding to the connected Dynkin diagrams of finite type are pairwise nonisomorphic (even though the corresponding Weyl groups are not all distinct — in particular, $\mathcal{W}_{\mathsf{B}_n} \cong \mathcal{W}_{\mathsf{C}_n}$).

Now relax the connectedness hypothesis for $\mathfrak{g}$ and $\mathfrak{h}$. Suppose a connected component $\mathfrak{g}'$ of $\mathfrak{g}$ has nodes indexed by a subset $J \subseteq I_n$. Let $V_1' = \mathrm{span}_{\mathbb{R}}(\{\alpha_i\}_{i \in J})$, so V_1' is a subspace of V with the induced inner product $\langle \cdot, \cdot \rangle_1'$. It is easy to see that the mapping $\phi' : \mathcal{W}_{\mathfrak{g}'} \to GL(V_1', \langle \cdot, \cdot \rangle_1')$ is a Euclidean representation of $\mathcal{W}_{\mathfrak{g}'}$. We say Euclidean representations ϕ and ψ of $\mathcal{W}_{\mathfrak{g}}$ and $\mathcal{W}_{\mathfrak{h}}$ are *isomorphic* if there is some one-to-one correspondence $\mathfrak{g}' \mapsto \mathfrak{h}'$ of connected components of $\mathfrak{g}$ and $\mathfrak{h}$ such that ϕ' and ψ' are isomorphic.

§2.11 Roots and root systems. Write $w.v$ for $\phi(w)(v)$ whenever $w \in \mathcal{W}$ and $v \in V$. As in [Hum2] and [BB], we define the *root system* $\mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i \in I_n})$ to be the set $\phi(\mathcal{W}_\mathfrak{g})(\{\alpha_i\}_{i \in I_n}) = \{w.\alpha_i\}_{i \in I_n, w \in \mathcal{W}}$. Set $\Phi := \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i \in I_n})$. Elements of Φ are *roots*. A root $\alpha = \sum k_i \alpha_i$ is *positive* if each $k_i \geq 0$ and is *negative* if each $k_i \leq 0$. The sets Φ^+ and Φ^- of positive and negative roots can be seen to partition Φ (see §3 of [Don7]). For any $i, j \in I_n$, by definition $s_j.\alpha_i = \alpha_i - M_{ij}\alpha_j$. Sine any $w \in \mathcal{W}$ is a product of s_j's, then by iterating the previous computation we see that $w.\alpha_i$ is an integral linear combination of simple roots. That is, when $\alpha = \sum k_i \alpha_i$, then each $k_i \in \mathbb{Z}$. Now, each $w \in \mathcal{W}$ permutes Φ. To see this, note that for any $w \in \mathcal{W}$ and $\alpha, \beta \in \Phi$, (1) $w.\alpha \in \Phi$ by definition so $w(\Phi) \subseteq \Phi$, (2) $\alpha = w.(w^{-1}.\alpha)$ so $\Phi \subseteq w(\Phi)$, and (3) if $w.\alpha = w.\beta$ then $w^{-1}.(w.\alpha) = w^{-1}.(w.\beta)$ so $\alpha = \beta$. So we have an induced action of $\mathcal{W}$ on Φ. Two root systems $\Phi := \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i \in I_n})$ and $\Psi := \mathcal{R}(\mathfrak{h}, \psi, \{\beta_j\}_{j \in J_m})$ are *isomorphic* (respectively, *isomorphic via* σ) if the Euclidean representations ϕ and ψ are isomorphic (respectively, isomorphic via σ).

For any $\alpha \in \Phi$, define $\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle}\alpha$. Observe that $\langle \alpha_i, \alpha_j^\vee \rangle = M_{ij}$ for all $i, j \in I_n$. Let $\Phi^\vee := \{\alpha^\vee\}_{\alpha \in \Phi}$. Based on the following lemma, we call $\Phi^\vee$ the *dual root system* for Φ.

Lemma 2.14 *We have $\Phi^\vee = \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i^\vee\}_{i \in I_n})$ (an equality of sets), and moreover $\alpha^\vee = w.\alpha_i^\vee$ for $w \in \mathcal{W}$ if and only if $\alpha = w.\alpha_i$.*

For this paragraph, assume that $\mathfrak{g}$ is connected. According to the discussion of the previous section, simple roots have two possible lengths, which we call *long* or *short*. (If only one simple root length is possible i.e. in the A-D-E cases, the adjectives "short" and "long" are interchangeable.) Note that if $\alpha \in \Phi$ with $\alpha = w.\alpha_i$ for some $w \in \mathcal{W}$ and simple root α_i, then $\langle \alpha, \alpha \rangle = \langle w.\alpha_i, w.\alpha_i \rangle = \langle \alpha_i, \alpha_i \rangle$. So α has the same length as α_i. With this in mind, we let $\Phi_{long} = \{\alpha \in \Phi \,|\, \alpha = w.\alpha_i \text{ for } w \in \mathcal{W} \text{ and } \alpha_i \text{ long}\}$ be the set of *long* roots, and analogously define the set Φ_{short} of *short* roots. We also have Φ^+_{long} (the set of positive roots that are long) and Φ^+_{short} (the set of positive roots that are short).

Lemma 2.15 *Suppose $\mathfrak{g} = (\Gamma_1, A = (A_{ij})_{i,j \in I_n})$ and $\mathfrak{h} = (\Gamma_2, B = (B_{ij})_{i,j \in J_m})$ are connected Dynkin diagrams with corresponding Weyl groups $\mathcal{W}_{\mathfrak{g}} = \langle s_i \rangle_{i \in I_n}$ and $\mathcal{W}_{\mathfrak{h}} = \langle t_j \rangle_{j \in J_m}$. Let $\phi : \mathcal{W}_{\mathfrak{g}} \to GL(V_1, \langle\cdot,\cdot\rangle_1)$ and $\psi : \mathcal{W}_{\mathfrak{h}} \to GL(V_2, \langle\cdot,\cdot\rangle_2)$ be isomorphic Euclidean representations of $\mathcal{W}_{\mathfrak{g}}$ and $\mathcal{W}_{\mathfrak{h}}$ respectively, with $V_1 := \mathrm{span}_{\mathbb{R}}(\{\alpha_i\}_{i \in I_n})$ and $V_2 := \mathrm{span}_{\mathbb{R}}(\{\beta_j\}_{j \in J_m})$ for simple roots $\{\alpha_i\}_{i \in I_n}$ and $\{\beta_j\}_{j \in J_m}$ respectively. As in §2.10, let $\sigma : I_n \to J_m$ be the associated bijection and $T : V_1 \to V_2$ the associated angle-preserving linear transformation. Let $\Phi := \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i \in I_n})$ and $\Psi := \mathcal{R}(\mathfrak{h}, \psi, \{\beta_j\}_{j \in J_m})$. Let $(i_1, i_2, \ldots, i_p)$ be a sequence of elements from I_n. (1) For all $i, j \in I_n$, $A_{ij} = B_{\sigma(i),\sigma(j)}$. (2) For all $v \in V_1$, $T(s_{i_1}s_{i_2}\cdots s_{i_p}.v) = t_{\sigma(i_1)}t_{\sigma(i_2)}\cdots t_{\sigma(i_p)}.T(v)$. (3) For $j \in I_n$, let $\alpha := s_{i_1}s_{i_2}\cdots s_{i_p}.\alpha_j$ and $\beta := t_{\sigma(i_1)}t_{\sigma(i_2)}\cdots t_{\sigma(i_p)}.\beta_{\sigma(j)}$. If α is positive in Φ (resp. long, short), then β is positive in Ψ (resp. long, short).*

For connected $\mathfrak{g}$, give Φ the following partial ordering: write $\alpha \leq \beta$ for roots α and β if and only if $\beta - \alpha = \sum k_i\alpha_i$ with each k_i nonnegative. View Φ^+, Φ^+_{long} and Φ^+_{short} as subposets of Φ in the induced order. If $\alpha \in \Phi^+$, write $\alpha = \sum k_i\alpha_i$ for nonnegative integers k_i. The *height* of α, denoted $ht(\alpha)$, is defined to be the quantity $\sum k_i$. The following facts can be understood by studying the so-called 'adjoint' and 'short adjoint' representations of the finite-dimensional complex simple Lie algebras.

Facts 2.16 *Keep the notation of the previous paragraph as well as the assumption that $\mathfrak{g}$ is connected. The posets of roots Φ^+ and Φ^+_{short} are ranked, connected posets with (in each* *case) rank function given by $\rho(\alpha) = ht(\alpha) - 1$. The minimal roots for Φ^+ (respectively, Φ^+_{short}) are the simple roots (resp. short simple roots). Each has a unique maximal root.*

In the setting of these results, the maximal root $\overline{\omega}$ for Φ^+ is called the *highest long root*. For Φ^+_{short} the maximal root $\overline{\omega}_{short}$ is the *highest short root*.

The *transpose* representation and root system defined next are helpful in explicitly identifying long and short roots. For this definition, however, $\mathfrak{g}$ need not be connected. Let $V^\top$ be the real vector space freely generated by $\{\alpha_i^\top\}_{i\in I_n}$, and define $\phi^\top : W_\mathfrak{g} \to V^\top$ by the rule $\phi^\top(s_i)(\alpha_j^\top) = \alpha_j^\top - M_{ji}^\top \alpha_i^\top$. Give $V^\top$ an inner product $\langle\cdot,\cdot\rangle_\top$ as in Theorem 2.13 above using the matrix $M^\top$. Then set $\Phi^\top := \mathcal{R}(\mathfrak{g}, \phi^\top, \{\alpha_i^\top\}_{i\in I_n})$. (Evidently, the root systems $\mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i\}_{i\in I_n})$ and $\mathcal{R}(\mathfrak{g}^\top, \phi^\top, \{\alpha_i^\top\}_{i\in I_n})$ are isomorphic via the identity bijection on I_n.)

Proposition 2.17 *Let $\mathfrak{g}$ be connected. For all $w \in \mathcal{W}$ and $j \in I_n$, it is the case that $w.\alpha_j$ is positive (resp. long, short) in Φ if and only if $w.\alpha_j^\vee$ is positive (resp. short, long) in $\Phi^\vee$ if and only if $w.\alpha_j^\top$ is positive (resp. short, long) in $\Phi^\top$. Moreover, $\Phi^\vee = \mathcal{R}(\mathfrak{g}, \phi, \{\alpha_i^\vee\}_{i\in I_n})$ and $\Phi^\top = \mathcal{R}(\mathfrak{g}, \phi^\top, \{\alpha_i^\top\}_{i\in I_n})$ are isomorphic via the identity bijection on I_n.*

§2.12 Weights. Some of the following recasts parts of §13 of [Hum1]. Using our inner product $\langle\cdot,\cdot\rangle$ we obtain another special basis for V, the basis of 'fundamental weights'. The following proposition shows how this basis is obtained and uniquely characterized.

Proposition 2.18 *Let $A = (A_{jk})_{j,k\in I_n}$ be a real $n \times n$ matrix. Define $\omega_j := \sum_{k\in I_n} A_{jk}\alpha_k$. Then $S_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$ for all $i, j \in I_n$ if and only if $A = M^{-1}$ if and only if $\langle\omega_j, \alpha_i^\vee\rangle = \delta_{ij}$ for all $i, j \in I_n$.*

In view of this result, we define the basis of *fundamental weights* $\{\omega_i\}_{i\in I_n}$ to be the unique basis for V satisfying the equivalent conditions of Proposition 2.18. As a consequence we see that for each $i \in I_n$, $\alpha_i = \sum_{j \in I_n} M_{ij}\omega_j$, i.e. the ith simple root is identified with the ith row of the Cartan matrix relative to the basis of fundamental weights. Let $\Lambda \subset V$ be the set of all vectors in the integer linear span of $\{\omega_i\}_{i \in I_n}$. Vectors in Λ are *weights*, and we call Λ the *lattice of weights*. (Here 'lattice' is used in the sense of the $\mathbb{Z}$-span of a basis.) A weight $\lambda \in \Lambda$ is *dominant* (*strongly dominant*) if $\lambda = \sum m_i\omega_i$ with each m_i nonnegative (positive). Denote by Λ^+ the set of dominant weights.

Lemma 2.19 *We have $\Phi \subset \Lambda$. Moreover each $w \in \mathcal{W}$ permutes Λ, and we have an induced action of $\mathcal{W}$ on Λ.*

Given a subset $J \subseteq I_n$, let $\mathcal{W}_J$ be the subgroup of $\mathcal{W}$ generated by $\{s_j\}_{j \in J}$. A dominant weight λ is J^c-*dominant* if when we write $\lambda = \sum_{i \in I_n} m_i\omega_i$, then $m_j > 0$ if and only if $j \notin J$. It can be shown that the results of [Hum2] §5.13 extend to the setting of our geometric representation of the Weyl group $\mathcal{W}$. It follows that $\mathcal{W}_J$ is the stablizer of λ under the action of $\mathcal{W}$ on Λ. So, by the 'orbit-stablizer' theorem, we have $|\mathcal{W}| = |\mathcal{W}\lambda||\mathcal{W}_J|$. When $\mathfrak{g}$ is connected, we apply this to the special cases of the sets Φ_{long} and Φ_{short} of long and short roots respectively. In §2.17 below we show how one can use a game played on the Dynkin diagram $\mathfrak{g}$ to determine the highest root and highest short root. Using this technique one can determine that for A_n, $\overline{\omega} = \omega_1 + \omega_n$. For B_n, $\overline{\omega} = \omega_2$ and $\overline{\omega}_{short} = \omega_1$. For C_n, $\overline{\omega} = 2\omega_1$ and $\overline{\omega}_{short} = \omega_2$. For D_n, $\overline{\omega} = \omega_2$. For E_6, $\overline{\omega} = \omega_2$. For E_7, $\overline{\omega} = \omega_2$. For E_8, $\overline{\omega} = \omega_2$. For F_4, $\overline{\omega} = \omega_1$ and $\overline{\omega}_{short} = \omega_4$. For G_2, $\overline{\omega} = \omega_2$ and $\overline{\omega}_{short} = \omega_1$. Therefore, the highest long and short roots are dominant weights. In fact, it can be seen that all roots of Φ_{long} (resp. Φ_{short}) are conjugate under the action of $\mathcal{W}$. We therefore obtain the following result, which gives us a nice way to compute the order of the Weyl group.

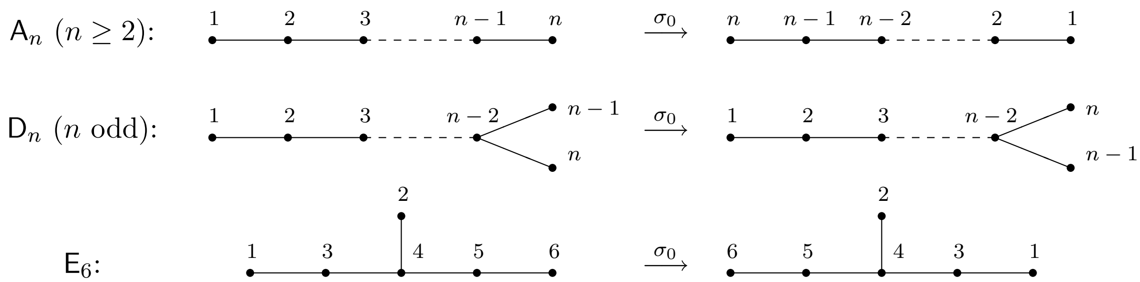
Theorem 2.20 *With $\mathfrak{g}$ connected, we have $\overline{\omega}$ (resp. $\overline{\omega}_{short}$) as the highest long (resp. short) root. Then $\overline{\omega}$ (resp. $\overline{\omega}_{short}$) is nonzero and dominant. Moreover, $\mathcal{W}\overline{\omega} = \Phi_{long}$ (resp. $\mathcal{W}\overline{\omega}_{short} = \Phi_{short}$). Suppose $\overline{\omega}$ (resp. $\overline{\omega}_{short}$) is J^c-dominant. Then $|\mathcal{W}| = |\Phi_{long}||\mathcal{W}_J|$ (resp. $|\mathcal{W}| = |\Phi_{short}||\mathcal{W}_J|$).*

§2.13 The longest element of the Weyl group. The material in this section is standard, see e.g. [Hum2] or [BB]. A finite Weyl group has a unique 'longest' element, where length is measured as follows: In any Weyl group, an element w may be written as a product $s_{i_1} \cdots s_{i_p}$. Any shortest such expression is a *reduced expression* for w, and the *length* of w is $\ell(w) := p$. Thus if $\mathcal{W}$ is finite, there is an upper bound on the lengths of group elements. The following result can be derived from standard facts (see e.g. Exercise 5.6.2 of [Hum2]).

Proposition 2.21 *In a finite Weyl group, there is exactly one longest element, denoted w_0. We have $w_0^2 = \varepsilon$. Moreover, the is a permutation $\sigma_0 : I_n \longrightarrow I_n$ such that for each $i \in I_n$, $w_0.\alpha_i = -\alpha_{\sigma_0(i)}$.*

Observe that since $\langle \alpha_i, \alpha_j^\vee \rangle = M_{ij}$ for all $i, j \in I_n$ then $\langle \alpha_{\sigma_0(i)}, \alpha_{\sigma_0(j)}^\vee \rangle = M_{\sigma_0(i),\sigma_0(j)}$. In particular, σ_0 is a symmetry of the Dynkin diagram $\mathfrak{g}$ in the sense that $\mathfrak{g} \cong \mathfrak{g}^{\sigma_0}$. Since $w_0^2 = \varepsilon$ in $\mathcal{W}$ then σ_0^2 is the identity permutation. It also follows from the proposition that when w_0 acts on Λ, then $\omega_i \mapsto -\omega_{\sigma_0(i)}$ for each $i \in I_n$: $\langle -w_0.\omega_i, \alpha_j^\vee \rangle = \langle \omega_i, -w_0.\alpha_j^\vee \rangle = \langle \omega_i, \alpha_{\sigma_0(j)}^\vee \rangle = \delta_{i,\sigma_0(j)}$, hence $w_0.\omega_i = -\omega_{\sigma_0(i)}$. Thus, $w_0.(\sum m_i\omega_i) = -\sum m_i\omega_{\sigma_0(i)}$. So once the action of w_0 on V is known (see §2.17 below) then one can compute σ_0. One finds that for connected Dynkin diagrams, σ_0 is trivial except in the cases A_n ($n \geq 2$), D_{2k+1} ($k \geq 2$), and E_6; see Figure 2.11.

Figure 2.11: Action of the permutation σ_0 when σ_0 is not the identity.



If R is a ranked poset with edges colored by the set I_n, then the σ_0-*recolored dual* $R^{\triangle}$ is the edge-colored poset $(R^{\sigma_0})^* \cong (R^*)^{\sigma_0}$. See Figure 2.12 for an example.

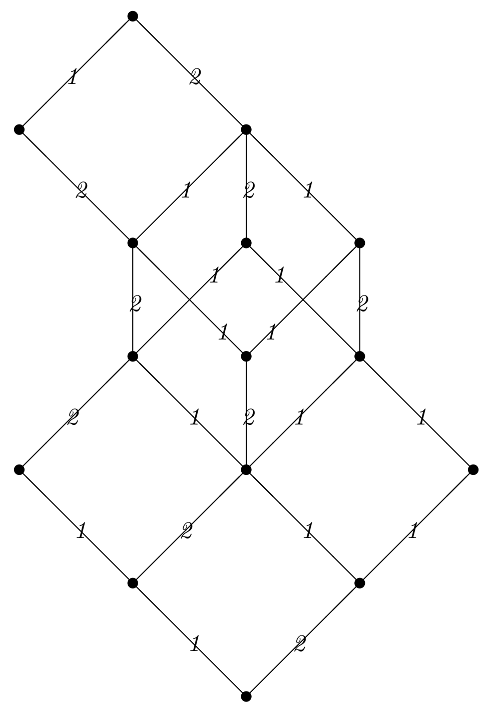
§2.14 The M-structure property (again). Let R be a ranked poset with edges colored by the set I_n. We say R has the $\mathfrak{g}$-*structure property* if R has the M-structure property for the Cartan matrix M associated to $\mathfrak{g}$ with weight function $wt_R : R \longrightarrow \Lambda$ such that $wt_R(\mathbf{s}) = \sum_{j \in I_n} m_j(\mathbf{s})\omega_j$. Thus R has the $\mathfrak{g}$-structure property if and only if for each simple root α_i we have $wt_R(\mathbf{s}) + \alpha_i = wt_R(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in R. This condition depends not only on $\mathfrak{g}$ (information from the corresponding Dynkin diagram) but also on the combinatorics of R.

Let us temporarily assume only that (Γ, M) is a GCM graph with nodes indexed by I_n. If R is a ranked poset with edges colored by the set I_n, then the edge-coloring function $\mathbf{edgecolor}_R : \mathcal{E}(R) \to I_n$ is *sufficiently surjective* if for each connected component of (Γ, M) there is a node γ_i and an edge $\mathbf{s} \to \mathbf{t}$ with $\mathbf{edgecolor}_R(\mathbf{s} \to \mathbf{t}) = i$. The following are some of the main results of [Don8].

Theorem 2.22 *Let $M = (M_{i,j})_{i,j \in I_n}$ be a real matrix. (1) If there is a diamond-colored distributive lattice L with surjective edge-coloring function* $\mathbf{edgecolor}_L : \mathcal{E}(L) \to I_n$ *and having the M-structure property, then M must be a generalized Cartan matrix. (2) Suppose*

Figure 2.12: $L^{\triangle}$ for the edge-colored lattice L from Figure 2.1.

Here regard L to be edge-colored by the nodes of A_2.



(Γ, M) *is a GCM graph with nodes indexed by* I_n. *Suppose* R *is a ranked poset with sufficiently surjective edge-coloring function* $\mathbf{edgecolor}_R : \mathcal{E}(R) \rightarrow I_n$. *If* R *has the* M*-structure property, then* $\mathbf{edgecolor}_R$ *is surjective and* (Γ, M) *is a Dynkin diagram of finite type.*

Combining both parts of the previous theorem we obtain:

Corollary 2.23 *Let* $M = (M_{i,j})_{i,j \in I_n}$ *be a real matrix. If there is a diamond-colored distributive lattice* L *with surjective edge-coloring function* $\mathbf{edgecolor}_L : \mathcal{E}(L) \rightarrow I_n$ *and having the* M*-structure property, then* M *must be a Cartan matrix.*

It is important to note that the condition "M is a Cartan matrix" in this corollary is necessary but not sufficient for there to be an M-structured diamond-colored distributive lattice.

Now return to the assumption that M is a Cartan matrix and $\mathfrak{g} = (\Gamma, M)$ is a Dynkin diagram. For a $\mathfrak{g}$-structured diamond-colored distributive lattice L, let λ be the weight of the unique maximal vertex of L. We say L is a $(\mathfrak{g}, \lambda)$-*structured* distributive lattice. A concept to be introduced in Chapter 3 (the 'distributive core') relates directly to the following question: For which Dynkin diagrams $\mathfrak{g}$ and weights λ is there a $(\mathfrak{g}, \lambda)$-structured distributive lattice? When a $(\mathfrak{g}, \lambda)$-structured distributive lattice exists, we have the following result concerning its unique minimal element. This result can be demonstrated using facts about the 'numbers game' as in [Don6].

Proposition 2.24 *Let R be an M-structured poset with a unique maximal element of weight λ, necessarily dominant. Then R has a minimal element of weight $w_0.\lambda$. In particular, if L is a $(\mathfrak{g}, \lambda)$-structured distributive lattice for some dominant weight λ, then the unique minimal element of L has weight $w_0.\lambda$.*

§2.15 Weyl characters. See [Hum1], [FH], or [Stem3] for discussions of the basic theory of Weyl characters, which we outline here without much reference to Lie representation theory. Observe that Λ is an abelian subgroup of V. Let $\mathbb{Z}[\Lambda]$ be the group ring over Λ: that is, $\mathbb{Z}[\Lambda]$ consists of finite integral linear combinations of elements of the basis $\{e_\mu \,|\, \mu \in \Lambda\}$. Multiplication in $\mathbb{Z}[\Lambda]$ is given by $e_\mu e_\nu = e_{\mu+\nu}$. We sometimes use 1 to denote e_0. The Weyl group $\mathcal{W}$ acts on $\mathbb{Z}[\Lambda]$ by the rule $w.e_\mu := e_{w.\mu}$. The *character ring* $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ for $\mathfrak{g}$ is the ring of $\mathcal{W}$-invariant elements of $\mathbb{Z}[\Lambda]$; elements of $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ are *characters for* $\mathfrak{g}$. For any weight μ, let $A_\mu := \sum_{w \in \mathcal{W}} \det(\phi(w)) e_{w.\mu}$. Using the fact that $S_i = \phi(s_i)$ is a reflection and hence $\det(S_i) = -1$, it follows that $s_i.A_\mu = -A_\mu$. So, A_μ is not in the character ring. Let $\varrho := \omega_1 + \cdots + \omega_n$, the sum of the fundamental weights. Part (1) of the following well-known theorem is the famous Weyl character formula, due to H. Weyl.

Theorem 2.25 (Weyl) *(1) For each dominant $\lambda \in \Lambda^+$, there exists a unique $\chi_\lambda \in \mathbb{Z}[\Lambda]$ such that $A_\varrho \chi_\lambda = A_{\varrho+\lambda}$, and moreover $\chi_\lambda \in \mathbb{Z}[\Lambda]^{\mathcal{W}}$. (2) The characters $\{\chi_\lambda\}_{\lambda \in \Lambda^+}$ are a basis for the character ring $\mathbb{Z}[\Lambda]^{\mathcal{W}}$. (3) The characters $\{\chi_{\omega_i}\}_{i \in I_n}$ are an algebraic basis for the character ring $\mathbb{Z}[\Lambda]^{\mathcal{W}}$.*

Weyl characters are nonnegative integral linear combinations of the characters $\{\chi_\lambda\}_{\lambda \in \Lambda^+}$. Elements of the basis $\{\chi_\lambda\}_{\lambda \in \Lambda^+}$ for the character ring are *irreducible Weyl characters*, and elements of $\{\chi_{\omega_i}\}_{i \in I_n}$ are *fundamental characters*. At times we use the nomenclature '$\mathfrak{g}$-character' to emphasize the connection to the Dynkin diagram $\mathfrak{g}$. For each $i \in I_n$, set $z_i := e_{\omega_i}$. If $\mu = \sum m_i \omega_i \in \Lambda$, set $z^\mu := z_1^{m_1} \cdots z_n^{m_n}$. Then for any $\lambda \in \Lambda^+$ we can write $\chi_\lambda = \sum_{\mu \in \Lambda} c_{\lambda,\mu} z^\mu$ for some $c_{\lambda,\mu} \in \mathbb{Z}$. So we can think of an irreducible Weyl character as a Laurent polynomial in the variables $\{z_i\}_{i \in I_n}$. At times we will emphasize this viewpoint by writing $\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)$ in place of χ_λ. The following facts about irreducible Weyl characters can be proved using the representation theory of semisimple Lie algebras. We record these here for future use.

Facts 2.26 *Keep the notation of the previous paragraph. (1) Each coefficient $c_{\lambda,\mu}$ is nonnegative. (2) Moreover, $c_{\lambda,\lambda} = 1$ and $c_{\lambda,w_0.\lambda} = 1$. (3) Partially order the set $\Pi(\lambda) := \{\mu \in \Lambda \,|\, c_{\lambda,\mu} \neq 0\}$ by the rule $\mu \leq \nu$ if and only if $\nu - \mu = \sum k_i \alpha_i$ with each $k_i \geq 0$. Then $\Pi(\lambda)$ is a connected ranked poset with unique maximal element λ and unique minimal* *element $w_0.\lambda$. (4) Moreover, $\mu \to \nu$ in $\Pi(\lambda)$ if and only if $\mu + \alpha_i = \nu$ for some simple root α_i. Therefore by giving each such edge $\mu \to \nu$ the color $i \in I_n$ of the appropriate simple root α_i, $\Pi(\lambda)$ is a $\mathfrak{g}$-structured poset.*

For example, to see that each coefficient $c_{\lambda,\mu}$ is nonnegative, one observes that $c_{\lambda,\mu}$ counts the dimension of a certain subspace of the highest weight λ irreducible representation of the corresponding semisimple Lie algebra. Subsequently one can see that if we evaluate $\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)$ at $z_1 = \cdots = z_n = 1$ we obtain the number $\sum_{\mu\in\Lambda} c_{\lambda,\mu}$, which is the dimension of the representing space. For this reason we will refer to the nonnegative integer $\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)|_{z_1=\cdots=z_n=1}$ as the *dimension* of χ_λ. More generally, the dimension of a Weyl character $\chi = \sum_{\lambda\in\Lambda^+} m_\lambda \chi_\lambda$ is the nonnegative integer $\sum_{\lambda\in\Lambda^+} m_\lambda \mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)|_{z_1=\cdots=z_n=1}$.

Example 2.27: Adjoint characters. Assume $\mathfrak{g}$ is connected. The highest long root ϖ and the highest short root ϖ_{short} are dominant weights. From [Don5] (for example) it follows that $\chi_{\varpi} = n\, e_0 + \sum_{\alpha\in\Phi} e_\alpha$ and that $\chi_{\varpi_{short}} = m\, e_0 + \sum_{\alpha\in\Phi_{short}} e_\alpha$, where m is the number of short simple roots. To see that these are both in the character ring, it suffices to observe that $\mathcal{W}$ permutes Φ (resp. Φ_{short}). We call χ_{ϖ} and $\chi_{\varpi_{short}}$ the *adjoint* and *short adjoint* characters, respectively. $\square$

§2.16 Our main goal: 'splitting posets' as combinatorial models for Weyl characters. Let R be a ranked poset with edges colored by the set I_n. We say R is a *splitting poset* for a Weyl character χ if (1) R has the $\mathfrak{g}$-structure property and (2) the weight-generating function on R is the Weyl character χ in the following sense: $\chi = \sum_{\mathbf{t}\in R} z^{wt_R(\mathbf{t})}$. If

R is a diamond-colored distributive lattice, then we say R is a *splitting distributive lattice* or *SDL*. The following is from Lemma 2.2 of [ADLMPPW].

Lemma 2.28 *Let $\lambda = \sum m_i\omega_i$ be dominant in the lattice of weights for $\mathfrak{g}$. Suppose R is a splitting poset for χ_{λ}. Then the dual R^* is a splitting poset for the irreducible $\mathfrak{g}$-Weyl character $\chi_{-w_0.\lambda}$. Given a one-to-one function $\sigma : I_n \to I_n$, the recolored poset R^σ is a splitting poset for the irreducible $\mathfrak{g}^\sigma$-Weyl character $\chi_{\sum m_i \omega_{\sigma(i)}}$. The σ_0-recolored dual $R^\triangle$ is also a splitting poset for the irreducible $\mathfrak{g}$-Weyl character χ_{λ}.*

If R is a connected splitting poset for an irreducible Weyl character χ_{λ}, then by Facts 2.26, R has a unique vertex **max** (respectively **min**) of maximal (resp. minimal) rank, and moreover we have $wt_R(\mathbf{max}) = \lambda$ and $wt_R(\mathbf{min}) = w_0.\lambda$. Set $\varrho^\vee := \sum_{i=1}^n \frac{2\omega_i}{\langle \alpha_i, \alpha_i \rangle}$. Observe that $\langle \alpha_i, \varrho^\vee \rangle = 1$ for $1 \leq i \leq n$. Using the vertices **max** and **min**, one now sees that the length of R is $\langle wt_R(\mathbf{max}) - wt_R(\mathbf{min}), \varrho^\vee \rangle = \langle \lambda - w_0.\lambda, \varrho^\vee \rangle$. This observation helps explain the appearance of the scaling factor $q^{-\langle w_0.\lambda, \varrho^\vee \rangle}$ in the next proposition, which shows how the rank generating function $RGF(R, q)$ for such a splitting poset R is obtained as a specialization of the irreducible Weyl character χ_{λ}.

Proposition 2.29 *Let R be a connected splitting poset for the irreducible Weyl character χ_{λ}. Then its rank generating function $RGF(R, q)$ can be obtained by specializing the Weyl character as follows:*

$$RGF(R, q) = q^{-\langle w_0.\lambda, \varrho^\vee \rangle} \mathbf{char}_{\mathfrak{g}}(\lambda; z_1, \ldots, z_n)\big|_{z_i := q^{\langle \omega_i, \varrho^\vee \rangle}}.$$

In view of this result, we will use $\ell(\lambda)$ to denote the length $\langle \lambda - w_0.\lambda, \varrho^\vee \rangle$ of any connected splitting poset for χ_{λ}. The following result (appearing as Proposition 2.4 in [ADLMPPW], based on Proctor's work in Section 6 of [Pro3] with the M-structure poset context contributed by Donnelly) shows that connected splitting posets for irreducible Weyl characters have certain salient combinatorial features.

Theorem 2.30 *Let R be a connected splitting poset for the irreducible Weyl character χ_λ. Then R is rank symmetric, rank unimodal, and has rank generating function*

$$RGF(R, q) = \prod_{\alpha \in \Phi^+} \frac{1 - q^{\langle \lambda + \varrho, \alpha^\vee \rangle}}{1 - q^{\langle \varrho, \alpha^\vee \rangle}}$$

Letting $q \to 1$ in the above expression gives:

Corollary 2.31 (Weyl Dimension Formula) *The dimension of χ_λ is*

$$\prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \varrho, \alpha^\vee \rangle}{\langle \varrho, \alpha^\vee \rangle}$$

Calculating the difference of the degrees of the numerator and denominator polynomials in Theorem 2.30 gives:

Corollary 2.32 *The length of any connected splitting poset for χ_λ is*

$$\ell(\lambda) = \sum_{\alpha \in \Phi^+} \langle \lambda, \alpha^\vee \rangle.$$

A crucial question at this point is: How does one obtain splitting posets? At present there are three general strategies. (1) *Impose 'natural' partial orders on combinatorial objects known to generate Weyl characters.* For example, the 'Littelmann' family of G_2-lattices shown in [Mc] to be SDL's for the irreducible G_2-characters were discovered by Donnelly by imposing a natural partial order on Littelmann's G_2 tableaux [Lit]. (2) *Apply Stembridge's product construction [Stem3].* For a given dominant weight λ, any resulting 'admissible system' is a 'minimal' splitting poset in the sense that it will not contain as a proper edge-colored subgraph a splitting poset for χ_λ. Further, one can sometimes show a given M-structure poset R is a splitting poset for χ_λ by locating an admissible system inside R as an edge-colored subgraph. This method is being employed right now by Alverson, Donnelly, Lewis, and Pervine to give another proof that the 'semistandard' lattices of [ADLMPPW] are SDL's for the irreducible Weyl characters for A_2, C_2, and G_2. (3) *Show that a given $\mathfrak{g}$-structured poset is a 'supporting graph' (cf. [Don4]) for a representation of the corresponding semisimple Lie algebra.* This method has been used in [Don3], [Don4], [Don5], and [DLP1] to produce/study many families of SDL's.

Example 2.33: The maximal splitting poset. Given an irreducible Weyl character χ_λ, consider the set of weights $\Pi(\lambda)$. By Facts 2.26, we may regard $\Pi(\lambda)$ as a ranked poset with edges colored by I_n, where $\mu \xrightarrow{i} \nu$ if and only if $\mu + \alpha_i = \nu$. We use $\Pi(\lambda)$ as the foundation for a new edge-colored ranked poset $\mathcal{M}(\lambda)$. As a set, we have

$$\mathcal{M}(\lambda) := \bigcup_{\mu \in \Pi(\lambda)} \left\{\mu^{(1)}, \ldots, \mu^{(c_{\lambda,\mu})}\right\},$$

where we have essentially extended each weight μ in $\Pi(\lambda)$ to a multiset of elements with weight μ using the coefficients $c_{\lambda,\mu}$. For $\mu^{(p)}$ and $\nu^{(q)}$ in $\mathcal{M}(\lambda)$, write $\mu^{(p)} \xrightarrow{i} \nu^{(q)}$ if and only if $\mu \xrightarrow{i} \nu$ in $\Pi(\lambda)$. In [Don4] it is observed that $\mathcal{M}(\lambda)$ is a supporting graph for the highest weight λ irreducible representation of the corresponding semisimple Lie algebra. In particular, $\mathcal{M}(\lambda)$ is a splitting poset for χ_λ. But this latter fact is easy enough to see directly from the definitions and Facts 2.26. It can be seen that $\mathcal{M}(\lambda)$ contains an isomorphic image of any other splitting poset R for χ_λ as a weak subposet. In effect, such an R has the same vertices as $\mathcal{M}(\lambda)$ but only a subset of its edges. We call $\mathcal{M}(\lambda)$ the *maximal splitting poset* for χ_λ. $\square$

Example 2.34: Splitting posets for adjoint characters. Let $\mathfrak{g}$ be connected. Define $\mathcal{A}$ to be the set $\{(i,j) | \langle \alpha_i, \alpha_j^\vee \rangle < 0\}_{i,j \in I_n}$ modulo the equivalence $(i,j) \equiv (j,i)$. For $k \in I_n$ set $\mathcal{A}^{(k)} := \mathcal{A} \cup \{(k,k)\}$, so $|\mathcal{A}^{(k)}| = n$. Let $L^{(k)}$ be the set $\Phi^+ \cup \mathcal{A}^{(k)} \cup \Phi^-$. Place directed edges with colors from the set I_n between the elements of $L^{(k)}$ as follows: Write $\alpha \xrightarrow{i} \beta$ if α and β are both roots in Φ^+ (or are both in Φ^-) and $\alpha + \alpha_i = \beta$. For each pair (i,j) in $\mathcal{A}^{(k)}$, include edges $-\alpha_r \xrightarrow{r} (i,j) \xrightarrow{r} \alpha_r$ if and only if $r = i$ or $r = j$. It is a consequence of Facts 2.16 that $L^{(k)}$ is the Hasse diagram for a ranked poset. We call $\mathcal{A}^{(k)}$ the *middle rank* of $L^{(k)}$. For reasons explained by Theorem 1.2 of [Don5], we call $L^{(k)}$ the *kth extremal* splitting poset for the adjoint character χ_{ϖ} for $\mathfrak{g}$. In that paper it is shown in Proposition 6.1 that the $L^{(k)}$'s are precisely the modular lattice supporting graphs for graphs for the adjoint representation of the simple Lie algebra $\mathfrak{g}$. It follows that each $L^{(k)}$ is a splitting modular lattice for the adjoint character χ_{ϖ} for $\mathfrak{g}$ (cf. Example 2.27). In [Don5] Corollary 6.2, it is also observed that an extremal splitting poset $L^{(k)}$ is a distributive lattice if and only if $\mathfrak{g}$ is one of A_n, B_n, C_n, F_4, or G_2 and γ_k is one of the end nodes for $\mathfrak{g}$.

There are similar objects for the short adjoint characters $\chi_{\varpi_{short}}$, cf. Example 2.27, [Don5]. Modify the constructions of the previous paragraph using only Φ_{short}. This results in splitting modular lattices $L^{(k)}_{short}$ for the short adjoint character, where each index $k \in I_n$ is such that α_k is short. As with the extremal splitting posets for the adjoint characters, we see that $L^{(k)}_{short}$ is a distributive lattice if and only if $\mathfrak{g}$ is one of A_n, B_n, C_n, F_4, or G_2 and α_k is a short simple root with at most one adjacent short simple root in the Dynkin diagram $\mathfrak{g}$. $\square$

§2.17 The numbers game and computations related to Weyl groups and roots systems. This subsection applies recent results of [Don6] in studying the combinatorial 'numbers game' of Mozes [Moz] and Eriksson [Erik1], [Erik2], [Erik3].

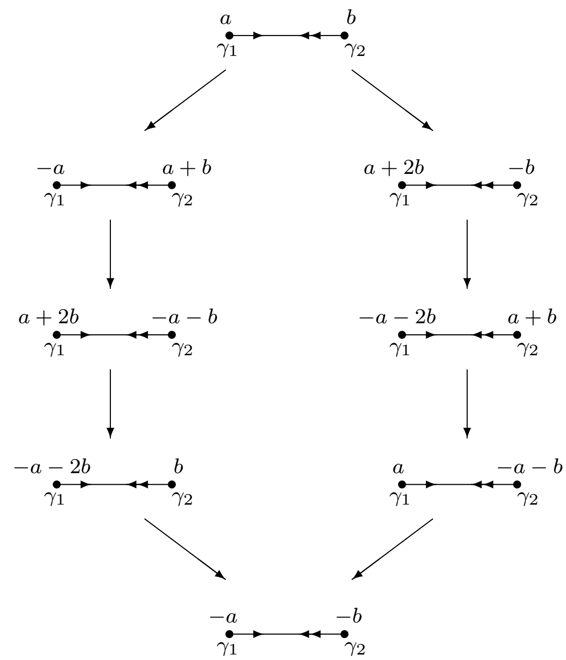
For the next two paragraphs, temporarily relax the finiteness hypothesis for $\mathcal{W} = \mathcal{W}_{\mathfrak{g}}$. For the game we describe next, a *position* λ is an assignment of numbers $(\lambda_i)_{i \in I_n}$ to the nodes of the GCM graph $\mathfrak{g} = (\Gamma, M)$. As with weights, say the position λ is *dominant* (respectively, *strongly dominant*) if $\lambda_i \geq 0$ (respectively $\lambda_i > 0$) for all $i \in I_n$; λ is *nonzero* if at least one $\lambda_i \neq 0$. Given a position λ on a GCM graph (Γ, M), to *fire* a node γ_i is to change the number at each node γ_j of Γ by the transformation

$$\lambda_j \longmapsto \lambda_j - M_{ij}\lambda_i,$$

provided the number at node γ_i is positive; otherwise node γ_i is not allowed to be fired. The *numbers game* is the one-player game on a GCM graph (Γ, M) in which the player (1) Assigns an initial position to the nodes of Γ; (2) Chooses a node with a positive number and fires the node to obtain a new position; and (3) Repeats step (2) for the new position if there is at least one node with a positive number.

Consider the GCM graph C_2. As we can see in Figure 2.13, the numbers game terminates in a finite number of steps for any initial position and any legal sequence of node firings, if it is understood that the player will continue to fire as long as there is at least one node with a positive number. In general, given a position λ, a *game sequence for* λ is the (possibly empty, possibly infinite) sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$, where γ_{i_j} is the jth node that is fired in some numbers game with initial position λ. More generally, a *firing sequence* from some position λ is an initial portion of some game sequence played from λ; the phrase *legal firing*

Figure 2.13: The numbers game for the Dynkin diagram C_2.



sequence is used to emphasize that all node firings in the sequence are known or assumed to be possible. Note that a game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$ is of finite length l (possibly with $l = 0$) if the number is nonpositive at each node after the lth firing; in this case we say the game sequence is *convergent* and the resulting position is the *terminal position* for the game sequence. We say a connected GCM graph (Γ, M) is *admissible* if there exists a nonzero dominant initial position with a convergent game sequence. Theorem 6.1 of [Don6] shows that a connected GCM graph is admissible if and only if it is a connected Dynkin diagram of finite type. In these cases, for any given initial position every game sequence will converge to the same terminal position in the same finite number of steps.

Return now to the assumption that $\mathcal{W} = \mathcal{W}_{\mathfrak{g}}$ is finite. The moves of the numbers game relate directly to the Euclidean representation $\phi : \mathcal{W}_{\mathfrak{g}} \to GL(V, \langle \cdot, \cdot \rangle)$, cf. §2.11. To see this, view a position $\lambda = (\lambda_i)_{i \in I_n}$ on $\mathfrak{g}$ as the weight $\sum \lambda_i \omega_i$. Now observe that firing node γ_i from weight λ on $\mathfrak{g}$ results in position $\phi(s_i)(\lambda)$: At each $j \in I_n$, $\langle s_i.\lambda, \alpha_j^\vee \rangle = \sum \lambda_k \langle \omega_k, \alpha_j^\vee \rangle - \lambda_i \langle \alpha_i, \alpha_j^\vee \rangle = \lambda_j - M_{ij}\lambda_i$. It follows from Eriksson's Reduced Word Result (see Theorem 2.8 of [Don6]) that $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$ is a game sequence for a numbers game played on $\mathfrak{g}$ from any given strongly dominant initial position if and only if $s_{i_l} \cdots s_{i_2} s_{i_1}$ is a reduced expression for w_0, the longest element of $\mathcal{W}$. For the rest of this subsection, let $s_{i_l} s_{i_{l-1}} \cdots s_{i_1}$ be a fixed reduced expression for w_0. The next result is an immediate application of Theorem 5.2 of [Don6] concerning the positive roots Φ^+.

Theorem 2.35 *For $1 \leq j \leq l$, set $\beta_j := s_{i_1} s_{i_2} \cdots s_{i_{j-1}}.\alpha_{i_j}$. Then $|\{\beta_j\}_{j=1}^{l}| = l$ and $\{\beta_j\} = \Phi^+$.*

Corollary 2.36 *Let $\lambda = \sum \lambda_i \omega_i \in \Lambda^+$. Keep the notation of the preceding theorem. For $1 \leq j \leq l$ let c_j be the number at the i_jth node when we play the legal firing sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{j-1}})$ from the initial position $(\lambda_i + 1)_{i \in I_n}$ on the Dynkin diagram $\mathfrak{g}$. Then $\langle \lambda + \varrho, \beta_j^\vee \rangle = c_j$. Moreover,*

$$\prod_{\alpha \in \Phi^+} (1 - q^{\langle \lambda + \varrho, \alpha^\vee \rangle}) = \prod_{j=1}^{l} (1 - q^{c_j}) \qquad \text{and} \qquad \prod_{\alpha \in \Phi^+} \langle \lambda + \varrho, \alpha^\vee \rangle = \prod_{j=1}^{l} c_j.$$

Proposition 2.37 *Keep the notation of Theorem 2.35. Assume $\mathfrak{g}$ is connected. Consider the transpose Euclidean representation $\phi^{\mathsf{T}} : \mathcal{W}_{\mathfrak{g}} \to GL(V^{\mathsf{T}}, \langle \cdot, \cdot \rangle_{\mathsf{T}})$ as in §2.11, with simple roots $\{\alpha_i^{\mathsf{T}}\}_{i \in I_n}$ and fundamental weights $\{\omega_i^{\mathsf{T}}\}_{i \in I_n}$. Suppose $\beta_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}.\alpha_{i_j} = \sum k_i \alpha_i \in \Phi^+$ is short (resp. long). Let $\beta_j^{\mathsf{T}} := s_{i_1} s_{i_2} \cdots s_{i_{j-1}}.\alpha_i^{\mathsf{T}}$, a root in Φ^{T}, with $(\beta_j^{\mathsf{T}})^\vee = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}.(\alpha_i^{\mathsf{T}})^\vee$ the corresponding root in $(\Phi^{\mathsf{T}})^\vee$, cf. Lemma 2.14. (1) Then $(\beta_j^{\mathsf{T}})^\vee$ is positive and short (resp. long). (2) For a strongly dominant weight $\mu = \sum \mu_i \omega_i^{\mathsf{T}}$, let d_j denote the number at the i_jth node after playing the legal sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{j-1}})$ from initial position $(\mu_i)_{i \in I_n}$ on the transpose graph $\mathfrak{g}^{\mathsf{T}}$. Then $\langle \mu, (\beta_j^{\mathsf{T}})^\vee \rangle_{\mathsf{T}} = d_j = \sum k_i \mu_i$.*

Remark 2.38 In view of the preceding results, the numbers game gives us simple iterative procedures for producing data concerning roots and Weyl group actions needed for example for the following computations. To compute the rank generating function of Theorem 2.30 above, observe that by Corollary 2.36 the exponents of the numerator and denominator in that formula are numbers appearing in a numbers game played from initial positions $(\lambda_i + 1)_{i \in I_n}$ and $(1)_{i \in I_n}$ on $\mathfrak{g}$ respectively. In combination, Theorem 2.35 and Proposition 2.37 show that if we play a numbers game on $\mathfrak{g}^{\mathsf{T}}$ from a generic strongly dominant position $(\mu_i)_{i \in I_n}$, then any positive root $\beta = \sum k_i \alpha_i$ in Φ will appear exactly once as the expression $\sum k_i \mu_i$ at node γ_{i_j} when it is fired. By Proposition 2.17 β will be short (resp. long) if and only if $\alpha_{i_j}^{\mathsf{T}}$ is long (resp. short) if and only if α_{i_j} is short (resp. long). Finally, to compute the action of w_0 on V, start with a generic strongly dominant weight $\lambda = \sum \lambda_i \omega_i$ as an initial position on $\mathfrak{g}$ and play the game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$. The terminal position is $s_{i_l} \cdots s_{i_2} s_{i_1}.\lambda = w_0.\lambda$. But since $w_0.\lambda = -\sum \lambda_i \omega_{\sigma_0(i)}$, one can now deduce how σ_0 permutes the elements of I_n. These techniques are applied to G_2 in the next subsection.

§2.18 An extended example: G_2. We now illustrate the main ideas of the preceding subsections with an example. We work with $\mathfrak{g} = \mathsf{G}_2$, which has Cartan matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ with inverse $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

$\boxed{\text{§2.8}}$ The Weyl group $\mathcal{W}$ is $\langle s_1, s_2 \,|\, s_1^2 = s_2^2 = (s_1 s_2)^6 = (s_2 s_1)^6 = \varepsilon \rangle$. This is easily seen to be the 12-element dihedral group. Its elements are $\{\varepsilon, s_1,\ s_2,\ s_1s_2,\ s_2s_1,\ s_1s_2s_1,\ s_2s_1s_2,\ s_1s_2s_1s_2,\ s_2s_1s_2s_1,\ s_1s_2s_1s_2s_1,\ s_2s_1s_2s_1s_2,\ s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1\}$.

$\boxed{\text{§2.10}}$ Let α_1 and α_2 be simple roots for the $\mathcal{W}$-module $V = \mathrm{span}_{\mathbb{R}}(\alpha_1, \alpha_2)$. We have $s_i.\alpha_j = \alpha_j - M_{ji}\alpha_i$ for $i, j = 1, 2$. Set $\langle \alpha_1, \alpha_1 \rangle = 2$. Then $\langle \alpha_2, \alpha_2 \rangle = \frac{M_{21}}{M_{12}} \langle \alpha_1, \alpha_1 \rangle = 3 \cdot 2 = 6$.

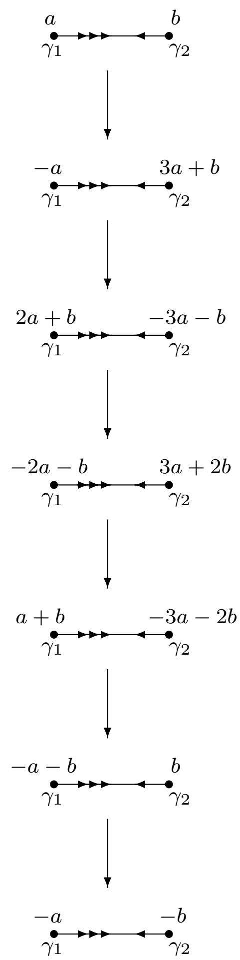
So α_1 is short and α_2 is long. Also, $\langle\alpha_1,\alpha_2\rangle = \frac{1}{2}\langle\alpha_2,\alpha_2\rangle M_{12} = \frac{1}{2}\cdot 6\cdot(-1) = -3$. Similarly see that $\langle\alpha_2,\alpha_1\rangle = -3$ as well. Then relative to the basis $\{\alpha_1,\alpha_2\}$ for V, the inner product $\langle\cdot,\cdot\rangle$ is represented by the matrix $\begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}$.

$\boxed{\S 2.11}$ Using $\mathfrak{g}^{\mathsf{T}}$, we compute the short and long roots in Φ^+. For the game sequence $(\gamma_1,\gamma_2,\gamma_1,\gamma_2,\gamma_1,\gamma_2)$ played in Figure 2.14 from a generic strongly dominant initial position (a,b) on $\mathfrak{g}^{\mathsf{T}}$, observe that the numbers at the fired nodes are a, $3a+b$, $2a+b$, $3a+2b$, $a+b$, and b respectively. Using Remark 2.38, it follows that $\Phi^+ = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$, $\Phi^+_{short} = \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2\}$, and $\Phi^+_{long} = \{3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_2\}$. Recall that α_1 corresponds to the first row of the Cartan matrix and α_2 corresponds to the second, relative to the basis of fundamental weights. That is, $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = -3\omega_1 + 2\omega_2$. Note that $3\alpha_1 + 2\alpha_2 = \omega_2$ is the highest root $\overline{\omega}$, and that $2\alpha_1+\alpha_2 = \omega_1$ is the highest short root $\overline{\omega}_{short}$. (Alternatively, these calculations are easily confirmed by directly computing the actions of the 12 elements of $\mathcal{W}$ on the simple roots α_1 and α_2.)

$\boxed{\S 2.12}$ At this point, we could use Theorem 2.20 to confirm that $|\mathcal{W}| = 12$, if we did not already know this by other means. The highest short root $\overline{\omega}_{short} = \omega_1$ is J^c-dominant for $J = \{2\}$. Then we have $|\Phi_{short}| = 2|\Phi^+_{short}| = 2\cdot 3 = 6$, and $|\mathcal{W}_J| = |\mathcal{W}_{\{2\}}| = 2$. Then $|\mathcal{W}| = 6\cdot 2 = 12$.

$\boxed{\S 2.13}$ From the numbers game played on $\mathfrak{g}^{\mathsf{T}}$ from the generic strongly dominant position (a,b), we see in Figure 2.14 that the terminal position is $(-a,-b)$. That is, $w_0.(a\omega_1^{\mathsf{T}} + b\omega_2^{\mathsf{T}}) = -a\omega_1^{\mathsf{T}} - b\omega_2^{\mathsf{T}}$. Since $\mathfrak{g}^{\mathsf{T}} \cong \mathfrak{g}$, we obtain that $w_0.(a\omega_1 + b\omega_2) = -a\omega_1 - b\omega_2$ for a generic strongly dominant weight $a\omega_1 + b\omega_2$. Then $w_0.\omega_1 = -\omega_1$ and $w_0.\omega_2 = -\omega_2$. In particular, the symmetry σ_0 of the Dynkin diagram $\mathfrak{g}$ is the identity.

Figure 2.14: The game sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ played on $\mathsf{G}_2^{\mathsf{T}}$ from a generic strongly dominant position (a, b).



$\boxed{\S 2.15}$ From Proposition 2.18 it follows that $s_1.\omega_1 = \omega_1 - \alpha_1 = -\omega_1 + \omega_2$, $s_1.\omega_2 = \omega_2$, $s_2.\omega_1 = \omega_1$, and $s_2.\omega_2 = \omega_2 - \alpha_2 = 3\omega_1 - \omega_2$. Let z_1 and z_2 denote the elements e_{ω_1} and e_{ω_2} of the group ring $\mathbb{Z}[\Lambda]$. In this notation, $s_1.z_1 = z_1^{-1}z_2$, $s_1.z_2 = z_2$, $s_2.z_1 = z_1$, and $s_2.z_2 = z_1^3 z_2^{-1}$.

Following Example 2.27, the adjoint and short adjoint characters are:

$$\begin{aligned}\chi_{\overline{\omega}} = \chi_{\omega_2} = \mathbf{char}(\omega_2; z_1, z_2) \quad &= \quad z_2 + z_1^3 z_2^{-1} + z_1 + z_1^{-1} z_2 + z_1^{-3} z_2^2 + z_1^2 z_2^{-1} \\ &\quad +2 + z_1^{-2} z_2 + z_1^3 z_2^{-2} + z_1 z_2^{-1} + z_1^{-1} + z_1^{-3} z_2 + z_2^{-1}\end{aligned}$$

$$\chi_{\overline{\omega}_{short}} = \chi_{\omega_1} = \mathbf{char}(\omega_1; z_1, z_2) \quad = \quad z_1 + z_1^{-1}z_2 + z_1^2 z_2^{-1} + 1 + z_1^{-2}z_2 + z_1 z_2^{-1} + z_1^{-1}$$

One can verify by hand that these polynomials are $\mathcal{W}$-invariant by using the prescribed action to see that s_1 and s_2 preserve each polynomial. For example,

$$\begin{aligned}
s_2.\chi_{\overline{\omega}_{short}} = s_2.\chi_{\omega_1} &= s_2.(z_1 + z_1^{-1}z_2 + z_1^2 z_2^{-1} + 1 + z_1^{-2}z_2 + z_1 z_2^{-1} + z_1^{-1}) \\
&= z_1 + z_1^{-1}(z_1^3 z_2^{-1}) + z_1^2(z_1^3 z_2^{-1})^{-1} + z_1^{-2}(z_1^3 z_2^{-1}) + z_1(z_1^3 z_2^{-1})^{-1} + z_1^{-1} \\
&= z_1 + z_1^2 z_2^{-1} + z_1^{-1}z_2 + 1 + z_1 z_2^{-1} + z_1^{-2}z_2 + z_1^{-1}
\end{aligned}$$

We note for the record that the alternating sums A_ϱ and $A_{\varrho+\lambda}$ can be written down directly using the definitions *since the Weyl group $\mathcal{W}$ is small for* G_2:

$$\begin{aligned}
A_\varrho &= z_1 z_2 - z_1^{-1}z_2^2 - z_1^4 z_2^{-1} + z_1^{-4}z_2^3 + z_1^5 z_2^{-2} - z_1^{-5}z_2^3 - z_1^5 z_2^{-3} + z_1^{-5}z_2^2 + z_1^4 z_2^{-3} \\
&\quad - z_1^{-4}z_2 - z_1 z_2^{-2} + z_1^{-1}z_2^{-1} \\
A_{\varrho+\lambda} &= z_1^{a+1}z_2^{b+1} - z_1^{-(a+1)}z_2^{a+b+2} - z_1^{a+3b+4}z_2^{-(b+1)} + z_1^{-(a+3b+4)}z_2^{a+2b+3} \\
&\quad + z_1^{2a+3b+5}z_2^{-(a+b+2)} - z_1^{-(2a+3b+5)}z_2^{a+2b+3} - z_1^{2a+3b+5}z_2^{-(a+2b+3)} \\
&\quad + z_1^{-(2a+3b+5)}z_2^{a+b+2} + z_1^{a+3b+4}z_2^{-(a+2b+3)} - z_1^{-(a+3b+4)}z_2^{b+1} \\
&\quad - z_1^{a+1}z_2^{-(a+b+2)} + z_1^{-(a+1)}z_2^{-(b+1)}
\end{aligned}$$

At this point, one could use a computer algebra system to quickly confirm that $A_\varrho \chi_{\omega_i} = A_{\varrho+\omega_i}$ for each $i = 1, 2$.

$\boxed{\S 2.16}$ We can compute the q-specialization of Proposition 2.29 for an irreducible $\mathfrak{g}$-character χ_λ as follows. Take $\lambda = a\omega_1 + b\omega_2 \in \Lambda^+$. Note that $\langle \omega_1, \varrho^\vee \rangle = \langle 2\alpha_1 + \alpha_2, \varrho^\vee \rangle = 3$ and $\langle \omega_2, \varrho^\vee \rangle = \langle 3\alpha_1 + 2\alpha_2, \varrho^\vee \rangle = 5$. Also, $-\langle w_0.\lambda, \varrho^\vee \rangle = -\langle -a\omega_1 - b\omega_2, \varrho^\vee \rangle = 3a + 5b$.

Then for any connected splitting poset for χ_λ we have

$$RGF(R,q) = q^{3a+5b}\mathbf{char}_{\mathfrak{g}}(\lambda; z_1, z_2)|_{z_1=q^3, z_2=q^5}$$

In the case of $\lambda = \omega_2$, we have

$$\begin{aligned} RGF(R,q) &= q^5\Big(q^5 + q^9q^{-5} + q^3 + q^{-3}q^5 + q^{-9}q^{10} + q^6q^{-5} \\ &\qquad +2 + q^{-6}q^5 + q^9q^{-10} + q^3q^{-5} + q^{-3} + q^{-9}q^5 + q^{-5}\Big) \\ &= q^{10} + q^9 + q^8 + q^7 + 2q^6 + 2q^5 + 2q^4 + q^3 + q^2 + q + 1 \end{aligned}$$

In the case of $\lambda = \omega_1$, we have

$$\begin{aligned} RGF(R,q) &= q^3(q^3 + q^{-3}q^5 + q^6q^{-5} + 1 + q^{-6}q^5 + q^3q^{-5} + q^{-3}) \\ &= q^6 + q^5 + q^4 + q^3 + q^2 + q + 1 \end{aligned}$$

Now on $\mathfrak{g}$ play the numbers game from initial position $(a+1, b+1)$, where a and b are nonnegative. For the game sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ played from this position, the numbers at the fired nodes are $a+1$, $a+b+2$, $2a+3b+5$, $a+2b+3$, $a+3b+4$, and $b+1$ respectively. See Figure 2.15. Then using Remark 2.38 together with Theorem 2.30, we have the following formula for the rank generating function for any connected splitting poset R for the $\mathfrak{g}$-character χ_λ with $\lambda = a\omega_1 + b\omega_2 \in \Lambda^+$:

$$RGF(R,q) = \frac{(1-q^{2a+3b+5})(1-q^{a+3b+4})(1-q^{a+2b+3})(1-q^{a+b+2})(1-q^{b+1})(1-q^{a+1})}{(1-q^5)(1-q^4)(1-q^3)(1-q^2)(1-q)(1-q)}$$

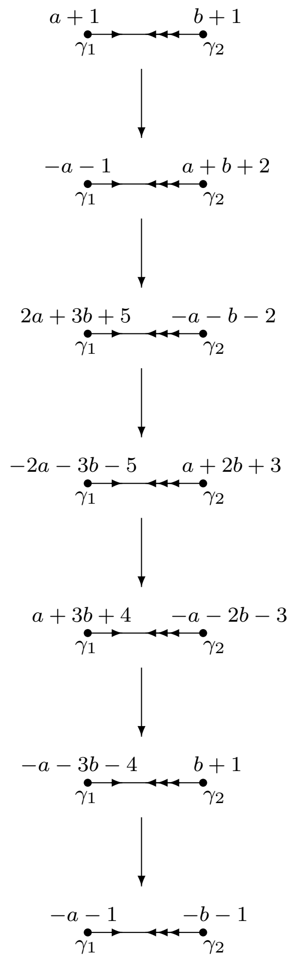
It follows from Corollary 2.31 that the dimension of χ_λ is

$$\frac{(2a+3b+5)(a+3b+4)(a+2b+3)(a+b+2)(b+1)(a+1)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}$$

and from Corollary 2.32 that the length of R is

$$\ell(\lambda) = (2a+3b) + (a+3b) + (a+2b) + (a+b) + a + b = 6a + 10b.$$

Figure 2.15: The game sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ played on G_2 from a position $(a+1, b+1)$ with a and b nonnegative.

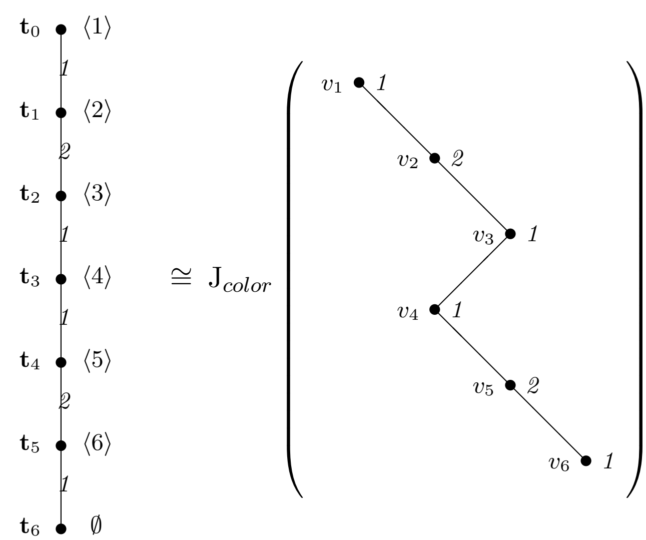


Now consider the short adjoint character, which is the fundamental character χ_{ω_1}. In this case, note from our computation above that each coefficient $c_{\omega_1,\mu}$ in the character polynomial is unity. From Facts 2.26 and Example 2.33 it follows that the maximal splitting poset $\mathcal{M}(\omega_1)$ coincides with $\Pi(\omega_1)$, as depicted in Figure 2.16. Check that in this case, no edges can be removed from $\mathcal{M}(\omega_1)$ without violating the $\mathfrak{g}$-structure property. Thus $\mathcal{M}(\omega_1)$ is the unique splitting poset for χ_{ω_1}. In particular, $\mathcal{M}(\omega_1)$ coincides with the SDL built from Φ_{short} in Example 2.34. The vertex-colored poset of irreducibles P_{ω_1} is also depicted

Figure 2.16: $\mathcal{M}(\omega_1) = \Pi(\omega_1)$ is edge-color isomorphic to $\mathrm{J}_{color}(P_{\omega_1})$.

Order ideals are notated as in Figure 2.7. A weight $p\omega_1 + q\omega_2$ is denoted (p, q).

Vertex	$\mathbf{t}_0$	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$	$\mathbf{t}_5$	$\mathbf{t}_6$
Weight	$(1,0)$	$(1,-1)$	$(-2,1)$	$(0,0)$	$(2,-1)$	$(-1,1)$	$(-1,0)$
Root	$2\alpha_1+\alpha_2$	$\alpha_1+\alpha_2$	α_1	NA	$-\alpha_1$	$-\alpha_1-\alpha_2$	$-2\alpha_1-\alpha_2$



in Figure 2.16. Next we consider the adjoint character χ_{ω_2}. In this case, we can build two SDL's using Example 2.34. These are depicted in Figures 2.17 and 2.18, along with their vertex-colored posets of irreducibles. The poset of irreducibles depicted in Figure 2.17 is designated as P_{ω_2} for reasons explained in the next paragraph.

Certain distributive lattice orderings of Littelmann's G_2-tableaux [Lit] were found by Donnelly. The main result of [Mc] was to confirm Donnelly's conjecture that these lattices are SDL's for the irreducible $\mathfrak{g}$-characters. Using ideas related to [DW], these G_2 lattices are constructed in [ADLMPPW] by 'stacking' the posets of irreducibles denoted P_{ω_1} and P_{ω_2}. For a dominant weight $\lambda = a\omega_1 + b\omega_2$, one 'stacks' a copies of P_{ω_1} 'on top of' b copies of P_{ω_2}, or alternatively one stacks b copies of P_{ω_2} on top of a copies of P_{ω_1}. (See Figures 2.19 and 2.20 for the $a = 2, b = 2$ cases.) These are posets of irreducibles for two 'G_2-semistandard' lattices denoted $L_{\mathsf{G}_2}^{\beta\alpha}(\lambda)$ and $L_{\mathsf{G}_2}^{\alpha\beta}(\lambda)$. These SDL's for χ_λ are related by the recolored dual: $L_{\mathsf{G}_2}^{\alpha\beta}(\lambda) \cong (L_{\mathsf{G}_2}^{\beta\alpha}(\lambda))^{\triangle}$.

Figure 2.17: An SDL for χ_{ω_2} identified as $\mathrm{J}_{color}(P_{\omega_2})$ for a vertex-colored poset P_{ω_2}. Order ideals are notated as in Figure 2.7. A weight $p\omega_1 + q\omega_2$ is denoted (p, q).

Vertex	$\mathbf{t}_0$	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$	$\mathbf{t}_5$	$\mathbf{t}_6$
Weight	$(0,1)$	$(3,-1)$	$(1,0)$	$(-1,1)$	$(-3,2)$	$(2,-1)$	$(0,0)$
Root	$3\alpha_1+2\alpha_2$	$3\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	$\alpha_1+\alpha_2$	α_2	α_1	NA

Vertex	$\mathbf{t}_7$	$\mathbf{t}_8$	$\mathbf{t}_9$	$\mathbf{t}_{10}$	$\mathbf{t}_{11}$	$\mathbf{t}_{12}$	$\mathbf{t}_{13}$
Weight	$(0,0)$	$(3,-2)$	$(-2,1)$	$(1,-1)$	$(-1,0)$	$(-2,1)$	$(0,-1)$
Root	NA	$-\alpha_2$	$-\alpha_1$	$-\alpha_1-\alpha_2$	$-2\alpha_1-\alpha_2$	$-3\alpha_1-\alpha_2$	$-3\alpha_1-2\alpha_2$

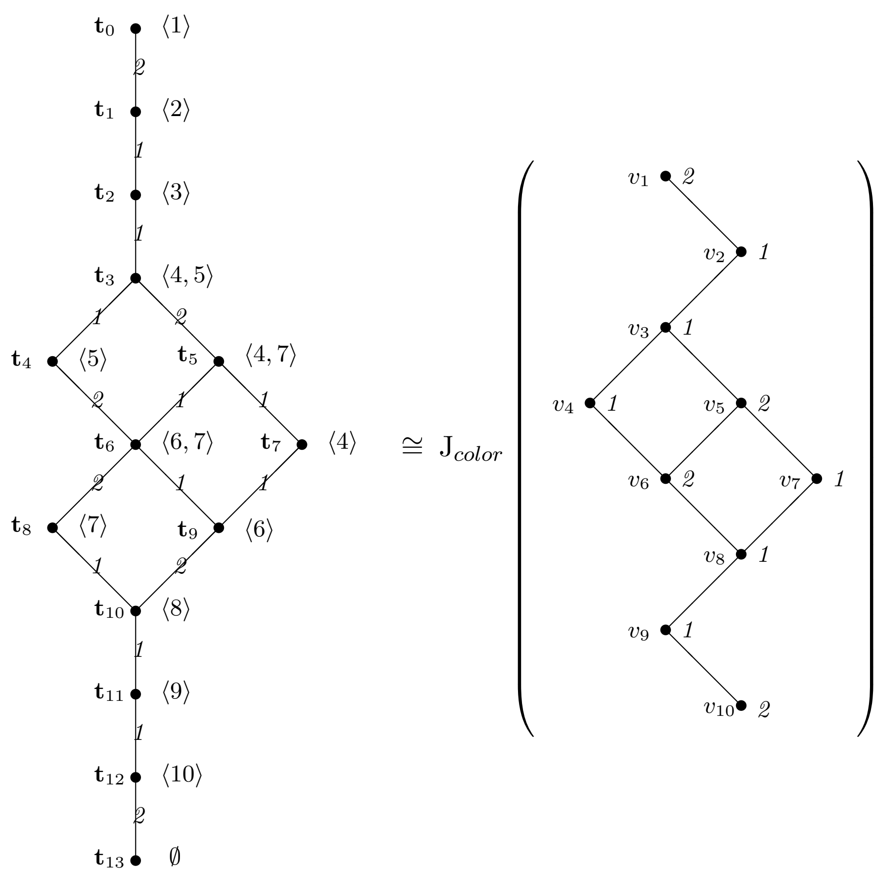


Figure 2.18: An SDL for χ_{ω_2} identified as $\mathrm{J}_{color}(Q)$ for a vertex-colored poset Q. Order ideals are notated as in Figure 2.7. A weight $p\omega_1 + q\omega_2$ is denoted (p, q).

Vertex	$\mathbf{t}_0$	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$	$\mathbf{t}_5$	$\mathbf{t}_6$
Weight	$(0,1)$	$(3,-1)$	$(1,0)$	$(-1,1)$	$(-3,2)$	$(2,-1)$	$(0,0)$
Root	$3\alpha_1+2\alpha_2$	$3\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	$\alpha_1+\alpha_2$	α_2	α_1	NA

Vertex	$\mathbf{t}_7$	$\mathbf{t}_8$	$\mathbf{t}_9$	$\mathbf{t}_{10}$	$\mathbf{t}_{11}$	$\mathbf{t}_{12}$	$\mathbf{t}_{13}$
Weight	$(0,0)$	$(3,-2)$	$(-2,1)$	$(1,-1)$	$(-1,0)$	$(-2,1)$	$(0,-1)$
Root	NA	$-\alpha_2$	$-\alpha_1$	$-\alpha_1-\alpha_2$	$-2\alpha_1-\alpha_2$	$-3\alpha_1-\alpha_2$	$-3\alpha_1-2\alpha_2$

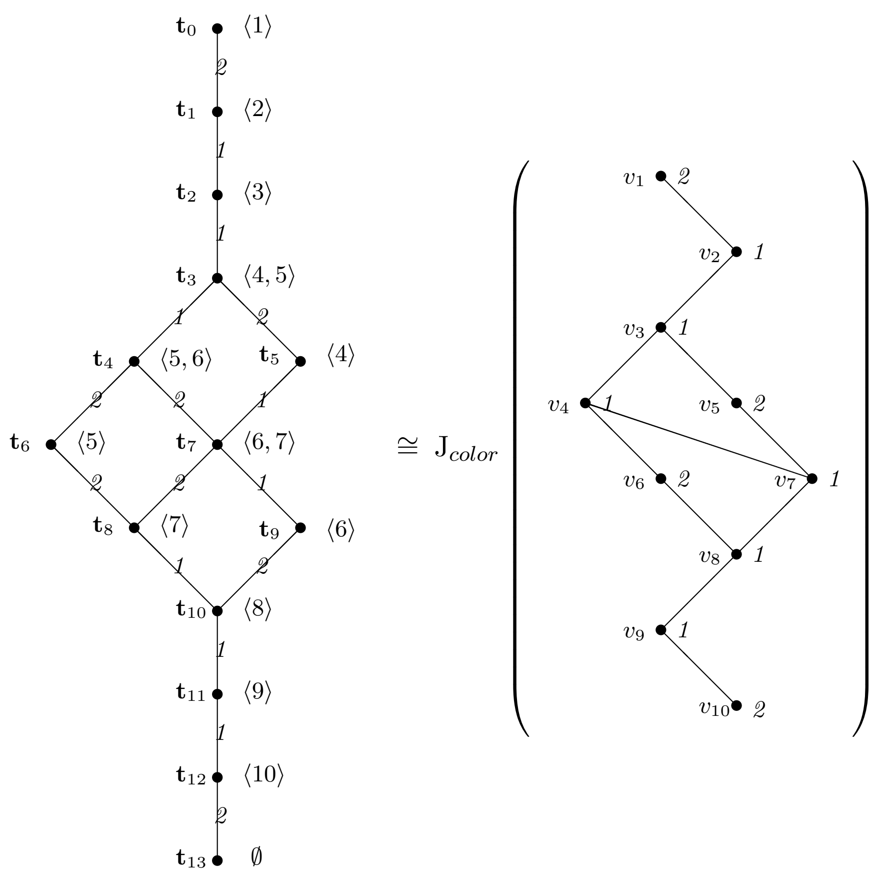


Figure 2.19: The stacking $P := P_{\omega_2} \lhd P_{\omega_2} \lhd P_{\omega_1} \lhd P_{\omega_1}$ of fundamental posets P_{ω_1} and P_{ω_2}. Theorem 5.3 of [ADLMPPW] shows that $\mathrm{J}_{color}(P)$ is an SDL for the G_2-character $\chi_{2\omega_1+2\omega_2}$.

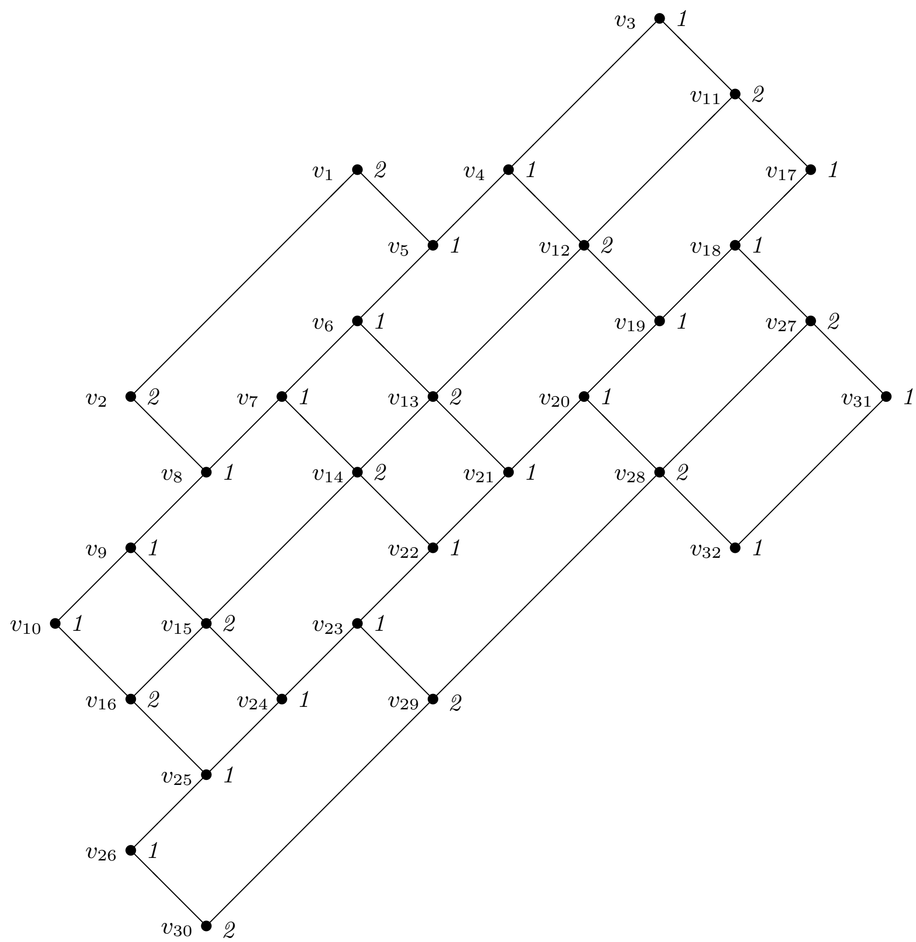
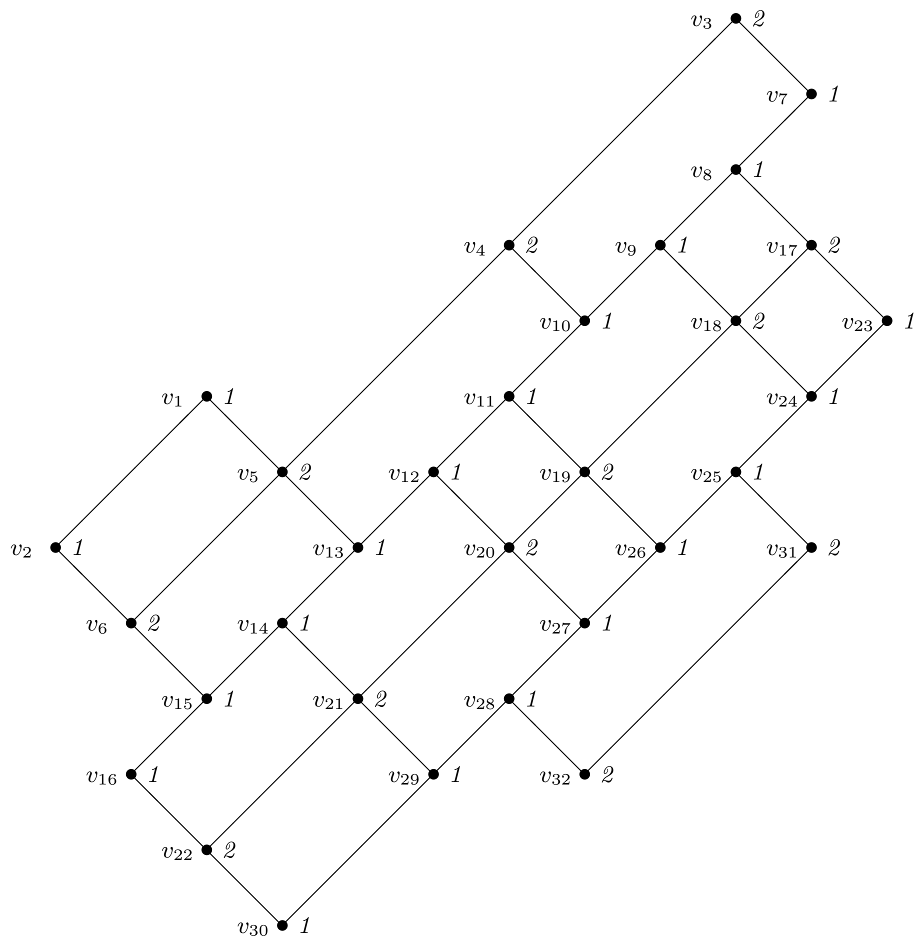


Figure 2.20: The stacking $Q := P_{\omega_1} \triangleleft P_{\omega_1} \triangleleft P_{\omega_2} \triangleleft P_{\omega_2}$ of fundamental posets P_{ω_1} and P_{ω_2}. Theorem 5.3 of [ADLMPPW] shows that $\mathrm{J}_{color}(Q)$ is an SDL for the G_2-character $\chi_{2\omega_1+2\omega_2}$.



CHAPTER 3: THE DISTRIBUTIVE CORE

In our search for splitting distributive lattices of type F_4, it was necessary to develop new methods. One such method was that of the 'distributive core'. For a given Dynkin diagram $\mathfrak{g} = (\Gamma, M)$ and a dominant weight λ (see Chapter 2 for definitions), the distributive core (when it exists) is a diamond-colored distributive lattice generated by a certain algorithm. When the algorithm returns nonempty output, the result is in fact a $(\mathfrak{g}, \lambda)$-structured distributive lattice obtained by 'working down' from a maximal vertex of weight λ in a natural way. Our goal was for this procedure to generate a 'minimal' $(\mathfrak{g}, \lambda)$-structured distributive lattice. Although we cannot prove minimality of the resulting object at this time, computational evidence suggests that the algorithm's output indeed has the desired minimality property. It is for this reason that we call the object a 'core'. (For further discussion of this point see Chapter 6.)

The distributive core concept has proven to be very useful for investigating SDL's of type F_4. In Chapter 5 we will generate two new splitting distributive lattices for irreducible F_4-characters with dimensions 324 and 1053. Both of the SDL's contain the distributive core as a full length distributive sublattice (cf. Theorem 2.6). In Chapter 5, we also use this idea of working down from a maximal vertex to show that two of the fundamental F_4-characters have no splitting distributive lattices.

By Proposition 2.24, a $(\mathfrak{g}, \lambda)$-structured distributive lattice L has $w_0.\lambda$ as the weight of its unique minimal element. In the notation of §2.14 and §2.16, the length of L is therefore $\ell(\lambda) = \langle \lambda - w_0.\lambda, \varrho^\vee \rangle = \sum_{\alpha \in \Phi^+} \langle \lambda, \alpha^\vee \rangle$. This explains one of the terminating conditions of Step 4 in the following algorithm. The other terminating condition of Step 4 is explained by Proposition 2.8: for any i-maximal $\mathbf{t} \in L$, $\mathbf{comp}_i(\mathbf{t})$ is distributive. Within this component, each descendant of $\mathbf{t}$ is meet irreducible. Since the length of a distributive lattice is the number of meet irreducible elements, then

$$m_i(\mathbf{t}) = l_i(\mathbf{t}) = |\{\text{meet irreducible elements of } \mathbf{comp}_i(\mathbf{t})\}| \geq |\{\mathbf{s} \in \mathbf{comp}_i(\mathbf{t}) | \mathbf{s} \xrightarrow{i} \mathbf{t}\}|.$$

Algorithm 3.1 (The Distributive Core)

Input: $\mathfrak{g} = (\Gamma, M)$, a connected Dynkin diagram of finite type with nodes indexed by an n-element set I_n, and $\lambda = \sum_{i \in I_n} a_i \omega_i$, a dominant weight.

Initialize: $Q := \emptyset$ and $K := \mathrm{M}_{color}(Q)$ with edge colors from I_n. For the unique $\mathbf{t}_0$ of K, $\nu_K(\mathbf{t}_0) := \lambda$, with $\nu_K^{(i)}(\mathbf{t}_0) := a_i = \langle \lambda, \alpha_i^\vee \rangle$. Let $\rho_K^{(i)}$ and $\delta_K^{(i)}$ be the rank and depth functions, respectively, for the color $i \in I_n$, and ρ_K and δ_K be the overall rank and depth functions in K. Let $d := 0$, the largest vertex index so far. For any $\mathbf{y}$ we will let $\mathcal{D}_K^{(i)}(\mathbf{y}) := \{\mathbf{x} | \mathbf{x} \xrightarrow{i} \mathbf{y}\}$.

Begin Procedure:

for l **from** 0 **by** 1 **do**

  let $\mathcal{K}_l$:=$\delta_K^{-1}(l)$

  if $\mathcal{K}_l$=$\emptyset$ **then** RETURN(K) **end if**

  for t in $\mathcal{K}_l$ **do**

    for $i \in I_n$ **do**

if $\mathcal{D}_K^{(i)}(\mathbf{t}) = \emptyset$ **and** $\rho_K^{(i)}(\mathbf{t}) - \delta_K^{(i)}(\mathbf{t}) \neq \nu_K^{(i)}(\mathbf{t})$ **then**

Step 1: *Create a new vertex-colored poset of irreducibles P.* The new vertex is v_{d+1}. The new poset is $P := Q \cup \{v_{d+1}\}$ with partial order $v \leq_P w$ if and only if v, w are in Q and $v \leq_Q w$ or $v = v_{d+1}$ and $w = v_j$ with $\mathbf{t} \leq_K \mathbf{t}_j$. Let $\mathbf{vertexcolor}_P(v)$ be $\mathbf{vertexcolor}_Q(v)$ if v is in Q and i if $v = v_{d+1}$.

Step 2: *Create a new lattice L corresponding to the new poset from Step 1.* Let $L := \mathrm{M}_{color}(P)$. Note that a filter from Q is a filter from P and thus K is a sublattice of L. Use $\mathbf{t}_0,\ldots,\mathbf{t}_d$ to denote those elements of L which coincide as subsets of P with the filters from Q comprising K, and let $\mathbf{t}_{d+1}$ be the filter with the unique minimal element v_{d+1}. The remaining elements of L are denoted $\mathbf{t}_{d+2},\ldots$, in some order.

Step 3: *Replace and recalculate the functions for the new lattice.* Now replace Q by the vertex-colored poset P and the edge-colored K by the new L. Extend ν_K to the new elements $\mathbf{t}_{d+1}, \ldots$ by declaring $\nu_K(\mathbf{t}_p) = \nu_L(\mathbf{t}_q) - \alpha_i$ where $\mathbf{t}_q$ is a filter from P for which $\mathbf{t}_p = \mathbf{t}_q \cup \{v_{d+1}\}$. Then $\nu_K^{(i)}(\mathbf{t}_p) := \langle \nu_K(\mathbf{t}_q), \alpha_i^\vee \rangle$, which is the coefficient of $\nu_K(\mathbf{t}_q)$ as a linear combination in the basis $\{\omega_i\}$ of fundamental weights. Recalculate $\rho_K(\mathbf{t}_p)$, $\delta_K(\mathbf{t}_p)$, $\mathcal{D}_K^{(i)}(\mathbf{t}_p)$, $\rho_K^{(i)}(\mathbf{t}_p)$, $\delta_K^{(i)}(\mathbf{t}_p)$ for all t_p in K and $i \in I_n$. Then reassign $d := |K|$.

Step 4: *Check two necessary conditions for K to be a $(\mathfrak{g}, \lambda)$-structured distributive lattice.*

if $|Q| > \ell(\lambda)$ **then**

RETURN($\emptyset$)

end if

for $\mathbf{t} \in K$ **do**

if $\mathbf{t}$ is i-maximal **and** $\nu_K^{(i)}(\mathbf{t}) < |\mathcal{D}_K^{(i)}(\mathbf{t})|$ **then**

RETURN($\emptyset$)

end if

end do

end if

end do

end do

end do

Output: The empty set or a diamond-colored distributive lattice.

Lemma 3.2 *For a given Dynkin diagram $\mathfrak{g}$ and dominant weight $\lambda = \sum_{i \in I_n} a_i\omega_i$, suppose the output $K := K(\mathfrak{g}, \lambda)$ of the distributive core algorithm is nonempty. (1) Whenever $\mathbf{x} \xrightarrow{i} \mathbf{y}$ in K it is the case that $\nu_K(\mathbf{x}) = \nu_K(\mathbf{y}) - \alpha_i$. (2) For the maximal element $\mathbf{t}_0$ of K, for each $i \in I_n$, $\mathbf{comp}_i(\mathbf{t}_0)$ is a chain of length a_i and each $\mathbf{t} \in \mathbf{comp}_i(\mathbf{t}_0)$ with $\mathbf{t} \neq \mathbf{t}_0$ is a meet irreducible element of K.*

Proof. For (1), we show by induction on $\delta_K(\mathbf{x})$ that whenever $\mathbf{x} \xrightarrow{i} \mathbf{y}$ in K we have $\nu_K(\mathbf{x}) = \nu_K(\mathbf{y}) - \alpha_i$. First consider the case that $\delta_K(\mathbf{x}) = 1$ and suppose that $\mathbf{x} \xrightarrow{i} \mathbf{y}$. The algorithm assigns $\nu_K(\mathbf{x})$ a vector value by finding a $\mathbf{z} \in K$ such that $\mathbf{x} \xrightarrow{j} \mathbf{z}$ and declaring that $\nu_K(\mathbf{x}) := \nu_K(\mathbf{z}) - \alpha_j$. But $\delta_K(\mathbf{x}) = 1$ means that $\delta_K(\mathbf{y}) = \delta_K(\mathbf{z}) = 0$, and hence $\mathbf{y} = \mathbf{z} = \mathbf{t}_0$, the unique maximal element of K. Then $j = i$ as well. In particular $\nu_K(\mathbf{x}) = \nu_K(\mathbf{y}) - \alpha_i$.

For our induction hypothesis, suppose that $\nu_K(\mathbf{s}) = \nu_K(\mathbf{t}) - \alpha_i$ whenever $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in K with $\delta_K(\mathbf{s}) \leq k$ for some positive integer k. If $\mathbf{x} \xrightarrow{i} \mathbf{y}$ in K with $\delta_K(\mathbf{x}) = k + 1$, let $\mathbf{z}$ be the element of K chosen by the algorithm such that $\mathbf{x} \xrightarrow{j} \mathbf{z}$ and $\nu_K(\mathbf{x}) := \nu_K(\mathbf{z}) - \alpha_j$. If $\mathbf{y} = \mathbf{z}$, then $j = i$ and $\nu_K(\mathbf{x}) := \nu_K(\mathbf{y}) - \alpha_i$ as desired. So assume $\mathbf{y} \neq \mathbf{z}$. Then in the diamond-colored distributive lattice K, there must be a unique element $\mathbf{w}$ such that we have the following diamond of edges:

$$\begin{tikzcd}[column sep=small, row sep=small] & \mathbf{w} & \\ \mathbf{y} \arrow[ur, "j"] & & \mathbf{z} \arrow[ul, "i"'] \\ & \mathbf{x} \arrow[ul, "i"] \arrow[ur, "j"'] & \end{tikzcd}.$$

Then $\nu_K(\mathbf{x}) = \nu_K(\mathbf{z}) - \alpha_j = \nu_K(\mathbf{w}) - \alpha_i - \alpha_j = \nu_K(\mathbf{y}) + \alpha_j - \alpha_i - \alpha_j = \nu_K(\mathbf{y}) - \alpha_i$. This completes the induction step, and the proof of (1).

For (2), first observe that after the algorithm finishes with a vertex $\mathbf{t}$ in the "**for** $\mathbf{t} \in \mathcal{K}_l$ **do**" loop, no subsequent steps of the algorithm will produce descendants for $\mathbf{t}$. Also observe that meet irreducible elements in K are only generated by Step 1 of the algorithm. Taken together this means that for any $\mathbf{t} \in K$ and $i \in I_n$, if $\mathcal{D}_K^{(i)}(\mathbf{t})$ contains a meet irreducible element $\mathbf{s}$, then $\mathcal{D}_K^{(i)}(\mathbf{t}) = \{\mathbf{s}\}$.

Now fix a color $i \in I_n$. By the observations of the previous paragraph, the maximal element $\mathbf{t}_0$ has a descendant $\mathbf{s}$ with $\mathbf{s} \xrightarrow{i} \mathbf{t}_0$ if and only if $a_i > 0$, in which case $\mathcal{D}_K^{(i)}(\mathbf{t}_0) = \{\mathbf{s}\}$ and $\mathbf{s}$ is generated by Step 1 as a meet irreducible element of K. In particular, if $a_i = 0$, then $\mathbf{comp}_i(\mathbf{t}_0) = \{\mathbf{t}_0\}$, and we are done. To complete the proof of (2), it suffices to consider the case that $a_i > 0$. We use induction on $\delta_K(\mathbf{s})$ to establish the following claim: if $\mathbf{s} \in \mathbf{comp}_i(\mathbf{t}_0)$ and $\mathbf{s} \neq \mathbf{t}_0$, then $\mathbf{s}$ is a meet irreducible element of K. We have already observed that the claim holds when $\delta_K(\mathbf{s}) = 1$. Now assume that the claim is true whenever $1 \leq \delta_K(\mathbf{s}) \leq k$ for some positive integer $k \leq l_i(\mathbf{t}_0)$. In this case, note that there is a unique element $\mathbf{t}_p$ in $\mathbf{comp}_i(\mathbf{t}_0)$ with $\delta_K(\mathbf{t}_p) = p$, for all $0 \leq p \leq k$. If $k < l_i(\mathbf{t}_0)$, consider $\mathbf{t}_k$. Suppose $\mathbf{x} \stackrel{i}{\to} \mathbf{t}_k$ for some $\mathbf{x}$ in K. We wish to show that $\mathbf{x}$ is meet irreducible in K, so for our contradiction hypothesis we suppose that $\mathbf{x}$ is not meet irreducible in K. So in particular, let $\mathbf{x} \stackrel{j}{\to} \mathbf{y}$ for some $\mathbf{y} \neq \mathbf{t}_k$. In the diamond-colored distributive lattice K, it must be the case that there is a unique $\mathbf{u}$ such that $\mathbf{t}_k \stackrel{j}{\to} \mathbf{u}$ and $\mathbf{y} \stackrel{i}{\to} \mathbf{u}$. But since $\mathbf{t}_k$ is meet irreducible in K, then $\mathbf{u} = \mathbf{t}_{k-1}$ and $i = j$. So now $\mathcal{D}_K^{(i)}(\mathbf{t}_{k-1})$ has at least two elements, namely $\mathbf{t}_k$ and $\mathbf{y}$. But this means that neither $\mathbf{t}_k$ nor $\mathbf{y}$ can be meet irreducible in K, a contradiction. Thus $\mathbf{x}$ is meet irreducible in K. This completes our induction argument for our claim.

Since all elements of $\mathbf{comp}_i(\mathbf{t}_0)$ other than $\mathbf{t}_0$ are meet irreducible, it follows that for each $\mathbf{s} \in \mathbf{comp}_i(\mathbf{t}_0)$ with $\delta_K(\mathbf{s}) < l_i(\mathbf{t}_0)$, $\mathcal{D}_K^{(i)}(\mathbf{s})$ has exactly one element (namely its meet irreducible descendant). In particular, $\mathbf{comp}_i(\mathbf{t}_0)$ is a chain. So when the algorithm encounters vertex $\mathbf{t}_p \in \mathbf{comp}_i(\mathbf{t}_0)$ with $0 \leq p < a_i$, it will generate a new meet irreducible $\mathbf{t}_{p+1}$ with $\mathbf{t}_{p+1} \stackrel{i}{\to} \mathbf{t}_p$. This will continue until $p = a_i$, at which point the algorithm encounters vertex $\mathbf{t}_{a_i}$ with no color i descendants and with no need to generate any. Therefore, the length of the chain $\mathbf{comp}_i(\mathbf{t}_0)$ is a_i. $\square$

Proposition 3.3 *The output of the distributive core algorithm does not depend on the ordering of elements in the level sets $\mathcal{K}_l$ or on the ordering of elements in the index set I_n.*

Proof. In the notation of Algorithm 3.1, suppose that the algorithm generates two meet irreducible elements $\mathbf{t}_q$ and $\mathbf{t}_r$ in succession as descendants of vertices $\mathbf{q}$ and $\mathbf{r}$ respectively, with $\mathbf{q}$ and $\mathbf{r}$ in the level set $\mathcal{K}_l$. The algorithm generates a new meet irreducible element at Steps 1 – 3, so in this discussion we closely follow what the algorithm does at these steps.

Assuming $\mathbf{t}_q$ is generated first and $\mathbf{t}_r$ is generated next, then $q < r$ for the indexing integers q and r. We allow the possibility that $\mathbf{q} = \mathbf{r}$. We assume that once the meet irreducible elements $\mathbf{t}_q$ and $\mathbf{t}_r$ are generated, they descend from $\mathbf{q}$ and $\mathbf{r}$ along edges of colors i_q and i_r respectively, i.e. $\mathbf{t}_q \xrightarrow{i_q} \mathbf{q}$ and $\mathbf{t}_r \xrightarrow{i_r} \mathbf{r}$.

Let P be the poset of meet irreducibles produced by the algorithm just before the meet irreducible $\mathbf{t}_q$ is generated, and let $L := \mathrm{J}_{color}(P)$. The algorithm uses the new meet irreducible element $\mathbf{t}_q$ to produce the following vertex-colored poset of meet irreducibles P' at Step 1: The elements of P' are $P \cup \{v_q\}$. The vertex color function is given by $\mathbf{vertexcolor}_{P'}(u) = \mathbf{vertexcolor}_P(u)$ if $u \in P$ and $\mathbf{vertexcolor}_{P'}(v_q) = i_q$. The partial order on P' is $u \leq_{P'} v$ if and only if $u, v \in P$ and $u \leq_P v$ or $u = v_q$, $v = v_p$ for some $p < q$, and $\mathbf{q} \leq_L \mathbf{t}_p$. Let $L' := \mathrm{J}_{color}(P')$, and perform the necessary updating of indices, functions, etc. as in Steps 2, 3. The algorithm then uses the new meet irreducible element $\mathbf{t}_r$ to produce the following vertex-colored poset of meet irreducibles P'' upon returning to Step 1: The elements of P'' are $P' \cup \{v_r\}$. The vertex color function is given by $\mathbf{vertexcolor}_{P''}(u) = \mathbf{vertexcolor}_{P'}(u)$ if $u \in P'$ and $\mathbf{vertexcolor}_{P''}(v_r) = i_r$. The partial order on P'' is $u \leq_{P''} v$ if and only if $u, v \in P'$ and $u \leq_{P'} v$ or $u = v_r$, $v = v_p$ for some $p < r$, and $\mathbf{r} \leq_{L'} \mathbf{t}_p$. If $\mathbf{r} \leq_{L'} \mathbf{t}_p$, we claim that $p < q$, $\mathbf{t}_p \in L$, and $\mathbf{r} \leq_L \mathbf{t}_p$. To see this, first note that $\mathbf{r} \in L$ since $\mathbf{r}$ is assumed to be in $\mathcal{K}_l$, all of whose elements were in place by the time the algorithm produced P. Suppose now that $p \geq q$. Then $\{v_q\} \subseteq \mathbf{t}_p$. And so $\mathbf{r} \leq_{L'} \mathbf{t}_p$ now means that $\mathbf{r} \supseteq \mathbf{t}_p \supseteq \{v_q\}$. However, $\mathbf{r} \in L$ implies that $\mathbf{r} \subseteq P$, and since $v_q \notin P$, we have a contradiction. Thus $p < q$, hence $\mathbf{t}_p \in L$, and so $\mathbf{r} \subseteq \mathbf{t}_p$ means that $\mathbf{r} \leq_L \mathbf{t}_p$.

We therefore have the following description of the vertex-colored poset of meet irreducibles P'': The elements of P'' are $(P \cup \{v_q\}) \cup \{v_r\}$. The vertex color function is given by $\mathbf{vertexcolor}_{P''}(u) = \mathbf{vertexcolor}_P(u)$ if $u \in P$, $\mathbf{vertexcolor}_{P''}(v_q) = i_q$, and $\mathbf{vertexcolor}_{P''}(v_r) = i_r$. The partial order on P'' is $u \leq_{P''} v$ if and only if (1) $u, v \in P$ and $u \leq_P v$; (2) $u = v_q$, $v = v_p$ for some $p < q$, and $\mathbf{q} \leq_L \mathbf{t}_p$; or (3) $u = v_r$, $v = v_p$ for some $p < q$, and $\mathbf{r} \leq_L \mathbf{t}_p$. Given this description of P'', it is evident that reversing the order in which the algorithm generates the meet irreducible elements $\mathbf{t}_q$ and $\mathbf{t}_r$, i.e. $\mathbf{t}_r$ first and then $\mathbf{t}_q$, results in an isomorphic vertex-colored poset of meet irreducibles. Therefore the output of the algorithm does not depend on the order in which the algorithm generates meet irreducibles as it traverses any given level set $\mathcal{K}_l$. □

Remark The above proof actually demonstrates something stronger than the statement of Proposition 3.3: *The output of the distributive core algorithm does not depend on the order in which meet irreducible elements are generated along a given level set* $\mathcal{K}_l$.

Theorem 3.4 *For a given Dynkin diagram* $\mathfrak{g}$ *and dominant weight* $\lambda = \sum_{i \in I_n} a_i \omega_i$, *suppose the output* $K := K(\mathfrak{g}, \lambda)$ *of the distributive core algorithm is nonempty. Then* K *is a* $(\mathfrak{g}, \lambda)$*-structured distributive lattice.*

Proof. Using induction on $\delta_K(\mathbf{x})$ we will show that $\nu_K(\mathbf{x}) = wt_K(\mathbf{x})$ and if $\mathbf{x} \xrightarrow{i} \mathbf{y}$ then $wt_K(\mathbf{x}) + \alpha_i = wt_K(\mathbf{y})$. If $\delta_K(\mathbf{x}) = 0$, then $\mathbf{x} = \mathbf{t}_0$, the unique maximal element of K. Lemma 3.2.2 shows that $\nu_K(\mathbf{x}) = wt_K(\mathbf{x})$ in this case. Since $\mathbf{x}$ is maximal, there is no $\mathbf{y} \in K$ such that $\mathbf{x} \xrightarrow{i} \mathbf{y}$.

For our induction hypothesis, suppose that for some nonnegative integer k and all $\mathbf{s}$ such that $\delta_K(\mathbf{s}) \leq k$, we have $\nu_K(\mathbf{s}) = wt_K(\mathbf{s})$ and $wt_K(\mathbf{s}) + \alpha_i = wt_K(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in K. Now let $\mathbf{x} \in K$ have depth $\delta_K(\mathbf{x}) = k + 1$, and suppose that $\mathbf{x} \xrightarrow{i} \mathbf{y}$. By the induction hypothesis, $\nu_K(\mathbf{y}) = wt_K(\mathbf{y})$. By Lemma 3.2.1, $\nu_K(\mathbf{x}) = \nu_K(\mathbf{y}) - \alpha_i$. At this point it is enough to show that for all $j \in I_n$,

$$(3) \qquad m_j(\mathbf{x}) + M_{ij} = m_j(\mathbf{y}).$$

Note that $m_i(\mathbf{x}) + M_{ii} = m_i(\mathbf{x}) + 2 = m_i(\mathbf{y})$, which verifies equation (3) when $j = i$. Now choose $j \neq i$. There are two cases to consider: (1) $\mathbf{x}$ is not j-maximal, and (2) $\mathbf{x}$ is j-maximal.

Case 1: Since $\mathbf{x}$ is not j-maximal then there is some vertex $\mathbf{z}$ such that $\mathbf{x} \xrightarrow{j} \mathbf{z}$. Since $\mathbf{y}$ and $\mathbf{z}$ have depth k they have $wt_K(\mathbf{z}) = \nu_K(\mathbf{z})$ and $\nu_K(\mathbf{x}) + \alpha_i = \nu_K(\mathbf{y})$. However, since $\mathbf{y}$ and $\mathbf{z}$ connect to $\mathbf{x}$ then there exists a $\mathbf{w}$ above $\mathbf{y}$ and $\mathbf{z}$ with connecting edges of color j and i respectively. Since $\mathbf{w}$ has depth $k-1$ then $wt_K(\mathbf{w}) = \nu_K(\mathbf{w})$ and $wt_K(\mathbf{z}) + \alpha_i = wt_K(\mathbf{w})$ and $wt_K(\mathbf{y}) + \alpha_j = wt_K(\mathbf{w})$. We know that $m_j(\mathbf{x}) + 2 = m_j(\mathbf{z})$, $m_j(\mathbf{z}) + M_{ij} = m_j(\mathbf{w})$, and $m_j(\mathbf{w}) = m_j(\mathbf{y}) + 2$. Putting these together we get $m_j(\mathbf{x}) + 2 + M_{ij} = m_j(\mathbf{y}) + 2$ which implies that $m_j(\mathbf{x}) + M_{ij} = m_j(\mathbf{y})$.

Case 2: Since $\mathbf{x}$ is j-maximal it suffices to show that $\nu_K^{(j)}(\mathbf{x}) = \rho_K^{(j)}(\mathbf{x})$. Then assume that $\nu_K^{(j)}(\mathbf{x}) \neq \rho_K^{(j)}(\mathbf{x})$. Then for all $\mathbf{z} \in \mathbf{comp}_j(\mathbf{x})$, one can see that $\nu_K^{(j)}(\mathbf{z}) \neq \rho_K^{(j)}(\mathbf{z}) - \delta_K^{(j)}(\mathbf{z})$. Since the j-component of $\mathbf{x}$ is finite then there exists a minimal vertex $\mathbf{t}$ with $\mathcal{D}_K^{(i)}(\mathbf{t}) = \emptyset$ and $\nu_K^{(j)}(\mathbf{t}) \neq \rho_K^{(j)}(\mathbf{t}) - \delta_K^{(j)}(\mathbf{t})$. However, Steps 1-3 of the procedure would at this point produce an edge of color j below $\mathbf{t}$, thus violating the minimality of $\mathbf{t}$. $\square$

CHAPTER 4: F_4 SPECIFICS

From the ideas in Chapter 2, we will compute some of the fundamental data (roots, Weyl group size, rank generating function, dimension formula, length formula) for the Dynkin diagram $\mathfrak{g} = \mathsf{F}_4$. (For some similar computations for G_2 see §2.18.) The Cartan matrix M for F_4 and its inverse M^{-1} are

$$M = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}$$

The Weyl group $\mathcal{W} := \mathcal{W}_{\mathsf{F}_4}$ has the following presentation by generators and relations: $\langle s_1, s_2, s_3, s_4 | s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1s_2)^3 = (s_2s_3)^4 = (s_3s_4)^3 = (s_1s_3)^2 = (s_1s_4)^2 = (s_2s_4)^2 = \varepsilon\rangle$.

Let $\alpha_1, \alpha_2, \alpha_3$, and α_4 be simple roots for the $\mathcal{W}$-module $V = \mathrm{span}_{\mathbb{R}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Recall, from § 2.10 that we have $s_i.\alpha_j = \alpha_j - M_{ji}\alpha_i$ for $i, j = 1, 2, 3, 4$. Set $\langle \alpha_1, \alpha_1\rangle = 4$. Then $\langle \alpha_2, \alpha_2\rangle = \frac{M_{21}}{M_{12}}\langle \alpha_1, \alpha_1\rangle = 4$, $\langle \alpha_3, \alpha_3\rangle = \frac{M_{32}}{M_{23}}\langle \alpha_2, \alpha_2\rangle = 2$, $\langle \alpha_4, \alpha_4\rangle = \frac{M_{43}}{M_{34}}\langle \alpha_3, \alpha_3\rangle = 2$. Thus α_3 and α_4 are short roots and α_1 and α_2 are long roots. Also, $\langle \alpha_1, \alpha_2\rangle = \frac{1}{2}\langle \alpha_2, \alpha_2\rangle M_{12} = \frac{1}{2}\cdot 4\cdot(-1) = -2$, $\langle \alpha_2, \alpha_3\rangle = \frac{1}{2}\langle \alpha_3, \alpha_3\rangle M_{23} = \frac{1}{2}\cdot 2\cdot(-2) = -2$, and $\langle \alpha_3, \alpha_4\rangle = \frac{1}{2}\langle \alpha_4, \alpha_4\rangle M_{34} = \frac{1}{2}\cdot 2\cdot(-1) = -1$ and similarly $\langle \alpha_2, \alpha_1\rangle = -2, \langle \alpha_3, \alpha_2\rangle = -2$ and $\langle \alpha_4, \alpha_3\rangle = -1$. Also, note since $M_{ij} = 0$ when $|i - j| > 1$, then $\langle \alpha_i, \alpha_j\rangle = 0$. Then relative to the basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

for V, the inner product $\langle\cdot,\cdot\rangle$ is represented by the matrix

$$\begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

We can find the short and long roots in Φ^+ (cf. §2.11) by playing the numbers game on $\mathfrak{g}^\mathsf{T}$ from the arbitrary strongly dominant starting position (a, b, c, d) (cf. §2.17, Remark 2.38). One can determine that this gives us $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_2+2\alpha_3, \alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+2\alpha_3, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_2+2\alpha_3+\alpha_4, \alpha_1+2\alpha_2+2\alpha_3, \alpha_2+2\alpha_3+2\alpha_4, \alpha_1+\alpha_2+2\alpha_3+\alpha_4, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+2\alpha_3+\alpha_4, \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+3\alpha_3+\alpha_4, \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+4\alpha_3+2\alpha_4, \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4, 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\}$, $\Phi^+_{short} = \{\alpha_3, \alpha_4, \alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_2+2\alpha_3+\alpha_4, \alpha_1+\alpha_2+2\alpha_3+\alpha_4, \alpha_1+2\alpha_2+2\alpha_3+\alpha_4, \alpha_1+2\alpha_2+3\alpha_3+\alpha_4, \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4\}$, and $\Phi^+_{long} = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_2+2\alpha_3, \alpha_1+\alpha_2+2\alpha_3, \alpha_1+2\alpha_2+2\alpha_3, \alpha_2+2\alpha_3+2\alpha_4, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+4\alpha_3+2\alpha_4, \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4, 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\}$. Since α_i corresponds to the i^{th} row of the Cartan matrix (relative to the basis of fundamental weights), then $\alpha_1 = 2\omega_1-\omega_2, \alpha_2 = -\omega_1+2\omega_2-2\omega_3, \alpha_3 = -\omega_2+2\omega_3-\omega_4$, and $\alpha_4 = -\omega_3+2\omega_4$. Note that $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = 2\cdot(2\omega_1-\omega_2) + 3\cdot(-\omega_1+2\omega_2-\omega_3) + 4\cdot(-\omega_2+2\omega_3-\omega_4) + 2\cdot(-\omega_3+2\omega_4) = \omega_1$ is the highest root $\overline{\omega}$, and that $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha 4 = (2\omega_1-\omega_2) + 2\cdot(-\omega_1+2\omega_2-\omega_3) + 3\cdot(-\omega_2+2\omega_3-\omega_4) + 2\cdot(-\omega_3+2\omega_4) = \omega_4$ is the highest short root $\overline{\omega}_{short}$. Note that the i^{th} row of the inverse matrix M^{-1} identifies the fundamental weight ω_i as an integral linear combination of simple roots, cf. Proposition 2.18.

From these results we can use Theorem 2.20 to determine $|\mathcal{W}|$. The highest short root $\overline{\omega}_{short} = \omega_4$ is J^c-dominant for $J = \{1, 2, 3\}$. Then we have $|\Phi_{short}| = 2|\Phi^+_{short}| = 2 \cdot 12 = 24$ and $|\mathcal{W}_J| = |\mathcal{W}_{\{1,2,3\}}| = |\mathcal{W}_{\mathsf{B}_3}|$. However, if we do not know the order of $\mathcal{W}_{\mathsf{B}_3}$ then we must perform a similar computation. The longest short root in the B_3 case is ω_1 (cf. §2.11), which is J_1^c-dominant for $J_1 = \{2, 3\}$. Also, $|\Phi_{short}(\mathsf{B}_3)| = 2|\Phi short^+(\mathsf{B}_3)| = 2 \cdot 3 = 6$ and $|W_{J_1}| = |\mathcal{W}_{\{2,3\}}| = |\mathcal{W}_{\mathsf{B}_2}|$. The longest short root for B_2 is ω_1, $J_2 = \{2\}$. Then $|\mathcal{W}_{\mathsf{B}_2}| = 2|\Phi^+_{short}(\mathsf{B}_2)| \cdot |\mathcal{W}_{\{2\}}| = 2 \cdot 2 \cdot 2 = 8$. This means that $|\mathcal{W}_{\mathsf{B}_3}| = 6 \cdot 8 = 48$. Therefore, $|\mathcal{W}| = |\Phi_{short}| \cdot |\mathcal{W}_{\mathsf{B}_3}| = 24 \cdot 48 = 1152$ is the order of the Weyl group. It is also of note that the Weyl group is isomorphic to the symmetry group of the 24-cell.

From the numbers game played on $\mathfrak{g}^{\mathsf{T}}$ from the generic strongly dominant starting position (a, b, c, d), the terminal position we end with is $(-a, -b, -c, -d)$. That is $\omega_0.(a\omega_1^{\mathsf{T}} + b\omega_2^{\mathsf{T}} + c\omega_3^{\mathsf{T}} + d\omega_4^{\mathsf{T}}) = -a\omega_1^{\mathsf{T}} - b\omega_2^{\mathsf{T}} - c\omega_3^{\mathsf{T}} - d\omega_4^{\mathsf{T}}$. Since, $\mathfrak{g} \cong \mathfrak{g}^{\mathsf{T}}$ then $\omega_0.(a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4) = -a\omega_1 - b\omega_2 - c\omega_3 - d\omega_4$ for a generic strongly dominant starting weight $a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4$. Then $\omega_0.\alpha_i = -\alpha_i$ for $i = 1, 2, 3, 4$. In particular, the symmetry σ_0 of the Dynkin diagram $\mathfrak{g}$ is the identity.

Next we work out the adjoint and short adjoint characters for F_4 following Example 2.27. To simplify the notation, we use z_i to denote e_{ω_i}.

$$\chi_{\overline{\omega}} = \chi_{\omega_1} = \textbf{char}(\omega_1; z_1, z_2, z_3, z_4) = z_1 + z_1^{-1}z_2 + z_2^{-1}z_3^2 + z_4 + z_3z_4^{-1} + z_2z_3^{-2}z_4^2 + z_2z_3^{-1} +$$

$$z_1z_2^{-1}z_4^2 + z_2z_4^{-2} + z_1z_2^{-1}z_3 + z_1^{-1}z_4^2 + z_1z_2^{-1}z_3^2z_4^{-2} + z_1z_3^{-1}z_4 + z_1^{-1}z_3 + z_1z_4^{-1} + z_1^{-1}z_3^2z_4^{-2} +$$

$$z_1^{-1}z_2z_3^{-1}z_4 + z_1z_2z_3^{-2} + z_1^{-1}z_2z_4^{-1} + z_2^{-1}z_3z_4 + z_1^2z_2^{-1} + z_1^{-1}z_2^2z_3^{-2} + z_2^{-1}z_3^2z_4^{-1} + z_3^{-1}z_4^2 +$$

$$4 + z_3z_4^{-2} + z_2z_3^{-2}z_4 + z_1z_2^{-2}z_3^2 + z_1^{-2}z_2 + z_2z_3^{-1}z_4^{-1} + z_1z_2^{-1}z_4 + z_1^{-1}z_2^{-1}z_3^2 + z_1z_2^{-1}z_3z_4^{-1} +$$

$$z_1z_3^{-2}z_4^2 + z_1^{-1}z_4 + z_1z_3^{-1} + z_1^{-1}z_3z_4^{-1} + z_1^{-1}z_2z_3^{-2}z_4^2 + z_1z_4^{-2} + z_1^{-1}z_2z_3^{-1} + z_2^{-1}z_4^2 +$$

$$z_1^{-1}z_2z_4^{-2} + z_2^{-1}z_3 + z_2^{-1}z_3^2z_4^{-2} + z_3^{-1}z_4 + z_4^{-1} + z_2z_3^{-2} + z_1z_2^{-1} + z_1^{-1}$$

$$\chi_{\overline{\omega}_{short}} = \chi_{\omega_4} = \mathbf{char}(\omega_4; z_1, z_2, z_3, z_4) = z_4 + z_3z_4^{-1} + z_2z_3^{-1} + z_1z_2^{-1}z_3 + z_1^{-1}z_3 + z_1z_3^{-1}z_4 +$$

$$z_1^{-1}z_2z_3^{-1}z_4 + z_1z_4^{-1} + z_2^{-1}z_3z_4 + z_1^{-1}z_2z_4^{-1} + z_3^{-1}z_4^2 + z_2^{-1}z_3^2z_4^{-1} + 2 + z_2z_3^{-2}z_4 + z_3z_4^{-2} +$$

$$z_1z_2^{-1}z_4 + z_2z_3^{-1}z_4^{-1} + z_1^{-1}z_4 + z_1z_2^{-1}z_3z_4^{-1} + z_1^{-1}z_3z_4^{-1} + z_1z_3^{-1} + z_1^{-1}z_2z_3^{-1} + z_2^{-1}z_3 +$$

$$z_3^{-1}z_4 + z_4^{-1}$$

From Proposition 2.18 and §2.15, we have that $s_i.e_{\omega_i} = e_{(\omega_j - \delta_{ij}\alpha_i)}$. Recall that $\alpha_i = \sum M_{ik}\omega_k$. So in the notation of the preceding paragraph we have for example $s_1.z_1 = s_1e_{\omega_1} = e_{(\omega_1-\alpha_1)} = e_{(\omega_1 - 2\omega_1 + \omega_2)} = e_{(-\omega_1+\omega_2)} = e_{-\omega_1}e_{\omega_2} = z_1^{-1}z_2$. Similarly, one can see that $s_1.z_2 = z_2$, $s_1.z_3 = z_3$, and $s_1.z_4 = z_4$. Then when s_1 is applied to the character χ_{ω_4} we get the following:

$$s_1.\chi_{\omega_4} = z_4 + z_3z_4^{-1} + z_2z_3^{-1} + (z_1^{-1}z_2)z_2^{-1}z_3 + (z_1^{-1}z_2)^{-1}z_3 +$$

$$(z_1^{-1}z_2)z_3^{-1}z_4 + (z_1^{-1}z_2)^{-1}z_2z_3^{-1}z_4 + (z_1^{-1}z_2)z_4^{-1} + z_2^{-1}z_3z_4 + (z_1^{-1}z_2)^{-1}z_2z_4^{-1} + z_3^{-1}z_4^2 +$$

$$z_2^{-1}z_3^2z_4^{-1} + 2 + z_2z_3^{-2}z_4 + z_3z_4^{-2} + (z_1^{-1}z2)z_2^{-1}z_4 + z_2z_3^{-1}z_4^{-1} + (z_1^{-1}z_2)^{-1}z_4 +$$

$$(z_1^{-1}z_2)z_2^{-1}z_3z_4^{-1} + (z_1^{-1}z_2)^{-1}z_3z_4^{-1} + (z_1^{-1}z_2)z_3^{-1} + (z_1^{-1}z_2)^{-1}z_2z_3^{-1} + z_2^{-1}z_3 + z_3^{-1}z_4 + z_4^{-1} =$$

$$z_4 + z_3z_4^{-1} + z_2z_3^{-1} + z_1^{-1}z_3 + z_1z_2^{-1}z_3 + z_1^{-1}z_2z_3^{-1}z_4 + z_1z_3^{-1}z_4 + z_1^{-1}z_2z_4^{-1} + z_2^{-1}z_3z_4 + z_1z_4^{-1}$$

$$+z_3^{-1}z_4^2 + z_2^{-1}z_3^2z_4^{-1} + 2 + z_2z_3^{-2}z_4 + z_3z_4^{-2} + z_1^{-1}z_4 + z_2z_3^{-1}z_4^{-1} + z_1z_2^{-1}z_4 + z_1^{-1}z_3z_4^{-1} +$$

$$z_1z_2^{-1}z_3z_4^{-1} + z_1^{-1}z_2z_3^{-1} + z_1z_3^{-1} + z_2^{-1}z_3 + z_3^{-1}z_4 + z_4^{-1} = \chi_{\omega_4}$$

In fact, it can be seen that s_1, s_2, s_3, and s_4 each preserve the character polynomial χ_{ω_4}. In this way, we can verify directly that χ_{ω_4} is $\mathcal{W}$-invariant.

In order to compute the expression of Theorem 2.30, we use the numbers game technique of Remark 2.38. We will play the numbers game on $\mathfrak{g}$ with the starting position $(a+1, b+1, c+1, d+1)$ where a, b, c,and d are nonnegative. After playing the game, the numbers at the fired nodes are $a+1, b+1, c+1, d+1, c+d+2, b+c+2, a+b+2, b+c+d+3, a+b+c+3, 2b+c+3, a+b+c+d+4, a+2b+c+4, 2b+c+d+4, 2a+2b+c+5, 2b+2c+d+5, a+2b+c+d+5, 2a+2b+c+d+6, a+2b+2c+d+6, a+3b+2c+d+7, 2a+2b+2c+d+7, 2a+3b+2c+d+8, 2a+4b+2c+d+9, 2a+4b+3c+d+10$, and $2a+4b+3c+2d+11$. By Theorem 2.30 and Remark 2.38, we find that for a character χ_λ with $\lambda = a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4$, the rank generating function for a connected splitting poset R is $RGF(R, q) =$

$$\frac{(1-q^{a+1})(1-q^{b+1})(1-q^{c+1})(1-q^{d+1})(1-q^{c+d+2})(1-q^{b+c+2})(1-q^{a+b+2})}{(1-q)(1-q)(1-q)(1-q)(1-q^2)(1-q^2)(1-q^2)}$$
$$\cdot\frac{(1-q^{b+c+d+3})(1-q^{a+b+c+3})(1-q^{2b+c+3})(1-q^{a+b+c+d+4})(1-q^{a+2b+c+4})}{(1-q^3)(1-q^3)(1-q^3)(1-q^4)(1-q^4)}$$
$$\cdot\frac{(1-q^{2b+c+d+4})(1-q^{2a+2b+c+5})(1-q^{2b+2c+d+5})(1-q^{a+2b+c+d+5})(1-q^{2a+2b+c+d+6})}{(1-q^4)(1-q^5)(1-q^5)(1-q^5)(1-q^6)}$$
$$\cdot\frac{(1-q^{a+2b+2c+d+6})(1-q^{a+3b+2c+d+7})(1-q^{2a+2b+2c+d+7})(1-q^{2a+3b+2c+d+8})}{(1-q^6)(1-q^7)(1-q^7)(1-q^8)}$$
$$\cdot\frac{(1-q^{2a+4b+2c+d+9})(1-q^{2a+4b+3c+d+10})(1-q^{2a+4b+3c+2d+11})}{(1-q^9)(1-q^{10})(1-q^{11})}.$$

The dimension of χ_λ determined by the Weyl Dimension Formula (Corollary 2.31) can now be obtained by letting $q \to 1$ in the above formula for $RGF(R, q)$:

$$\frac{(a+1)(b+1)(c+1)(d+1)(c+d+2)(b+c+2)(a+b+2)(b+c+d+3)}{1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3}$$
$$\cdot\frac{(a+b+c+3)(2b+c+3)(a+b+c+d+4)(a+2b+c+4)(2b+c+d+4)}{3\cdot 3\cdot 4\cdot 4\cdot 4}$$
$$\cdot\frac{(2a+2b+c+5)(2b+2c+d+5)(a+2b+c+d+5)(2a+2b+c+d+6)}{5\cdot 5\cdot 5\cdot 6}$$

$$\cdot \frac{(a+2b+2c+d+6)(a+3b+2c+d+7)(2a+2b+2c+d+7)(2a+3b+2c+d+8)}{6\cdot 7\cdot 7\cdot 8}$$

$$\cdot \frac{(2a+4b+2c+d+9)(2a+4b+3c+d+10)(2a+4b+3c+2d+11)}{9\cdot 10\cdot 11}.$$

Also, computing the difference of the degrees of the numerator and denominator polynomials in our expression for $RGF(R,q)$, we obtain the length of any connected splitting poset R for χ_λ (cf. Corollary 2.32):

$$\ell(\lambda) = a+b+c+d+(c+d)+(b+c)+(a+b)+(b+c+d)+(a+b+c)+(2b+c)+(a+b+c+d)+$$

$$(a+2b+c)+(2b+c+d)+(2a+2b+c)+(2b+2c+d)+(a+2b+c+d)+(2a+2b+c+d)+$$

$$(a+2b+2c+d)+(a+3b+2c+d)+(2a+2b+2c+d)+(2a+3b+2c+d)+(2a+4b+2c+d)+$$

$$(2a+4b+3c+d)+(2a+4b+3c+2d) = 22a+42b+30c+16d$$

Performing the preceding calculation for a character will not only provide us with the length of any connected splitting poset but will also give the number of vertices needed for the poset of irreducibles for any SDL of that character.

The rank generating function for the character χ_{ω_4} determines that any splitting poset R with this character is:

$$RGF(R,q) = \frac{(1-q)(1-q)(1-q)(1-q^2)(1-q^3)(1-q^2)(1-q^2)(1-q^4)(1-q^3)(1-q^3)}{(1-q)(1-q)(1-q)(1-q)(1-q^2)(1-q^2)(1-q^2)(1-q^3)(1-q^3)(1-q^3)}$$

$$\frac{(1-q^5)(1-q^4)(1-q^5)(1-q^5)(1-q^6)(1-q^6)(1-q^7)(1-q^7)(1-q^8)(1-q^8)(1-q^9)}{(1-q^4)(1-q^4)(1-q^4)(1-q^5)(1-q^5)(1-q^5)(1-q^6)(1-q^6)(1-q^7)(1-q^7)(1-q^8)}$$

$$\frac{(1-q^{10})(1-q^{11})(1-q^{13})}{(1-q^9)(1-q^{10})(1-q^{11})} = \frac{(1-q^8)(1-q^{13})}{(1-q^1)(1-q^4)} = (1+q^4)(1+q+q^2+\cdots+q^{12}) =$$

$$1+q+q^2+q^3+2q^4+2q^5+2q^6+2q^7+2q^8+2q^9+2q^{10}+2q^{11}+2q^{12}+q^{13}+q^{14}+q^{15}+q^{16}$$

One can verify by inspection that this is the rank generating function for the χ_{ω_4}-splitting posets of Figures 5.1 and 5.3. Similar computations can be used to find rank generating functions for other F_4 characters.

We will now use these formulas to determine the dimension and length of all of the weights that we will explore in Chapter 5. We use the 4-tuple (a, b, c, d) to denote the weight $a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4$. First let us take the weight $\omega_4 = (0, 0, 0, 1)$. Then the dimension formula gives the necessary dimension of χ_{ω_4} to be as follows,

$$\frac{1 \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 4 \cdot 3 \cdot 3 \cdot 5 \cdot 4 \cdot 5 \cdot 5 \cdot 6 \cdot 6 \cdot 7 \cdot 7 \cdot 8 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 13}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 4 \cdot 5 \cdot 5 \cdot 5 \cdot 6 \cdot 6 \cdot 7 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} = \frac{8 \cdot 13}{4} = 26.$$

The length formula applied to this gives

$$\ell(0, 0, 0, 1) = 16.$$

This is the smallest dimension and length for any F_4 character. Next we consider the character $\chi_{2\omega_4}$. This has highest weight $2\omega_4 = (0, 0, 0, 2)$ The dimension formula gives

$$\frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 4 \cdot 2 \cdot 2 \cdot 5 \cdot 3 \cdot 3 \cdot 6 \cdot 4 \cdot 6 \cdot 5 \cdot 7 \cdot 7 \cdot 8 \cdot 8 \cdot 9 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 15}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 4 \cdot 5 \cdot 5 \cdot 5 \cdot 6 \cdot 6 \cdot 7 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} =$$

$$\frac{8 \cdot 9 \cdot 12 \cdot 15}{2 \cdot 4 \cdot 5} = 324.$$

Then the length of any connected splitting poset for this character is

$$\ell(0, 0, 0, 2) = 2 \cdot 16 = 32.$$

For the character χ_{ω_1} with highest weight $\omega_1 = (1, 0, 0, 0)$, the dimension is

$$\frac{2 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 3 \cdot 5 \cdot 5 \cdot 4 \cdot 7 \cdot 5 \cdot 6 \cdot 8 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 4 \cdot 5 \cdot 5 \cdot 5 \cdot 6 \cdot 6 \cdot 7 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} = \frac{8 \cdot 12 \cdot 13}{4 \cdot 6} = 52.$$

The corresponding length is

$$\ell(1, 0, 0, 0) = 22.$$

Further, $\chi_{2\omega_1}$ with $2\omega_1 = (2,0,0,0)$ has dimension

$$\frac{3\cdot 1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 4\cdot 3\cdot 5\cdot 3\cdot 6\cdot 6\cdot 4\cdot 9\cdot 5\cdot 7\cdot 10\cdot 8\cdot 9\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15}{1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 4\cdot 4\cdot 4\cdot 5\cdot 5\cdot 5\cdot 6\cdot 6\cdot 7\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11} =$$

$$\frac{9\cdot 12\cdot 13\cdot 14\cdot 15}{2\cdot 4\cdot 5\cdot 7} = 1053.$$

The length for this character is

$$\ell(2,0,0,0) = 44.$$

The character χ_{ω_3} has highest weight $\omega_3 = (0,0,1,0)$ and dimension

$$\frac{1\cdot 1\cdot 2\cdot 1\cdot 3\cdot 3\cdot 2\cdot 4\cdot 4\cdot 4\cdot 5\cdot 5\cdot 5\cdot 6\cdot 7\cdot 6\cdot 7\cdot 8\cdot 9\cdot 9\cdot 10\cdot 11\cdot 13\cdot 14}{1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 4\cdot 4\cdot 4\cdot 5\cdot 5\cdot 5\cdot 6\cdot 6\cdot 7\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11} = \frac{9\cdot 13\cdot 14}{2\cdot 3} = 273.$$

The associated length is

$$\ell(0,0,1,0) = 30.$$

The other fundamental character χ_{ω_2} has $\omega_2 = (0,1,0,0)$ and dimension

$$\frac{1\cdot 2\cdot 1\cdot 1\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4\cdot 5\cdot 5\cdot 6\cdot 6\cdot 7\cdot 7\cdot 7\cdot 8\cdot 8\cdot 10\cdot 9\cdot 11\cdot 13\cdot 14\cdot 15}{1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 4\cdot 4\cdot 4\cdot 5\cdot 5\cdot 5\cdot 6\cdot 6\cdot 7\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11} =$$

$$\frac{\cdot 7\cdot 8\cdot 13\cdot 14\cdot 15}{2\cdot 3\cdot 4\cdot 5} = 1274.$$

The associated length is

$$\ell(0,1,0,0) = 42.$$

This information will be used and can be checked in the figures of the next chapter.

CHAPTER 5: SPLITTING DISTRIBUTIVE LATTICES FOR F_4-CHARACTERS

§5.1 Characters Relating to the Fourth Fundamental Weight

The irreducible Weyl character for the fourth fundamental weight of F_4 has the smallest dimension (26, cf. Chapter 4) amongst all the fundamental characters. It is for this reason we explore χ_{ω_4} before any other irreducible characters. Among the splitting posets for χ_{ω_4} we will exhibit a minimal splitting poset, a maximal splitting poset, and two splitting distributive lattices.

Using a partial ordering on the short roots for F_4 one can obtain a poset that meets Stembridge's requirements [Stem3] for an admissible system. An admissible system for an irreducible Weyl character is minimal in the sense that it cannot properly contain (as an edge-colored subgraph) any other splitting poset for the same character. It can be seen that the poset of Figure 5.1 is the unique minimal splitting poset for χ_{ω_4}. The maximal splitting poset (cf. Example 2.33) of Figure 5.1 can also be obtained from the partial ordering on short roots for F_4.

Splitting distributive lattices for χ_{ω_4} can be similarly obtained. Since ω_4 is the highest short root, there are precisely two of these (cf. Example 2.34). Their posets and lattices are given in Figures 5.2 and 5.3. It is of interest that the only difference between the two lattices appears at the middle level. We note that the distributive core for (F_4, ω_4) is a distributive lattice that is similar to those of Figure 5.3, with the difference being that one middle

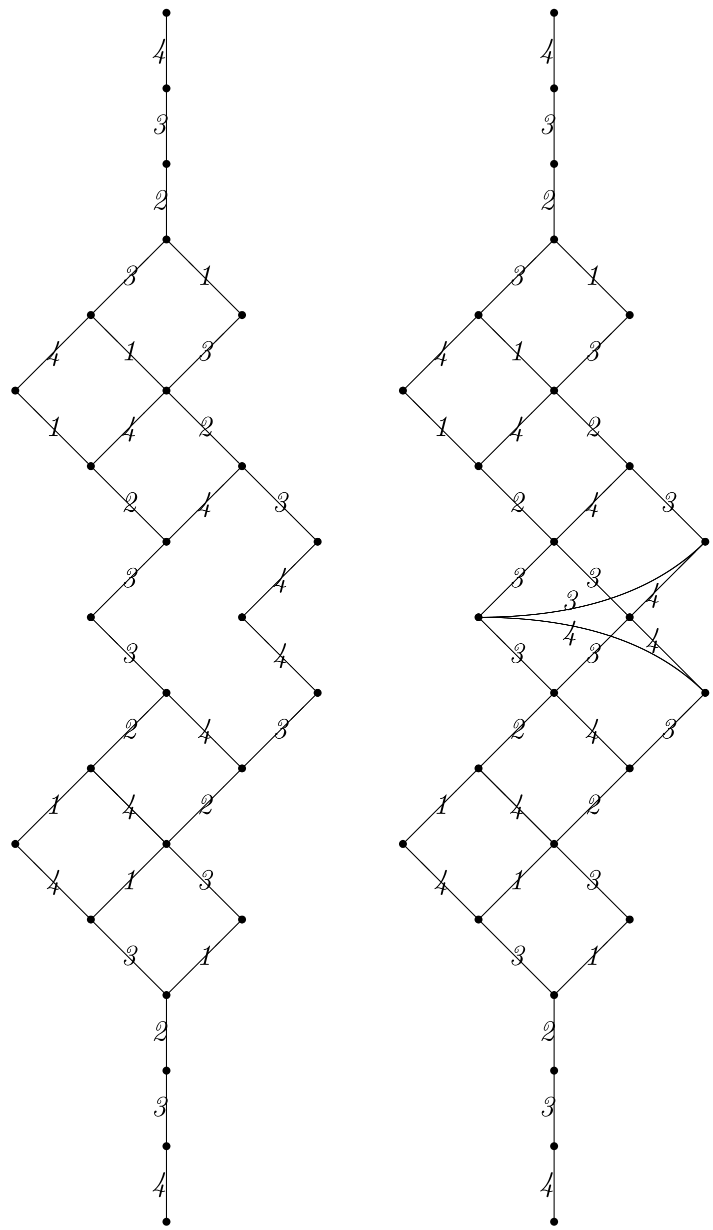


Figure 5.1: Stembridge's Minimal Splitting Poset and the Maximal Splitting Poset for χ_{ω_4}

weight vertex is excluded. Since the distributive core is a full length sublattice of both splitting distributive lattices, it follows from Theorem 2.6 that the posets of irreducibles for the distributive lattices (Figure 5.2) are weak vertex-colored subposets of the poset of irreducibles of the distributive core.

We now turn our attention to the 324-dimensional (cf. Chapter 4) irreducible F_4-character $\chi_{2\omega_4}$. The maximal splitting poset for $\chi_{2\omega_4}$ can be produced using Eveland's **maximal_support** command for the 'Supporting Graphs' Maple package, see [Eve]. We have applied Stembridge's product construction [Stem3] to the product of two SDL's for χ_{ω_4} in order to get a minimal splitting poset for $\chi_{2\omega_4}$. Since this is not needed for the subsequent development, we will not reproduce that poset here.

However, finding a splitting distributive lattice for $\chi_{2\omega_4}$ turned out to be a more difficult task. Eventually we looked at a splitting distributive lattice K obtained in [DW] for the 351-dimensional E_6-character $\chi_{2\omega_1}$. Let σ be the set mapping $\{1,2,3,4,5,6\} \xrightarrow{\sigma} \{1,2,3,4\}$ given by $1 \overset{\sigma}{\mapsto} 4$, $2 \overset{\sigma}{\mapsto} 1$, $3 \overset{\sigma}{\mapsto} 3$, $4 \overset{\sigma}{\mapsto} 2$, $5 \overset{\sigma}{\mapsto} 3$, $6 \overset{\sigma}{\mapsto} 4$. The recoloring K^σ afforded by σ turns out to be an $(\mathsf{F}_4, 2\omega_4)$-structured distributive lattice. We subsequently 'peeled' out of K^σ the desired SDL for the F_4-character $\chi_{2\omega_4}$ as a 324-element full-length distributive sublattice. Using Theorem 2.6 and our knowledge of the distributive core, we were able to find a nice way to present the poset of irreducibles for this SDL. The result is the vertex-colored poset of Figure 5.4 and the following theorem.

Figure 5.4 also appeared in another remarkable and unexpected way when we ran the distributive core algorithm on E_6-character $\chi_{2\omega_1}$. The output was an edge-colored distribu-

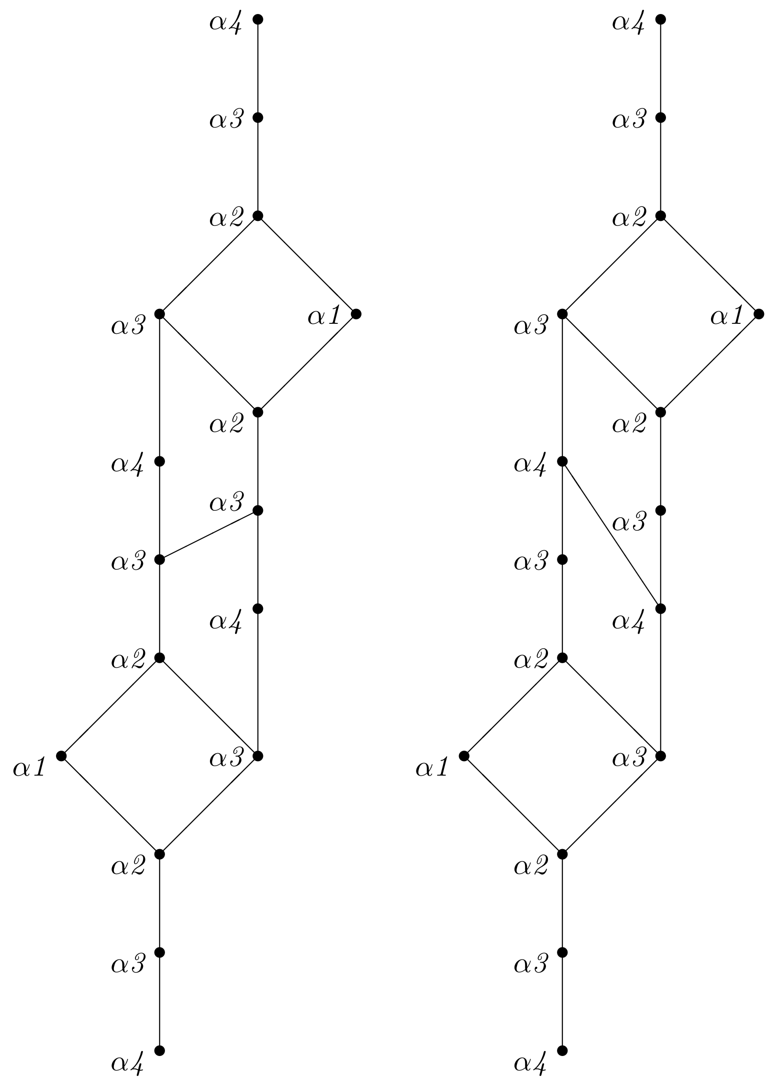


Figure 5.2: Posets of irreducibles for the two SDL's for χ_{ω_4}

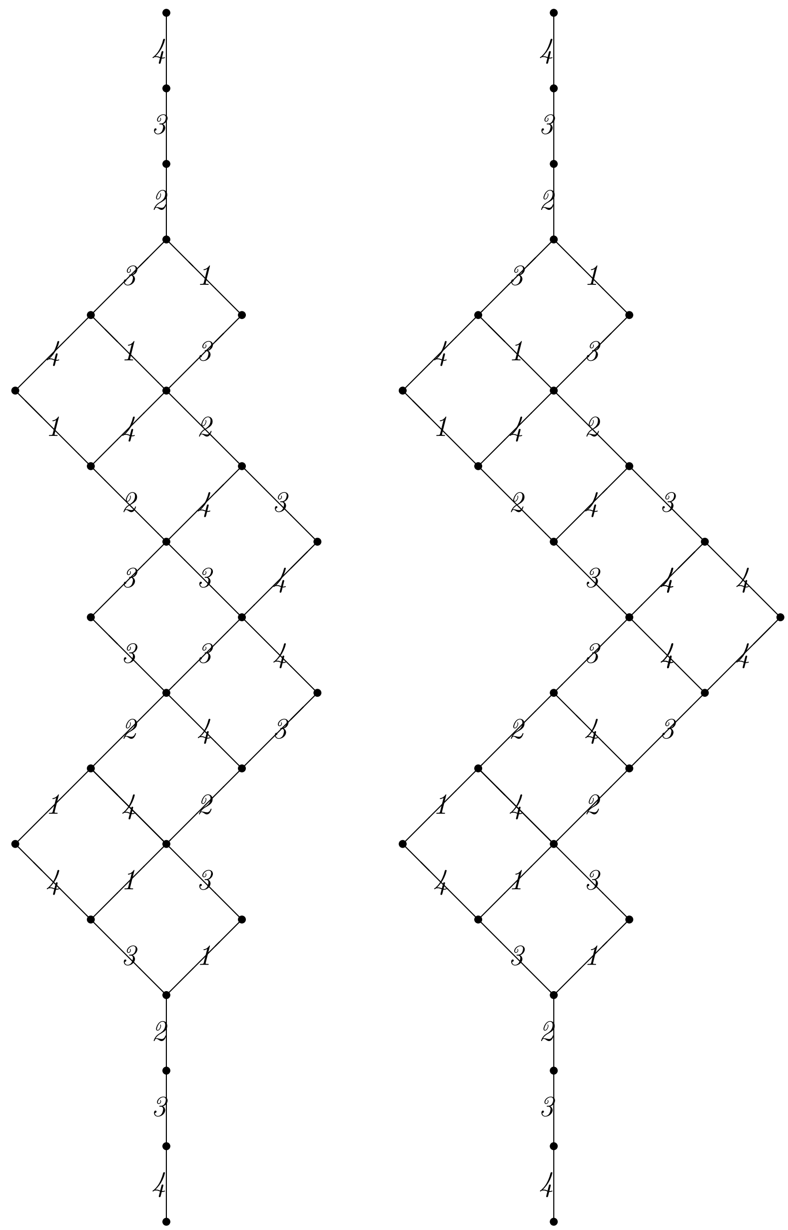


Figure 5.3: The two SDL's for χ_{ω_4}

tive lattice with precisely 324-elements, and after the same recoloring K^σ we observed that it was in fact the same $(\mathsf{F}_4, 2\omega_4)$-structured distributive lattice as the previous paragraph.

Theorem 5.1 *Let* $L := \mathrm{J}_{color}(P)$ *for the vertex-colored poset* P *of Figure 5.4. Then* L *is a splitting distributive lattice for the* F_4*-character* $\chi_{2\omega_4}$*.*

The proof is below. By inspection of Figure 5.4, one sees that if $P^* \cong P$, an isomorphism of vertex-colored posets, then $L^* \cong L$ by Proposition 2.4. Since σ_0 is the identity permutation for F_4 (see Chapter 4), it follows that for the σ_0-recolored dual (cf. §2.13), $L^\Delta \cong L$.

The hardest part of obtaining Theorem 5.1 was finding a candidate SDL for $\chi_{2\omega_4}$. The SDL of our theorem was only found after many prior abortive attempts using other methods. Much like the problem of factoring large numbers, verifying that a candidate distributive lattice is indeed an SDL is a straightforward iterative process, in contrast to the problem of finding good candidates in the first place. Our 324-element SDL is not so large that it is inconceivable to perform these verifications by hand. However, our proof of Theorem 5.1 uses iterative procedures implemented in the computer algebra system Maple to carry out this process.

Proof of Theorem 5.1. In the computer algebra system Maple, we use an iterative procedure **J_color** to compute $L = \mathrm{J}_{color}(P)$. Another iterative procedure calculates the weight generating function $WGF(L; z_1, z_2, z_3, z_4) = \sum_{\mathbf{t}\in L} z_1^{m_1(\mathbf{t})} z_2^{m_2(\mathbf{t})} z_3^{m_3(\mathbf{t})} z_4^{m_4(\mathbf{t})}$ as a simple sum over the elements of L. Next, we apply a **weyl_character** command based on Stembridge's **weight_mults** command from his Coxeter/Weyl Maple package [Stem4] in order to calculate $\mathbf{char}_{\mathsf{F}_4}(2\omega_4; z_1, z_2, z_3, z_4)$. Stembridge calculates weight multiplicities using an iterative procedure based on Freudenthal's multiplicity formula (see e.g. §22.3 of [Hum1]). Alternatively, consult [BMP]. Finally, we use Maple to check that $WGF(L; z_1, z_2, z_3, z_4) = \mathbf{char}_{\mathsf{F}_4}(2\omega_4; z_1, z_2, z_3, z_4)$ is an identity. $\square$

In Chapter 4, we worked out an explicit quotient-of-products expression for rank generating functions for F_4 character splitting posets. Using this result (with $a = b = c = 0$ and $d = 2$) along with Theorem 2.30, we have the following Corollary of Theorem 5.1:

Corollary 5.2 *For the vertex-colored poset P of Figure 5.4, the distributive lattice $L = J_{color}(P)$ is rank symmetric and rank unimodal, and moreover*

$$RGF(L, q) = \frac{(1-q^8)(1-q^9)(1-q^{12})(1-q^{15})}{(1-q)(1-q^2)(1-q^4)(1-q^5)} =$$

$$(1+q^2+q^4+q^6)(1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8)(1+q^4+q^8)(1+q^5+q^{10}) =$$

$$1+q+2q^2+2q^3+4q^4+5q^5+7q^6+8q^7+10q^8+11q^9+14q^{10}+15q^{11}+17q^{12}+17q^{13}+19q^{14}+19q^{15}$$

$$+20q^{16}+19q^{17}+19q^{18}+17q^{19}+17q^{20}+15q^{21}+14q^{22}+11q^{23}+10q^{24}+8q^{25}+7q^{26}+5q^{27}+4q^{28}$$

$$+2q^{29}+2q^{30}+q^{31}+q^{32}.$$

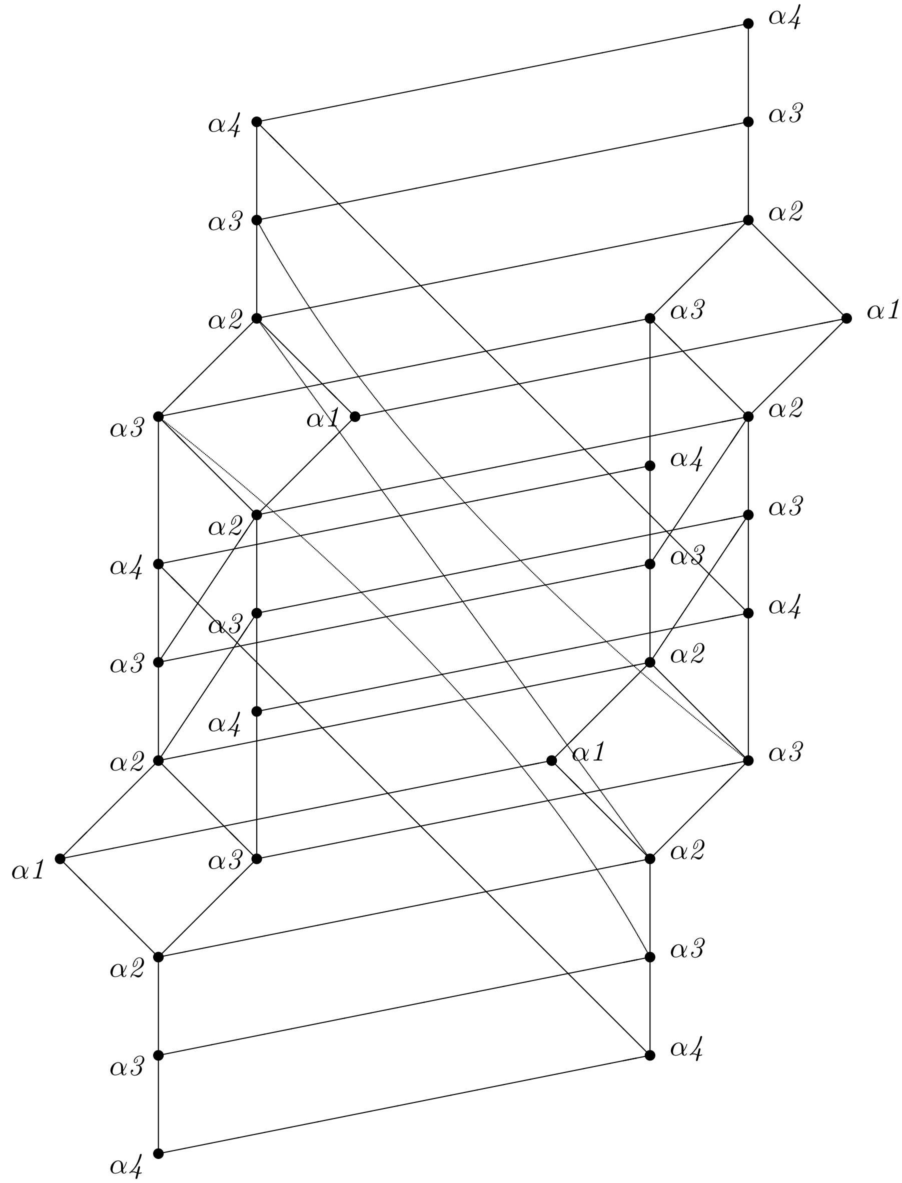


Figure 5.4: Poset of irreducibles for the SDL for $\chi_{2\omega_4}$ of Theorem 5.1

§5.2 Characters Relating to the First Fundamental Weight

The first fundamental character for F_4 has dimension 52 (cf. Chapter 4), which is second smallest amongst the irreducible F_4-characters. Similar to the fourth fundamental weight there exist two splitting distributive lattices which differ only at the middle level (Figures 5.8 and 5.9). In Figures 5.5 and 5.6, we depict a minimal splitting poset and a maximal splitting poset which were obtained using the same methods of §5.1.

To find a $\chi_{2\omega_1}$ splitting distributive lattice, our approach was to extend our result for the $\chi_{2\omega_4}$ case (cf. Theorem 5.1). To do so, we examined the similarities between the distributive cores for $(\mathsf{F}_4, 2\omega_1)$ and $(\mathsf{F}_4, 2\omega_4)$ in order to build a poset of irreducibles for an SDL for $\chi_{2\omega_1}$. Remarkably, this method of 'analogizing' worked. Our procedure in both cases can be loosely described as follows: take a 'modified' poset P (modified in the sense that both of the SDL's for the given fundamental case are full length sublattices of $\mathrm{J}_{color}(P)$ place two copies of P (denoted P_1 and P_2) together, connect the top of each 'monochromatic' chain in P_1 to the bottom of the corresponding chain in P_2 and connect the bottom vertex of P_2 to the highest non-maximal vertex of similar color in P_1. In particular, these relations determine the poset of irreducibles for an SDL for the $\chi_{2\omega_1}$ character. The resulting poset is depicted in Figure 5.10. We omit the proof of the following theorem since it is entirely similar to the proof of Theorem 5.1.

Theorem 5.3 *Let $L := \mathrm{J}_{color}(P)$ for the vertex-colored poset P of Figure 5.10. Then L is a splitting distributive lattice for the F_4-character $\chi_{2\omega_1}$.*

Figure 5.5: Minimal Splitting Poset for χ_{ω_1}.

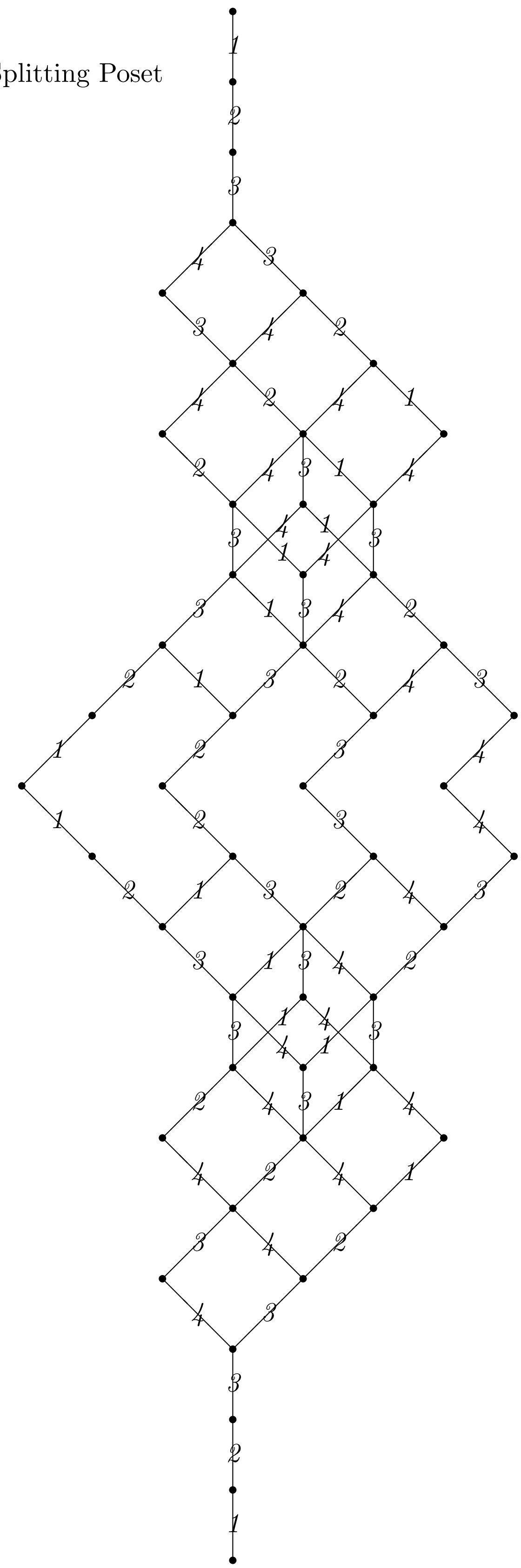
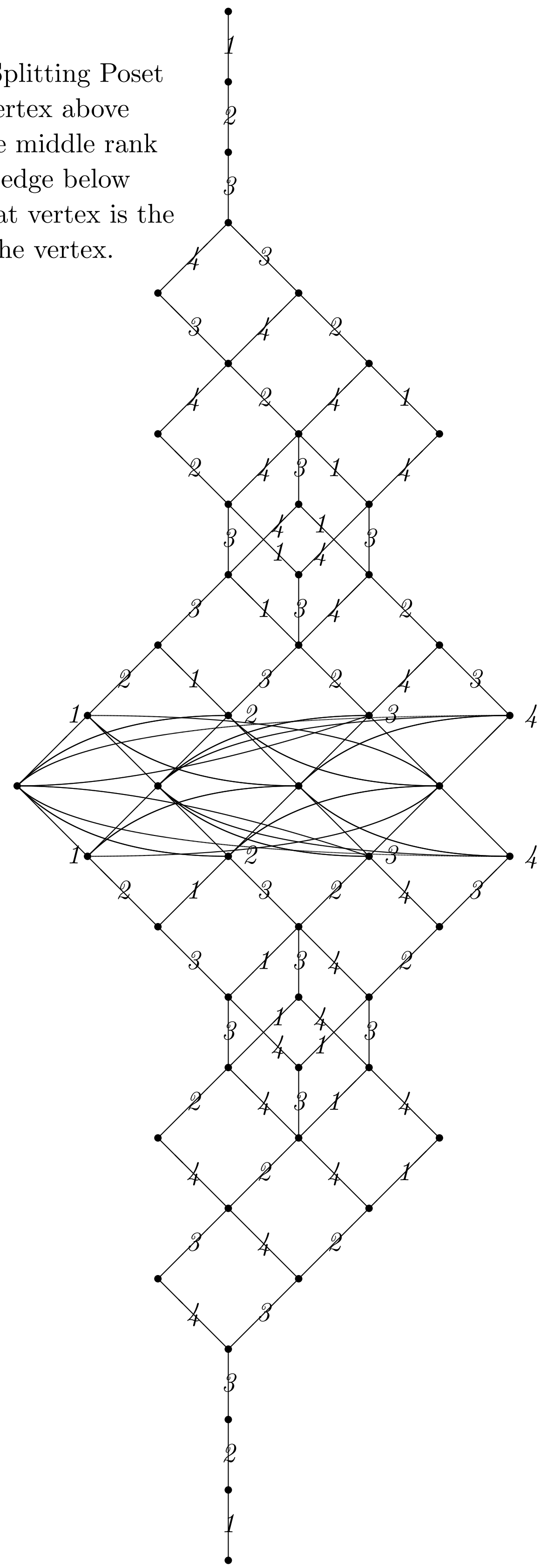


Figure 5.6: Maximal Splitting Poset for χ_{ω_1}. For a vertex above (resp. below) the middle rank the color of any edge below (resp. above) that vertex is the number beside the vertex.



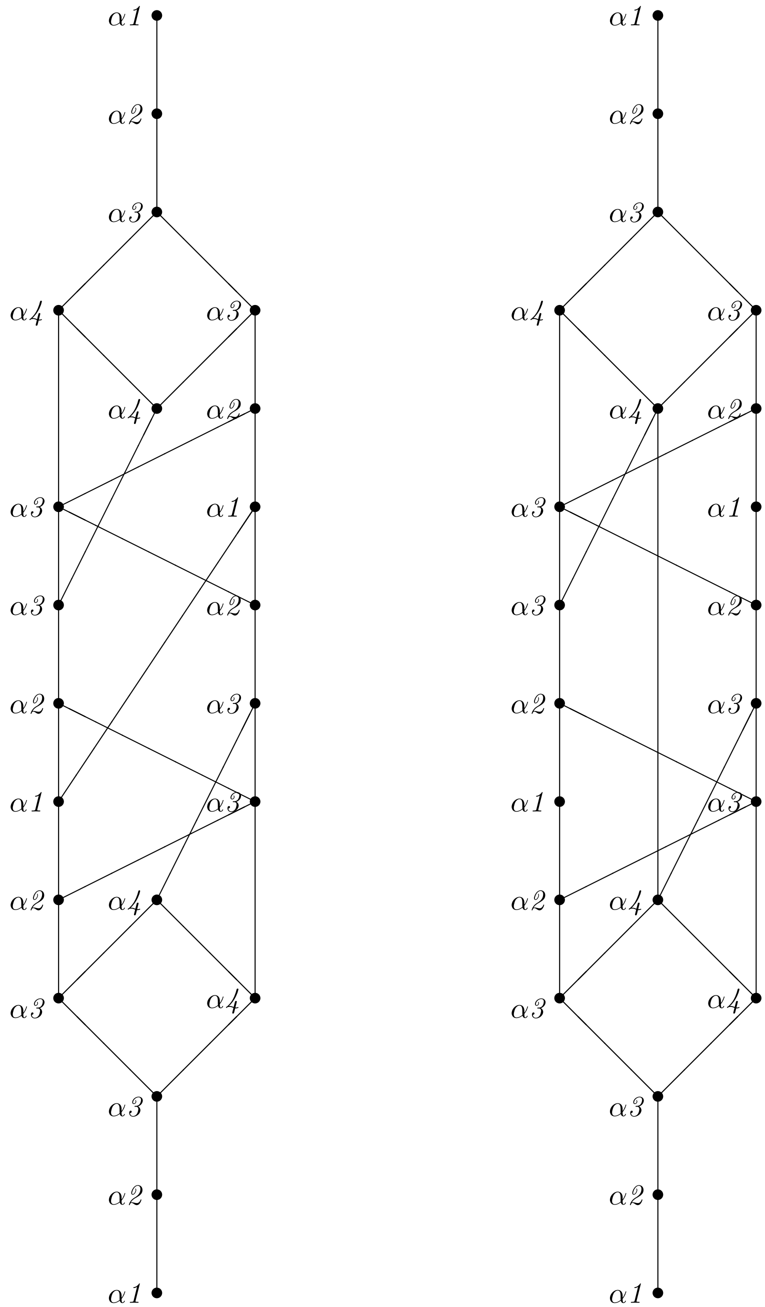


Figure 5.7: Posets of irreducibles for the two SDL's for χ_{ω_1}

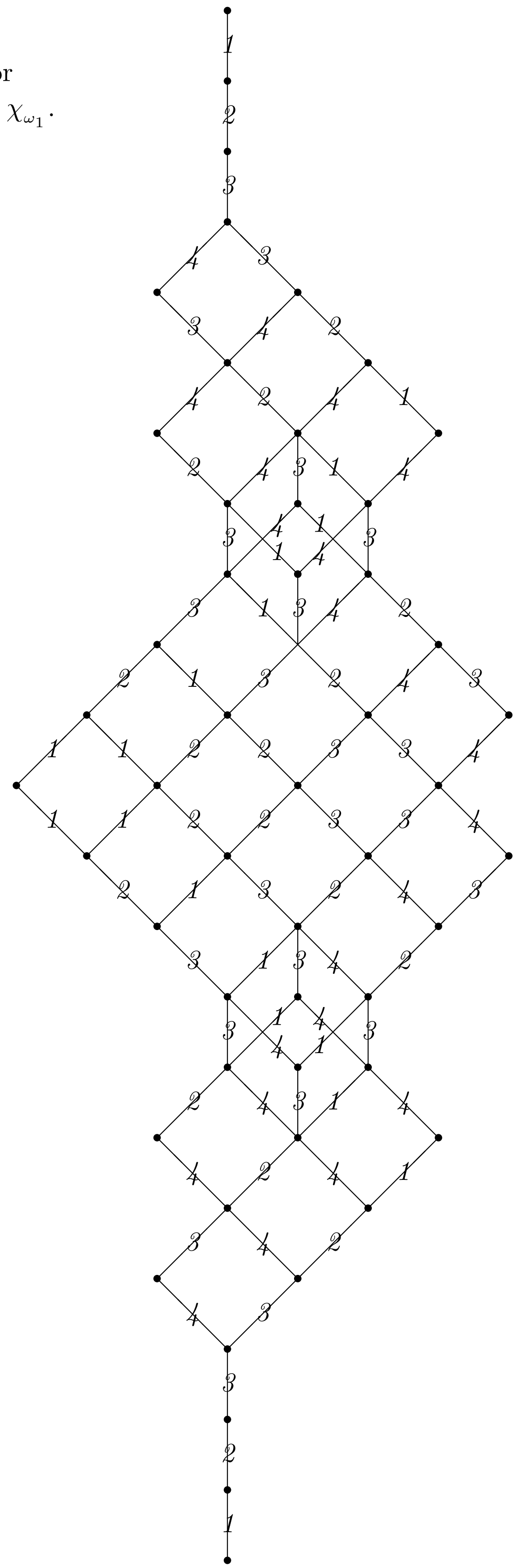
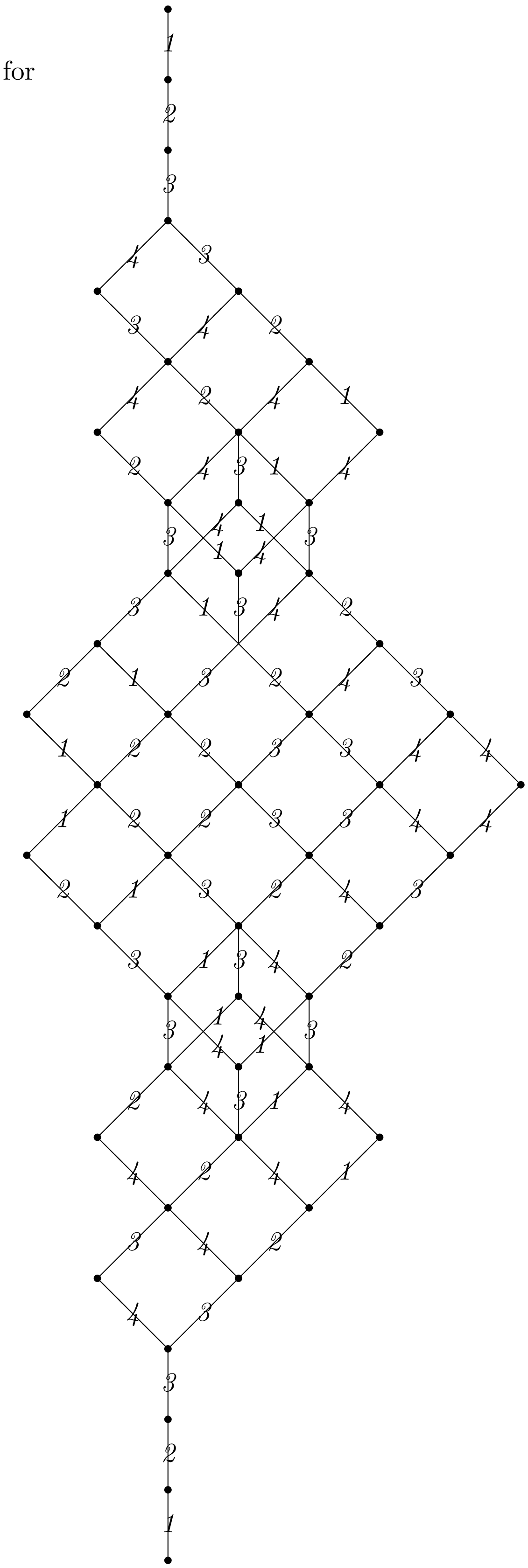


Figure 5.8: An SDL for the F_4-character χ_{ω_1}.

Figure 5.9: The other SDL for the F_4-character χ_{ω_1}.



As in §5.1, we note that by inspection one can see that for the poset P of Figure 5.10, $P^* \cong P$. Then by Proposition 2.4, $L^* \cong L$. Since the permutation σ_0 for F_4 is the identity permutation (see Chapter 4), it follows that for the σ_0-recolored dual (cf §2.13), $L^\Delta \cong L$.

Similar to Corollary 5.2, using the rank generating function (with $a = 2$ and $b = c = d = 0$) and Theorem 2.30 we get the following Corollary of Theorem 5.3:

Corollary 5.4 *For the vertex-colored poset P of Figure 5.10, the distributive lattice $L = J_{color}(P)$ is rank symmetric and rank unimodal, and moreover*

$$RGF(L,q) = \frac{(1-q^9)(1-q^{12})(1-q^{13})(1-q^{14})(1-q^{15})}{(1-q)(1-q^2)(1-q^4)(1-q^5)(1-q^7)} =$$

$$(1+q+q^2)(1+q+q^2)(1+q+q^2)(1-q+q^2)(1+q^3+q^6)(1-q^2+q^4)(1-q+q^2-q^3+q^4-q^5+q^6)$$

$$(1-q+q^3-q^4+q^5-q^7+q^8)(1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}+q^{11}+q^{12}) =$$

$$1+q+2q^2+2q^3+4q^4+5q^5+7q^6+9q^7+12q^8+14q^9+18q^{10}+21q^{11}+26q^{12}+28q^{13}+33q^{14}+36q^{15}+$$

$$41q^{16}+43q^{17}+47q^{18}+48q^{19}+51q^{20}+51q^{21}+53q^{22}+51q^{23}+51q^{24}+48q^{25}+47q^{26}+43q^{27}+$$

$$41q^{28}+36q^{29}+33q^{30}+28q^{31}+26q^{32}+21q^{33}+18q^{34}+14q^{35}+12q^{36}+9q^{37}+7q^{38}+5q^{39}+4q^{40}+$$

$$2q^{41}+2q^{42}+q^{43}+q^{44}.$$

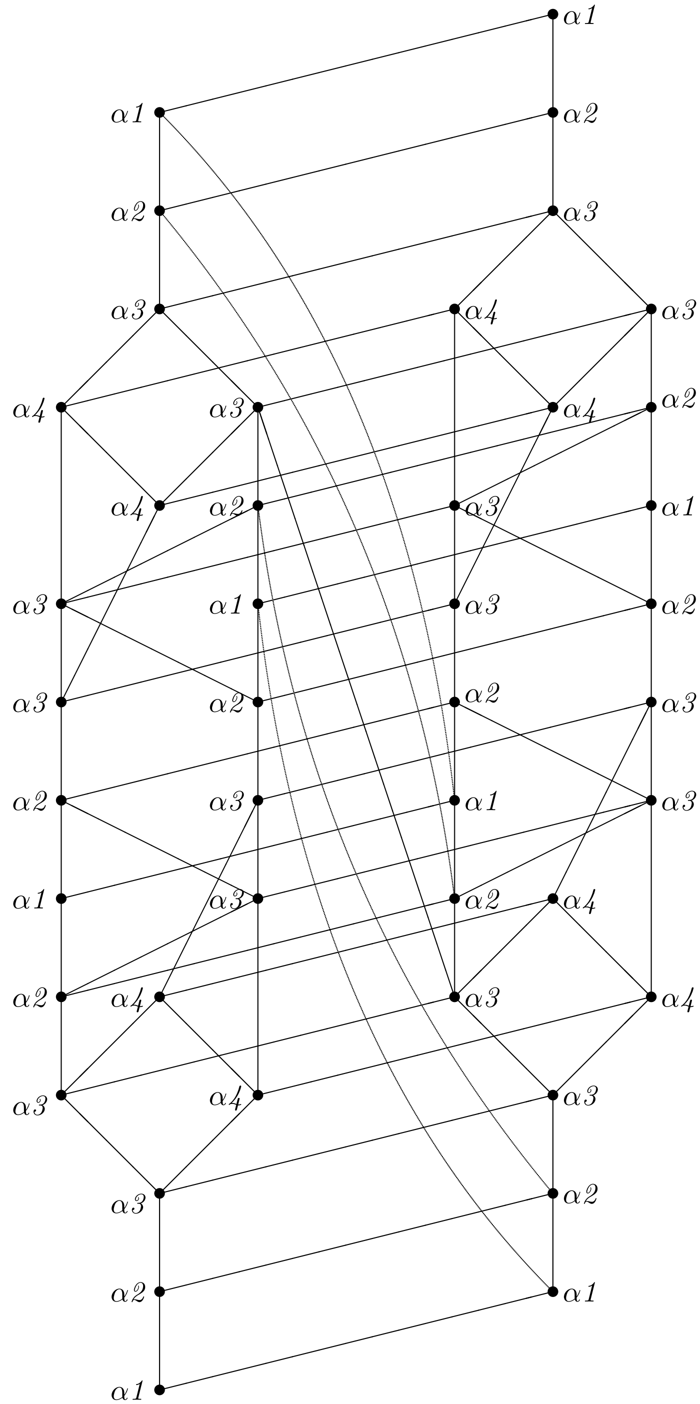


Figure 5.10: Poset of irreducibles for the SDL for $\chi_{2\omega_1}$ of Theorem of 5.3

§5.3 The Third Fundamental Character

Starting this thesis, the main goal was to find a splitting distributive lattice for the 273-dimensional (cf. Chapter 4) F_4-character χ_{ω_3}. This task was more challenging than expected and gave rise to the obvious question: Does χ_{ω_3} have a splitting distributive lattice?

Among the methods we tried were looking for a χ_{ω_3} splitting poset inside $\bigwedge^2 L$, the second exterior power of a splitting distributive lattice for χ_{ω_4} (cf. §2.3). The splitting poset found by this method was not even a lattice. We also applied Stembridge's product construction [Stem3]. This meant looking for a χ_{ω_3} splitting poset inside of $L \times L$ (a product of a χ_{ω_4}-SDL with itself). We obtained Stembridge's corresponding admissible system R, which has too few edges to be a distributive lattice. We then attempted to 'repair' this splitting poset by adding edges induced by the inclusion of R in $L \times L$. While this technique works in other settings (e.g. the SDL's for the rank two cases studied in [ADLMPPW]), the result in this case was a splitting poset which is not a lattice. One of the first methods we tried was to build a poset of irreducibles for an SDL directly from numbers game information similar to an idea from [DW]. While this gave us no affirmative information, it did lead us to suspect that χ_{ω_3} might not have a splitting distributive lattice. Eventually, we began to focus on the requirements for distributivity. This resulted in the notion of the distributive core (see Chapter 3). The distributive core algorithm gave insight on exactly where the problem was and lead to the proof of the following answer to our motivating question.

Theorem 5.5 *The F_4-character χ_{ω_3} has no splitting distributive lattice.*

This result is not entirely surprising since it was previously known that some fundamental characters for other Dynkin diagrams have no splitting distributive lattices.

Proof of Theorem 5.5. In our proof we use the notation (a, b, c, d) as shorthand for the weight $a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4$. Any vertex here denoted $\mathbf{t}_n$ corresponds to the vertex of Figure 5.11 with the number n adjacent to it. The symbols $\alpha\mathit{1}$, $\alpha\mathit{2}$, $\alpha\mathit{3}$, $\alpha\mathit{4}$ used in this proof and in Figure 5.11 should be simultaneously thought of as edge colors and as simple roots.

We use a contradiction argument. Assume there exists a splitting distributive lattice L for the F_4-character χ_{ω_3}. Working down from the maximal vertex $\mathbf{t}_1$ of L which has weight $(0, 0, 1, 0)$, we will show that L necessarily contains the structure of Figure 5.11 as an edge-colored subgraph. The structure of Figure 5.11 has three vertices of weight $(1, 1, -3, 2)$. However, this gives a contradiction since it is known (using [Stem4] or [BMP]) that any splitting poset for χ_{ω_3} has exactly one vertex of weight $(1, 1, -3, 2)$.

Our reasoning is based on the following facts concerning any $(\mathfrak{g}, \lambda)$-structured distributive lattice K and fixed element $\mathbf{t} \in K$. (1) For any color i, the i-component $\mathbf{comp}_i(\mathbf{t})$ is the Hasse diagram for a distributive lattice and its length is the same as the color i weight $m_i(\mathbf{x})$ of its unique maximal element $\mathbf{x}$ (cf. Proposition 2.8, Proposition 2.9). In particular, if $m_i(\mathbf{t}) \neq -m_i(\mathbf{x})$ then there is an edge of color i below $\mathbf{t}$. (2) The 'descendants' part of Proposition 2.7 tells us how distributive lattice properties 'fill in' the structure when we find a set of descendants of $\mathbf{t}$: Let $\mathbf{vertexcolor}_D(\mathbf{s}) := \mathbf{edgecolor}_L(\mathbf{s} \to \mathbf{t})$ for each $\mathbf{s}$ in some given subset D of the set of descendants of $\mathbf{t}$. Let $\mathbf{r} := \wedge_{\mathbf{s} \in D}(\mathbf{s})$. Then $[\mathbf{r}, \mathbf{t}] \cong \mathrm{M}_{color}(D)$.

Since elements of D are pairwise incomparable, the interval $[\mathbf{r}, \mathbf{t}]$ is a Boolean lattice cf. Example 2.2.

Since the maximal element $\mathbf{t}_1$ for L has weight $\omega_3 = (0, 0, 1, 0)$, then by fact (1) above, $\mathbf{t}_1$ necessarily has an edge of color $\alpha 3$ below it leading to a vertex $\mathbf{t}_2$ with weight $\omega_3 - \alpha_3 = (0, 0, 1, 0) - (0, -1, 2, -1) = (0, 1, -1, 1)$. Apply fact (1) to see that below $\mathbf{t}_2$ is an edge of color $\alpha 4$ connecting to a vertex $\mathbf{t}_3$ with weight $(0, 1, 0, -1)$ and an edge of color $\alpha 2$ connecting to a vertex $\mathbf{t}_4$ with weight $(1, -1, 1, 1)$. Fact (2) now applies to the descendants $\mathbf{t}_3$ and $\mathbf{t}_4$ of $\mathbf{t}_2$ to guarantee that there is a vertex $\mathbf{t}_5$ below both $\mathbf{t}_3$ and $\mathbf{t}_4$ as depicted in Figure 5.11. From this point on, the reader should follow Figure 5.11 and consult Table 5.1 for weights of the new vertices. By (1), off of $\mathbf{t}_4$ are edges colored $\alpha 3$ and $\alpha 1$ to vertices $\mathbf{t}_6$ and $\mathbf{t}_7$ respectively. From this, we get $\mathbf{t}_8, \mathbf{t}_9, \mathbf{t}_{10}$, and $\mathbf{t}_{11}$ from fact (2). From vertex $\mathbf{t}_{10}$ an edge color of $\alpha 2$ is needed to vertex $\mathbf{t}_{12}$ by (1). Fact (2) generates $\mathbf{t}_{13}$. From vertex $\mathbf{t}_8$ edges of color $\alpha 4$ and $\alpha 3$ to vertices $\mathbf{t}_{14}$ and $\mathbf{t}_{17}$ respectively are created by fact (1). Fact (2) generates $\mathbf{t}_{15}, \mathbf{t}_{16}, \mathbf{t}_{18}, \mathbf{t}_{19}, \mathbf{t}_{20}, \mathbf{t}_{21}$, and $\mathbf{t}_{22}$ along with the edges above them. Below $\mathbf{t}_{12}$ an edge of color $\alpha 3$ is needed which goes to vertex $\mathbf{t}_{23}$, by fact (1). Fact (2) now generates $\mathbf{t}_{25}, \mathbf{t}_{26}$, and $\mathbf{t}_{27}$. Vertex $\mathbf{t}_{17}$ needs an edge of color $\alpha 2$ to vertex $\mathbf{t}_{28}$, by (1). Then fact (2) generates $\mathbf{t}_{29}$ through $\mathbf{t}_{35}$. By (1), vertex $\mathbf{t}_{31}$ has an edge $\alpha 3$ to vertex $\mathbf{t}_{36}$; (2) now generates $\mathbf{t}_{37}, \mathbf{t}_{38}$, and $\mathbf{t}_{39}$. Vertex $\mathbf{t}_{34}$ needs an edge of color $\alpha 3$ to a vertex $\mathbf{t}_{30}$, by (1). Vertices $\mathbf{t}_{41}$ and $\mathbf{t}_{42}$ are generated by (2). Now at vertex $\mathbf{t}_{27}$ we need an edge of color $\alpha 3$ to connect to a new vertex $\mathbf{t}_{43}$. 'Filling in' the structure using distributivity (fact(2)) creates vertices $\mathbf{t}_{44}, \mathbf{t}_{45}$ and $\mathbf{t}_{46}$. Thus the structure of Figure 5.11 is necessarily an edge-colored subgraph of L. One can check that $\mathbf{t}_{42}, \mathbf{t}_{45}$, and $\mathbf{t}_{46}$ have weight $(1, 1, -3, 2)$. But from the known information from the χ_{ω_3} character polynomial, the splitting poset L can only have one vertex of weight (1,1,-3,2). This is a contradiction. Therefore, there does not exist a splitting distributive lattice for the third fundamental character for F_4. $\square$

weight	vertices	weight	vertices	weight	vertices
$(0, 0, 1, 0)$	1	$(1, 0, 1, -2)$	14	$(0, 0, 1, -1)$	$22, 25, 32$
$(0, 1, -1, 1)$	2	$(1, 1, -2, 1)$	17	$(0, 1, -2, 2)$	26
$(0, 1, 0, -1)$	3	$(0, -1, 2, 0)$	13	$(1, -2, 2, 1)$	30
$(1, -1, 1, 1)$	4	$(-1, 1, 1 - 2)$	15	$(2, 0, -1, 0)$	36
$(1, -1, 2, -1)$	5	$(-1, 2, -2, 1)$	18	$(0, 1, -1, 0)$	$27, 37$
$(1, 0, -1, 2)$	6	$(1, 1, -1, -1)$	20	$(1, -2, 3, -1)$	33
$(-1, 0, 1, 1)$	7	$(0, 0, -1, 3)$	23	$(1, -1, 0, 2)$	34
$(1, 0, 0, 0)$	8	$(2, -1, 0, 1)$	28	$(1, -1, 1, 0)$	$35, 38$
$(-1, 0, 2, -1)$	9	$(0, -1, 3, -2)$	16	$(1, 0, -2, 3)$	40
$(-1, 1, -1, 2)$	10	$(0, 0, 0, 1)$	$19, 24, 29$	$(0, 2, -3, 1)$	43
$(-1, 1, 0, 0)$	11	$(-1, 2, -1, -1)$	21	$(1, 0, -1, 1)$	$39, 41, 44$
$(0, -1, 1, 2)$	12	$(2, -1, 1, -1)$	31	$(1, 1, -3, 2)$	$42, 45, 46$

Table 5.1: Weights corresponding to Figure 5.11

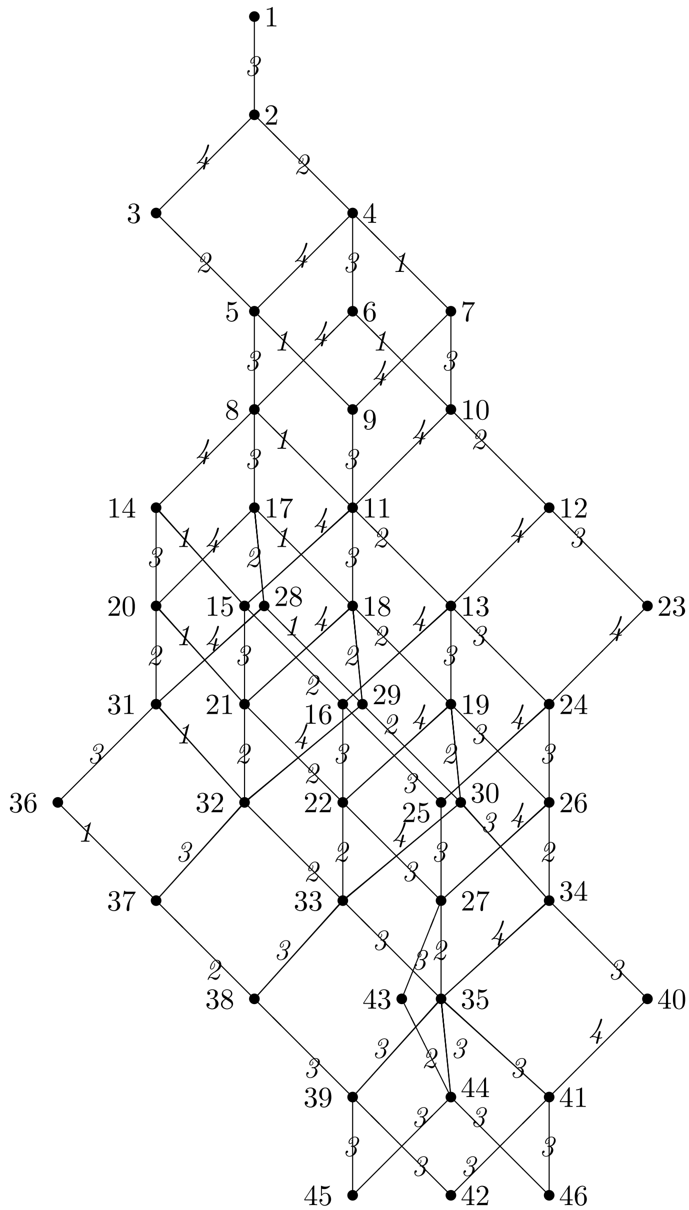


Figure 5.11: Necessary subgraph of any SDL for χ_{ω_3}

§5.4 The Second Fundamental Character

Following the last section, one might ask if F_4-character χ_{ω_2} has a splitting distributive lattice. In fact, it does not.

Theorem 5.6 *The F_4-character χ_{ω_2} has no splitting distributive lattice.*

Proof. The argument here is similar to the proof of Theorem 5.5, so we will use facts (1) and (2) from that proof. Assume that there is a splitting distributive lattice for χ_{ω_2}. Let $\mathbf{t}_1$ be a maximal vertex of weight $(0,1,0,0)$(see Figure 5.12 and Table 5.2). Then below $\mathbf{t}_1$ there is an edge of color $\alpha\mathit{2}$ to vertex $\mathbf{t}_2$ by fact (1). Vertex $\mathbf{t}_2$ is above edges of color $\alpha\mathit{3}$ and $\alpha\mathit{1}$ to vertices $\mathbf{t}_3$ and $\mathbf{t}_4$ respectively by fact (1). Fact (2) creates $\mathbf{t}_5$ below $\mathbf{t}_3$ and $\mathbf{t}_4$. Vertex $\mathbf{t}_3$ is above edges of color $\alpha\mathit{3}$ and $\alpha\mathit{4}$ to vertices $\mathbf{t}_7$ and $\mathbf{t}_6$ respectively. Fact (2) accounts for vertices $\mathbf{t}_8, \mathbf{t}_9, \mathbf{t}_{10}$, and $\mathbf{t}_{11}$. Then by fact (1) vertex $\mathbf{t}_{12}$ comes from an edge of color $\alpha\mathit{2}$ below vertex $\mathbf{t}_5$. Fact (2) generates $\mathbf{t}_{13}, \mathbf{t}_{14}$, and $\mathbf{t}_{15}$. Vertex $\mathbf{t}_{16}$ descends from vertex $\mathbf{t}_7$ via an edge of color $\alpha\mathit{2}$ by fact (1). We get vertices $\mathbf{t}_{17}$ through $\mathbf{t}_{21}$ by fact (2). Fact (1) makes vertex $\mathbf{t}_{22}$ below $\mathbf{t}_{14}$ along an edge of color $\alpha\mathit{3}$. Fact (2) then gives us $\mathbf{t}_{23}, \mathbf{t}_{24}$, and $\mathbf{t}_{25}$. Vertex $\mathbf{t}_{26}$ descends from $\mathbf{t}_{18}$ via an edge of color $\alpha\mathit{4}$, by fact (1). Then fact (2) necessitates $\mathbf{t}_{27}, \mathbf{t}_{28}$, and $\mathbf{t}_{29}$. By fact (1) we need an edge of color $\alpha\mathit{3}$ below $\mathbf{t}_{24}$ to a vertex $\mathbf{t}_{30}$. Then fact (2) gives $\mathbf{t}_{31}$ and $\mathbf{t}_{32}$. From fact (1) we see that $\mathbf{t}_{23}$ needs a descendant $\mathbf{t}_{33}$ along an edge of color $\alpha\mathit{3}$. Fact (2) then gives $\mathbf{t}_{34}, \mathbf{t}_{35}$, and $\mathbf{t}_{36}$. Note at this point that the structure of Figure 5.11 is a necessary edge-colored subgraph of L. Note also that vertices $\mathbf{t}_{32}, \mathbf{t}_{35}$, and $\mathbf{t}_{36}$ have the same weight $(1,1,-3,3)$. But by [Stem4] or [BMP] a splitting poset for χ_{ω_2} should only have one vertex of weight $(1,1,-3,3)$. From this contradiction we deduce that there are no splitting distributive lattices for χ_{ω_2}. $\square$

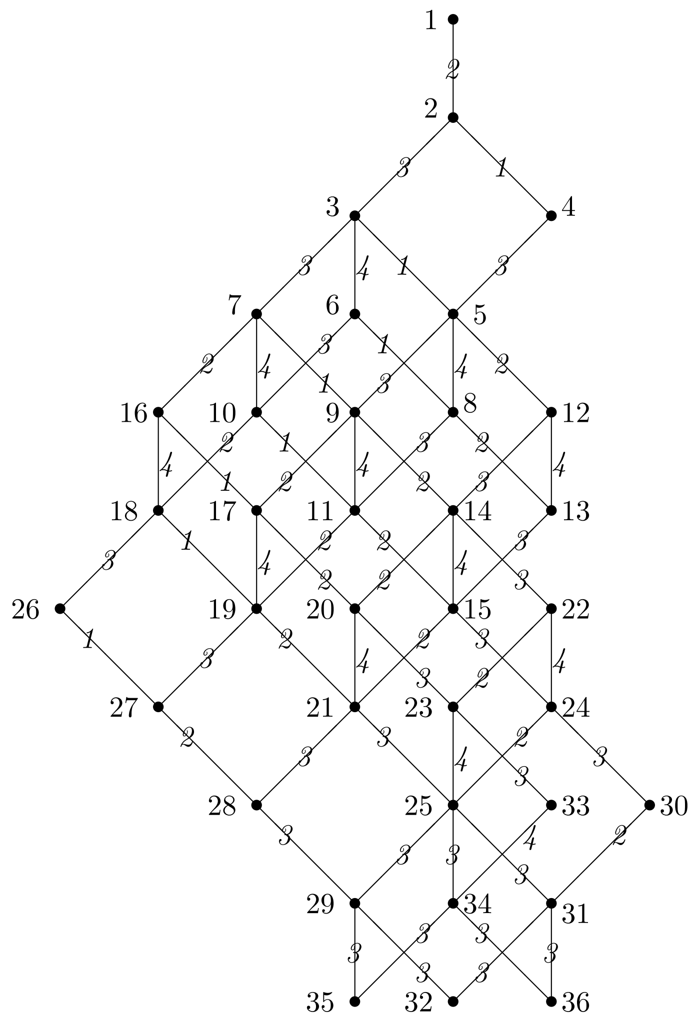


Figure 5.12: Necessary subgraph of any SDL for χ_{ω_2}

weight	vertices	weight	vertices	weight	vertices
$(0,1,0,0)$	1	$(0,-1,2,1)$	12	$(1,-2,3,0)$	21
$(1,-1,2,0)$	2	$(2,-1,0,2)$	16	$(1,-1,0,3)$	23
$(1,0,0,1)$	3	$(-1,2,-1,0)$	11	$(0,1,-1,1)$	$24,27$
$(-1,0,2,0)$	4	$(0,-1,3,-1)$	13	$(1,-1,1,1)$	$25,28$
$(-1,1,0,1)$	5	$(0,0,0,2)$	$14,17$	$(0,2,-3,2)$	30
$(1,0,1,-1)$	6	$(2,0,-1,1)$	18	$(1,0,-2,4)$	33
$(1,1,-2,2)$	7	$(0,0,1,0)$	$15,19$	$(1,0,-1,2)$	$29,31,34$
$(-1,1,1,-1)$	8	$(1,-2,2,2)$	20	$(1,1,-3,3)$	$32,35,36$
$(-1,2,-2,2)$	9	$(0,1,-2,3)$	22		
$(1,1,-1,0)$	10	$(2,0,-1,1)$	26		

Table 5.2: Weights corresponding to Figure 5.12

CHAPTER 6: REMARKS AND SPECULATIONS

We begin with some remarks and speculations concerning the distributive core of Chapter 3. The main result of Chapter 3 was Theorem 3.4, which showed that if the output $K(\mathfrak{g}, \lambda)$ of the distributive core algorithm is nonempty for some input Dynkin diagram $\mathfrak{g}$ and dominant weight λ, then $K(\mathfrak{g}, \lambda)$ is a $(\mathfrak{g}, \lambda)$-structured distributive lattice. Our use of the word 'core' in reference to this object $K(\mathfrak{g}, \lambda)$ is not suggested by the defining algorithm itself or the subsequent results of Chapter 3, but rather by the following conjecture.

Conjecture 6.1 *For a Dynkin diagram $\mathfrak{g}$, let λ be a dominant weight. If there exists a $(\mathfrak{g}, \lambda)$-structured distributive lattice L, then L contains a full length edge-colored sublattice that is isomorphic to the output $K(\mathfrak{g}, \lambda)$ of the distributive core algorithm. In particular, if the distributive core algorithm returns empty output for some input $\mathfrak{g}$ and λ, then there does not exist a $(\mathfrak{g}, \lambda)$-structured distributive lattice.*

This conjecture has been confirmed (by computer) for most of the known SDL's for small dimension irreducible Weyl characters. Moreover, the conjecture has been confirmed for those small dimension irreducible Weyl characters for which it is known that there are no SDL's. These are two key pieces of evidence which lead us to this conjecture. We have the following consequence of Conjecture 6.1.

Corollary 6.2 *Assume Conjecture 6.1 and that the distributive core output $K(\mathfrak{g}, \lambda)$ is nonempty. Let L be any $(\mathfrak{g}, \lambda)$-structured distributive lattice. Then the poset of irreducibles*

$\mathrm{j}_{color}(K(\mathfrak{g},\lambda))$ *has the same number of vertices as* $\mathrm{j}_{color}(L)$ *and contains a weak subposet which is vertex-color isomorphic to* $\mathrm{j}_{color}(L)$.

This follows from Conjecture 6.1 by applying Theorem 2.6. Based partly on our hypothesis that this corollary is true, we were able to use the distributive core together with our poset of irreducibles for the SDL of Theorem 5.1 in order to obtain the poset of irreducibles for the SDL of Theorem 5.3. Two other immediate consequences of Conjecture 6.1 are:

Corollary 6.3 *Assume Conjecture 6.1 and that the distributive core output* $K(\mathfrak{g},\lambda)$ *is nonempty. Let* L *be any* $(\mathfrak{g},\lambda)$*-structured distributive lattice. Then* L *has at least as many elements as* $K(\mathfrak{g},\lambda)$*. Moreover,* L *has the same number of elements as* $K(\mathfrak{g},\lambda)$ *if and only if* $L \cong K(\mathfrak{g},\lambda)$.

Corollary 6.4 *Assume Conjecture 6.1 and that the distributive core output* $K(\mathfrak{g},\lambda)$ *is empty. Then there does not exist a splitting distributive lattice for the irreducible* $\mathfrak{g}$*-character* χ_λ.

Corollary 6.4 follows directly from the second part of Conjecture 6.1 because an SDL for the $\mathfrak{g}$-character χ_λ is a $(\mathfrak{g},\lambda)$-structured distributive lattice. In fact, it was in the spirit of this corollary that we modified the distributive core approach to prove that χ_{ω_2} and χ_{ω_3} have no SDL's. More generally, we hope that Conjecture 6.1 and its corollaries might be used someday to help classify for all connected Dynkin diagrams of finite type which irreducible Weyl characters have SDL's.

Paired with our 'nonexistence results' for χ_{ω_2} and χ_{ω_3} are the existence results for $\chi_{2\omega_4}$ and $\chi_{2\omega_1}$ (Theorems 5.1 and 5.3). Inspired in part by analogy with the G_2 case (see §2.18), we ask: Can the construction of the posets of irreducibles for our SDL's for $\chi_{2\omega_4}$ and $\chi_{2\omega_1}$

be generalized to the $\chi_{a\omega_1+b\omega_4}$ cases, where a and b are arbitrary nonnegative integers? The fact that the distributive core algorithm returns nonempty output for the 1053-dimensional $\chi_{\omega_1+\omega_4}$ and the 2652-dimensional $\chi_{3\omega_4}$ cases gives us further reason to hope that SDL's might exist for the $\chi_{a\omega_1+b\omega_4}$ cases. It is worth investigating whether there are other 324-element $\chi_{2\omega_4}$ SDL's contained inside the 351-element SDL for the E_6-character $\chi_{2\omega_1}$ mentioned in §5.1. Other such SDL's for the F_4-character $\chi_{2\omega_4}$ might be useful in pursuing a generalization of our existence results to the $\chi_{a\omega_1+b\omega_4}$ cases.

For the SDL's we have in hand or those which might arise as generalizations of our work here, it is appropriate to ask whether these are/will be supporting graphs for irreducible representations of the simple Lie algebra of type F_4, or whether these have/might have the strong Sperner property or symmetric chain decompositions. For the new SDL's of Theorems 5.1 and 5.3, these are open questions.

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