On Kepler's Laws of Planetary Motion
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NOTE: All section references are to the text Calculus with Early Transcendentals, 4th ed., by James Stewart.

It is possible to use purely mathematical reasoning to prove that planets have elliptical orbits. This can be done by solving a sequence of simple problems. In what follows, $\mathbf{r}(t)$ is a smooth, arbitrary speed curve, $\mathbf{v}(t) := \mathbf{r}'(t)$, and $\mathbf{a}(t) := \mathbf{r}''(t)$. The assertions in #1 through #7 below make no reference to physics whatsoever, but it still might be helpful to think of $\mathbf{r}(t)$ as the curve traced out in space by the orbit of a planet. Although $\mathbf{r}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$ are functions of time and depend on t, we will sometimes omit t and just write $\mathbf{r}$, $\mathbf{v}$, or $\mathbf{a}$ when referring to these vector functions.

1. Suppose that the acceleration $\mathbf{a}(t)$ is parallel to $\mathbf{r}(t)$ for all time t. Prove that the torsion $\tau(t)$ is zero for all t (and hence the curve lies in a plane).

   DISCUSSION: A curve $\mathbf{r}(t)$ lies in a plane if and only if its torsion $\tau(t)$ is identically zero for all time t (cf. Exercise #43 (d) from §13.3.) Now by Exercise #45 (d) from §13.3, we have

   $$\tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

   We have $\mathbf{a}(t) = \mathbf{r}''(t) = \lambda(t)\mathbf{r}(t)$ for some scalar function $\lambda(t)$ since $\mathbf{a}$ is parallel to $\mathbf{r}$ at any time t. Thus, $\mathbf{r}'''(t) = \mathbf{a}'(t) = \frac{d}{dt}(\lambda(t)\mathbf{r}(t)) = \lambda'(t)\mathbf{r}(t) + \lambda(t)\mathbf{r}'(t)$. So we have:

   $$\tau(t) = \frac{(\mathbf{r}' \times \lambda\mathbf{r}) \cdot (\lambda'\mathbf{r} + \lambda\mathbf{r}')}{|\mathbf{r}' \times \lambda\mathbf{r}|^2}$$

   Why does it follow that $\tau(t) = 0$ for all time t?

2. Under these same assumptions, show that $\mathbf{r}(t) \times \mathbf{v}(t)$ is some constant vector $\mathbf{c}$.

   DISCUSSION: Using the product rule for derivatives of cross products, we have

   $$\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{v}(t)) = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a}$$

   Why are both of the cross products on the right-hand side of this identity equal to $\mathbf{0}$ (the zero vector)? Once this has been established, it follows that the derivative of $\mathbf{r}(t) \times \mathbf{v}(t)$ is $\mathbf{0}$, whence $\mathbf{r}(t) \times \mathbf{v}(t)$ is some constant vector $\mathbf{c}$.

3. If $\mathbf{r}$ and $\mathbf{a}$ are parallel for all time t, prove that

   $$\frac{d}{dt}\Big(\mathbf{v} \times (\mathbf{r} \times \mathbf{v})\Big) = \mathbf{a} \times (\mathbf{r} \times \mathbf{v}).$$

   DISCUSSION: Using the product rule for derivatives of cross products, we have

   $$\frac{d}{dt}\Big(\mathbf{v} \times (\mathbf{r} \times \mathbf{v})\Big) = \mathbf{a} \times (\mathbf{r} \times \mathbf{v}) + \mathbf{v} \times \Big(\frac{d}{dt}(\mathbf{r} \times \mathbf{v})\Big)$$

   Why is $\dfrac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{0}$?

4. If $\mathbf{a} = -\dfrac{k}{r^3}\mathbf{r}$ (where k is some constant and $r = r(t) = |\mathbf{r}(t)|$), show that

   $$\mathbf{a} \times (\mathbf{r} \times \mathbf{v}) = \frac{d}{dt}\Big(\frac{k}{r}\mathbf{r}\Big).$$

DISCUSSION: On the right-hand side, we see that $\frac{d}{dt}\left(\frac{k}{r}\mathbf{r}\right) = -\frac{k\,r'}{r^2}\mathbf{r} + \frac{k}{r}\mathbf{r}'$. An identity from Theorem 8 of §12.4 says that $\mathbf{a} \times (\mathbf{r} \times \mathbf{v}) = (\mathbf{a} \cdot \mathbf{v})\mathbf{r} - (\mathbf{a} \cdot \mathbf{r})\mathbf{v}$. So it must be shown that $\mathbf{a} \cdot \mathbf{v} = -\frac{k\,r'}{r^2}$ and that $\mathbf{a} \cdot \mathbf{r} = -\frac{k}{r}$. The last identity is easy, since $\mathbf{a} \cdot \mathbf{r} = -\frac{k}{r^3}\mathbf{r} \cdot \mathbf{r} = -\frac{k}{r^3}r^2 = -\frac{k}{r}$. So why does $\mathbf{a} \cdot \mathbf{v} = -\frac{k\,r'}{r^2}$? HINT: Start with the identity $r^2 = \mathbf{r} \cdot \mathbf{r}$ and differentiate both sides with respect to t.

5. Again assume that $\mathbf{a} = -\dfrac{k}{r^3}\mathbf{r}$. Show that there exists a constant vector $\mathbf{b}$ such that:

$$\mathbf{v} \times (\mathbf{r} \times \mathbf{v}) = \frac{k}{r}\mathbf{r} + \mathbf{b}.$$

DISCUSSION: By #3 and #4, we see that the vector functions $\mathbf{v}(t) \times (\mathbf{r}(t) \times \mathbf{v}(t))$ and $\dfrac{k}{r(t)}\mathbf{r}(t)$ have the same derivative. Then they must differ by a constant, i.e. there is a constant vector $\mathbf{b}$ such that $\mathbf{v}(t) \times (\mathbf{r}(t) \times \mathbf{v}(t)) = \dfrac{k}{r(t)}\mathbf{r}(t) + \mathbf{b}$ for all time t.

6. Show that the (polar coordinates) function $r = \dfrac{p}{1 + e\cos\theta}$ is an ellipse. (Here, p and e are constants, but e isn't meant to be confused with the base of the natural exponential function.)

DISCUSSION: It's worth trying to work this out by hand on one's own. For brevity, however, I'll just refer you to Theorem 6 of §10.7.

7. Now again assume that $\mathbf{a} = -\dfrac{k}{r^3}\mathbf{r}$. Show that $\mathbf{r}(t)$ moves along an ellipse with equation (in polar coordinates) $r = \dfrac{p}{1 + e\cos\theta}$, where $p = \frac{\|\mathbf{c}\|^2}{k}$ and $e = \frac{\|\mathbf{b}\|}{k}$.

DISCUSSION: Let P be the plane through the origin that is orthogonal to the vector $\mathbf{c}$. Since $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$ for all time t, it follows that $\mathbf{r}(t)$ lies in P. Also, $\mathbf{v}(t) \times \mathbf{c}$ is orthogonal to $\mathbf{c}$ for all time t, so $\mathbf{v}(t)$ lies in P. Since $\mathbf{b} = \mathbf{v} \times \mathbf{c} - \frac{k}{r}\mathbf{r}$, then $\mathbf{b} \cdot \mathbf{c} = 0$, so $\mathbf{b}$ is in P. Let $\theta = \theta(t)$ be the angle in the plane P between the vectors $\mathbf{r}(t)$ and $\mathbf{b}$ as $\mathbf{r}(t)$ sweeps through space. Then:

$$\begin{aligned}
\mathbf{c} \cdot \mathbf{c} &= (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{c} \\
&= \mathbf{r} \cdot (\mathbf{v} \times \mathbf{c}) \quad \text{(Why?) (See Theorem 8 §12.4)} \\
&= \mathbf{r} \cdot (\frac{k}{r}\mathbf{r} + \mathbf{b}) \\
&= \frac{k}{r}\mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{b} \\
&= \frac{k}{r}r^2 + |\mathbf{r}||\mathbf{b}|\cos\theta
\end{aligned}$$

So we see that $|\mathbf{c}|^2 = kr + r|\mathbf{b}|\cos\theta$. Solve for r to get $r = \dfrac{p}{1 + e\cos\theta}$, where $p = \frac{\|\mathbf{c}\|^2}{k}$ and $e = \frac{\|\mathbf{b}\|}{k}$.

8. **Application to celestial mechanics:** Let $\mathbf{r}(t)$ be the space curve describing the orbit of a planet moving through the gravitational field of the sun. Prove that the orbit is elliptical by putting together Newton's Second Law of motion ($\mathbf{F} = m\mathbf{a}$) with Newton's Universal Law of Gravitation ($\mathbf{F} = -\dfrac{GMm}{r^2}\mathbf{u}$). (In this last equation, G is the universal gravitational constant, M is the mass of the sun, m is the mass of the planet, and $\mathbf{u}$ is a unit vector in the direction of $\mathbf{r}$.)

DISCUSSION: All we need to do now is show that $\mathbf{a} = -\dfrac{k}{r^3}\mathbf{r}$ for some constant k, and then we can invoke #7. Notice that in Newton's Universal Law of Gravitation, $\mathbf{u} = \frac{1}{r}\mathbf{r}$. Now set $m\mathbf{a} = -\frac{GMm}{r^2}\mathbf{u} = -\frac{GMm}{r^2}\frac{1}{r}\mathbf{r} = -\frac{GMm}{r^3}\mathbf{r}$.