The Geometry of Curves, Part II

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From Murray State University's Calculus III, Fall 2001

NOTE: This material supplements Sections 13.3 and 13.4 of the text
Calculus with Early Transcendentals, 4th ed., by James Stewart.

V. Arbitrary Speed Curves

We've been taking great care to make sure that all of our curves are unit speed, so what happens if we relax this assumption? Can we still define curvature and torsion in the same way? What will the Frenet frame look like in this case?

We want to associate an orthonormal frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ to each point $\mathbf{r}(t)$ of our curve, as before. In fact, we'd like this frame to be the same as the frame defined in Section III, if $\mathbf{r}(t)$ happens to be unit speed. While it isn't obvious at first, if we define the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ by the following formulas, we get what we want:

$$\mathbf{T}(t) \stackrel{\text{def}}{=} \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\mathbf{N}(t) \stackrel{\text{def}}{=} \frac{\|\mathbf{r}'(t)\|}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}\mathbf{r}''(t) - \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}\mathbf{T}(t)$$

$$\mathbf{B}(t) \stackrel{\text{def}}{=} \mathbf{T}(t) \times \mathbf{N}(t) \stackrel{\text{exercise}}{=} \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}$$

Exercise 1. Prove directly from the definitions that

$$\mathbf{B} = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}.$$

Prove that $\mathbf{N} \cdot \mathbf{T} = 0$ and that $\mathbf{N} \cdot \mathbf{N} = 1$, and then prove that the frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is orthonormal.

With some patience, one can also prove the following facts (compare to Exercise III.2):

$\mathbf{T}'$ relations:	$\mathbf{T}' \cdot \mathbf{T} = 0$	and	$\mathbf{T}' \cdot \mathbf{B} = 0$
$\mathbf{N}'$ relations:	$\mathbf{N}' \cdot \mathbf{N} = 0$		
$\mathbf{B}'$ relations:	$\mathbf{B}' \cdot \mathbf{T} = 0$	and	$\mathbf{B}' \cdot \mathbf{B} = 0$

Combining these with Exercise 1 above allows us to write down exact analogs of the Frenet formulas. But this time (as you might have suspected) these new formulas will include speed in some non-trivial way. So let $v(t) = \|\mathbf{r}'(t)\|$ be the speed of $\mathbf{r}(t)$, and let $\kappa(t) = \dfrac{\mathbf{T}' \cdot \mathbf{N}}{v}$ and $\tau(t) = \dfrac{\mathbf{N}' \cdot \mathbf{B}}{v}$. Here are the new Frenet formulas:

$$\begin{aligned} \mathbf{T}' &= \kappa v \mathbf{N} \\ \mathbf{N}' &= -\kappa v \mathbf{T} + \tau v \mathbf{B} \\ \mathbf{B}' &= -\tau v \mathbf{N} \end{aligned}$$

Are there more direct formulas that we can use to compute curvature and torsion, without having to first compute **T**, **N**, and **B**? The good news is that there are such formulas. The bad news is that they are a little messy:

$$\begin{aligned}
\kappa(t) &= \frac{\mathbf{T}' \cdot \mathbf{N}}{v} && \text{(By definition)} \\
&= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} && \text{(Messy to prove)}
\end{aligned}$$

$$\begin{aligned}
\tau(t) &= \frac{\mathbf{N}' \cdot \mathbf{B}}{v} && \text{(By definition)} \\
&= \frac{\big(\mathbf{r}'(t) \times \mathbf{r}''(t)\big) \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2} && \text{(Even messier)}
\end{aligned}$$

Exercise 2. The curve $\mathbf{r}(t) = \langle r\cos t^2, r\sin t^2, 0\rangle$ traces along a circle of radius r in the xy-plane (why?). Even though it is not unit speed (check this), its curvature should still be the same as the curvature for a unit-speed parameterization. Use the above formulas to show that the curvature of $\mathbf{r}(t)$ is $\kappa(t) = 1/r$ for time $t \neq 0$.

Exercise 3. If $\mathbf{r}(t)$ is a unit speed curve (i.e. $\|\mathbf{r}'(t)\| = 1$ for all t), show that the formulas for κ and τ in the box above reduce to the boxed formulas for κ and τ of Section IV.

Exercise 4. Let $y = f(x)$ be a differentiable function. Describe the graph of f as a curve in parametric form. Then find a formula for the curvature in terms of derivatives of f. (Hint: Even though the curve is in the xy-plane, it's best to use 3 coordinates to describe the curve (with $z = 0$ of course). That is, let $\mathbf{r}(t) = \langle t, f(t), 0\rangle$.)

Exercise 5. Compute the curvature and torsion for the twisted cubic

$$\mathbf{r}(t) = \langle t, \alpha t^2 + \beta t + \gamma, at^3 + bt^2 + ct + d\rangle.$$

Under what conditions will the twisted cubic be contained in a plane? (Remember: a curve is in a plane if and only if its torsion is identically zero at each point.)

VI. Isometries

How do we measure the distance between points? In the plane, of course, we have the distance formula (which you've known about since high school), and now we even have a distance formula for points in space. These two distance formulas are actually *exactly* the same if we write them in vector form (yet another advantage of vector notation). Specifically, let $\mathbf{v}$ and $\mathbf{w}$ be vectors (in the plane or in space). Then

$$\textit{distance from } \mathbf{v} \textit{ to } \mathbf{w} = \|\mathbf{v} - \mathbf{w}\|.$$

Exercise 1. Use the distance formulas to confirm this claim.

The title for this section might be a word that is unfamiliar to you. The word *isometry* literally means "same measure." Mathematically, an "isometry of the plane" is a function I that takes $\mathbb{R}^2$ to $\mathbb{R}^2$ in such a way that the distance from $\mathbf{v}$ to $\mathbf{w}$ is the same as the distance from $I(\mathbf{v})$ to $I(\mathbf{w})$. An "isometry of $\mathbb{R}^3$" does the same thing. In other words, for us *an isometry is a distance- or length-preserving function.* Returning once again to the title of this handout, we can now say with more precision that geometry is the study of properties (of curves) that are invariant under (i.e. don't change under) isometries. We'll use isometries to give a precise meaning to the word "congruence": we say that two curves $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are *congruent* if there is an isometry I that takes $\mathbf{r}(t)$ and places it exactly on top of $\mathbf{s}(t)$. That is, $\mathbf{s}(t) = I\big(\mathbf{r}(t)\big)$ for all t.

The problem is: can you think of any examples of isometries?

Exercise 2. Isometries of the plane.

(a) Suppose a function $F_{\mathbf{a}} : \mathbb{R}^2 \to \mathbb{R}^2$ flips (i.e. *reflects*) points in the plane across a line through the origin in the direction of $\mathbf{a}$. (In the case $\mathbf{a} = \mathbf{0}$, we define $F_{\mathbf{a}}$ to be the "identity transformation": $F_{\mathbf{a}}(\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v}$.) If $\mathbf{a} \neq \mathbf{0}$, then let $\mathbf{p}$ be *any* unit vector perpendicular to $\mathbf{a}$. Notice that there is more than one possibility for $\mathbf{p}$. Now define the function $F_{\mathbf{a}}$ by the rule

$$F_{\mathbf{a}}(\mathbf{v}) = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{p})\mathbf{p}.$$

Verify that this is exactly the reflection we're looking for by checking these things:

1. $F_{\mathbf{a}}(\mathbf{v}) = \mathbf{v}$ if and only if $\mathbf{v}$ is on the same line as $\mathbf{a}$.
2. $F_{\mathbf{a}}(\mathbf{v})$ has the same length as $\mathbf{v}$. (Hint: Use dot products.)
3. $F_{\mathbf{a}}(\mathbf{v})$ makes the same angle with $\mathbf{a}$ as $\mathbf{v}$ does.
4. If $\mathbf{q}$ is *any* other unit vector perpendicular to $\mathbf{a}$, then $F_{\mathbf{a}}(\mathbf{v}) = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{q})\mathbf{q}$. (Thus, if two people choose different unit vectors $\mathbf{p}$ and $\mathbf{q}$ both orthogonal to $\mathbf{a}$, then they still get the same reflection $F_{\mathbf{a}}$.)
5. Why do all these facts imply that $F_{\mathbf{a}}$ is exactly the reflection we're looking for?

Now prove that $F_{\mathbf{a}}$ is an isometry.

(b) Suppose the function $T_{\mathbf{b}} : \mathbb{R}^2 \to \mathbb{R}^2$ shifts (i.e. *translates*) the points of the plane by the rule:

$$T_{\mathbf{b}}(\mathbf{v}) = \mathbf{v} + \mathbf{b}.$$

(Notice that $T_{\mathbf{b}}$ is the "identity transformation": $T_{\mathbf{b}}(\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v}$.) Prove that $T_{\mathbf{b}}$ is an isometry.

(c) One other kind of isometry in the plane is a *rotation*. Recall that if $\mathbf{v} = \langle x, y\rangle$ has length r and makes an angle of θ with the positive x-axis, then $\mathbf{v} = \langle r\cos\theta, r\sin\theta\rangle$. Conceptually, a *rotation through an angle* α should take the point with polar coordinates (r, θ) to the point with polar coordinates $(r, \theta+\alpha)$. Formally, the rotation $R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is the function that takes the vector $\langle r\cos\theta, r\sin\theta\rangle$ to the vector $\langle r\cos(\theta+\alpha), r\sin(\theta+\alpha)\rangle$. (Notice that R_0 is the "identity transformation": $R_0(\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v}$.) Prove that $R_\alpha(\langle x, y\rangle) = \langle x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha\rangle$.

Set $\mathbf{v} = \langle r\cos\theta, r\sin\theta\rangle$ and set $\mathbf{w} = \langle s\cos\psi, s\sin\psi\rangle$. Express $\|\mathbf{v}-\mathbf{w}\|$ in terms of r, s, θ, and ψ. Then prove that R_α is an isometry by showing that $\|\mathbf{v}-\mathbf{w}\| = \|R_\alpha(\mathbf{v})-R_\alpha(\mathbf{w})\|$.

(d) With $\mathbf{a} = \langle -2, 3\rangle$, $\mathbf{b} = \langle 1, -2\rangle$, and $\alpha = 45°$, let I be the isometry $I = T_{\mathbf{b}} \circ R_\alpha \circ F_{\mathbf{a}}$. Sketch what happens to the parabola $\mathbf{r}(t) = \langle t, t^2\rangle$ when we apply the isometry I.

Exercise 3. If I is an isometry, and if $\mathbf{r}(t)$ is a curve defined on a closed interval $[a, b]$, then we can define a new curve $\mathbf{s}(t) \stackrel{\text{def}}{=} I(\mathbf{r}(t))$. Note that the domain of $\mathbf{s}$ is also $[a, b]$. Now suppose that $\mathbf{r}(t)$ is a unit speed curve in the plane. For each isometry I below, prove that $\mathbf{s}(t) = I(\mathbf{r}(t))$ is also a unit speed curve, and has the same curvature and torsion as $\mathbf{r}(t)$.

(a) $I = F_{\mathbf{a}}$ from part (a) of the previous problem.

(b) $I = T_{\mathbf{b}}$ from part (b) of the previous problem.

(c) $I = R_\alpha$ from part (c) of the previous problem.

So now we know how to describe reflections, translations, and rotations in the plane. But what about reflections, translations, and rotations in space? In $\mathbb{R}^3$, a reflection flips points across a *plane* instead of a line. A translation of $\mathbb{R}^3$ can be defined exactly as in part (b) of the previous exercise (yet another advantage of vector notation). Rotations are only slightly more complicated: this time you pick an axis (say the line through the origin with direction $\langle 1, 2, -3\rangle$) and then rotate all points around this axis. We won't write down the formula for rotations, but we leave it as an exercise for you to write down a formula for a reflection in $\mathbb{R}^3$.

We have emphasized translations, rotations, and reflections in our discussion of isometries because these are essentially the *only* isometries (along with any combination of these), according to the following proposition:

Proposition. *Any isometry I of the plane or of space can be written as the composition of a reflection F followed by a rotation R and then a translation T:*

$$I = T \circ R \circ F.$$

(Note: Any of T, R, or F might be the identity transformation (defined in Exercise 2 above), and thus have no real effect.)

VII. The Fundamental Theorem of Curves

Do you recognize the following theorem?

Theorem. *Let T be a triangle, and let $\{T_1, T_2, T_3\}$ be the set of the lengths of the sides of T. Let S be another triangle, with "side length set" $\{S_1, S_2, S_3\}$. Then S is congruent to T if and only if $\{T_1, T_2, T_3\} = \{S_1, S_2, S_3\}$ (i.e. S and T have the same "side length set.")*

This is the famous Side-Side-Side (SSS) criterion for the congruence of triangles, which you probably learned about in high school geometry. Can you prove this theorem? There are also several other criteria for the congruence of triangles: the Side-Angle-Side (SAS) and the Angle-Side-Angle (ASA) criteria are among them. (Alas, I hate to disappoint any of the overgrown Bart Simpsons who are reading this right now, but the (infamous) Angle-Side-Side criterion does NOT imply congruence of triangles. Can you produce a counterexample?)

It might be worth your while at this point to re-read the first few paragraphs from the Introduction (Section I). There we define a *geometric property* of a curve (also called a *geometric invariant*) to be a property that does not depend on the particular placement of the curve within a coordinate system. Now, one consequence of the SSS criterion for triangles is that *if* two triangles are congruent, *then* they have the same "side length set." In other words, the side length set of a triangle is a geometric invariant: the side length set won't change no matter where you put the triangle. What does *congruent* mean? In the language of Section VI, two triangles S and T are congruent if there is an isometry that takes S to T.

With this (re-)orientation to geometric invariants in mind, we are now in a position to state the culminating result from all our work on curvature and torsion. This is called the Fundamental Theorem of Curves, and has two parts. We will state this beautiful result, but we will not prove it here.

Fundamental Theorem of Curves. *Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be arbitrary speed curves, defined on the same closed interval $[a, b]$. Assume that $\mathbf{r}$ and $\mathbf{s}$ have the same speed, i.e. $\|\mathbf{r}'(t)\| = \|\mathbf{s}'(t)\|$. Then:*

(a) *If $\mathbf{r}$ and $\mathbf{s}$ are congruent (i.e. they differ by an isometry), then they have the same curvature and torsion.*

(b) *If $\mathbf{r}$ and $\mathbf{s}$ have the same curvature and torsion, then they are congruent.*

One aspect of this Fundamental Theorem (Part (a)) is that curvature and torsion are geometric properties. Take a look again at Exercise VI.3: in that problem you basically showed that for curves in the plane, curvature is a geometric invariant. But to me, the other aspect of this Theorem (Part (b)) is the most striking. In practice, it tells us that if we compute the curvature κ and torsion τ of a given curve, and it turns out that κ and τ are the same as for some other known curve, then the two curves must be congruent. There are several applications of this in the exercise set.

The Fundamental Theorem of Curves has effectively liberated us from coordinates; this great theorem has given to us a coordinate-free characterization of curves. It says that all of the geometric information about a curve is encoded in two functions, the curvature and the torsion. To this extent, the Fundamental Theorem of Curves can be viewed as a vast generalization of the SSS criterion for triangles.

Exercise 1. If $\mathbf{r}(t)$ is a unit speed curve with constant positive curvature κ and with torsion $\tau(t) = 0$ for all t, prove that $\mathbf{r}(t)$ must be (part of) a circle. HINT: What is the curvature and torsion of a circle?

Exercise 2. If $\mathbf{r}(t)$ is a unit speed curve with constant positive curvature κ and with constant non-zero torsion τ, prove that $\mathbf{r}(t)$ must be (part of) a circular helix. HINT: See hint for Exercise 1 above.

Exercise 3. Consider the curve

$$\mathbf{r}(t) = \left(\frac{6}{\sqrt{17}}\cos\left(\frac{t}{3}\right) + \frac{3}{\sqrt{2}}\sin\left(\frac{t}{3}\right)\right)\mathbf{i} + \left(\frac{-9}{\sqrt{17}}\cos\left(\frac{t}{3}\right)\right)\mathbf{j} + \left(\frac{6}{\sqrt{17}}\cos\left(\frac{t}{3}\right) - \frac{3}{\sqrt{2}}\sin\left(\frac{t}{3}\right)\right)\mathbf{k}$$

Prove that $\mathbf{r}(t)$ is (part of) a circle. What is its radius?

Exercise 4. Consider the curve

$$\begin{aligned}\mathbf{r}(t) = &\left(\frac{5}{2}\cos\left(\frac{t}{5}\right) + \frac{3}{2}\sin\left(\frac{t}{5}\right) + \frac{2\sqrt{2}t}{15}\right)\mathbf{i}\\ &+\left(-\frac{3}{2}\cos\left(\frac{t}{5}\right) + \frac{3}{2}\sin\left(\frac{t}{5}\right) + \frac{2\sqrt{2}t}{5}\right)\mathbf{j}\\ &+\left(\frac{\sqrt{2}}{2}\cos\left(\frac{t}{5}\right) - \frac{3\sqrt{2}}{2}\sin\left(\frac{t}{5}\right) + \frac{8t}{15}\right)\mathbf{k}\end{aligned}$$

Prove that $\mathbf{r}(t)$ is (part of) a circular helix (cf. Exercises III.4 and VII.2).

Exercise 5. Compute the curvature and torsion for the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$. Then use the Fundamental Theorem of Curves to prove that the curve

$$\mathbf{s}(t) = \left\langle \frac{4}{3\sqrt{2}}t^2 + \frac{1}{3}t^3\ ,\ \frac{1}{\sqrt{2}}t - \frac{1}{3\sqrt{2}}t^2 + \frac{2}{3}t^3\ ,\ -\frac{1}{\sqrt{2}}t - \frac{1}{3\sqrt{2}}t^2 + \frac{2}{3}t^3 \right\rangle$$

is congruent to the twisted cubic $\mathbf{r}(t)$.

VIII. More Exercises

Exercise 1. List at least three situations in which it is to our advantage to describe a given phenomenon using the language of vectors. (HINT: one advantage was listed in the

Introduction (Section I), and two more were given in Section VI.)

Exercise 2. Consider the curve $\mathbf{r}(t) = \langle a\cos t, b\sin t, 5\rangle$. Notice that the z-coordinate is the constant 5.

(a) Prove that $\mathbf{r}(t)$ is contained entirely within a plane. Show that the torsion $\tau(t)$ is zero for all time t. (In fact, this curve is a *conic*... but what kind of conic?)

(b) Compute the curvature of $\mathbf{r}(t)$. Where is the curvature maximized? minimized? (NOTE: Without some assumptions about a and b, the curve probably won't be unit speed, so you'll have to compute curvature using Section V.)

Exercise 3. Suppose that $\mathbf{r}(t)$ is a unit speed curve whose image lies on a sphere of radius r centered at the origin. In other words, $\|\mathbf{r}(t)\| = r$ for all time t. (For example, our curve might be the equator of the sphere.) Now, it seems to me that the smallest the curvature of the curve could be is $1/r$ (in the case that the curve is a great circle for example). So I would suspect that $\kappa(t) \geq 1/r$ for all t. Prove this.

Exercise 4. *The circle of curvature.* Let $\mathbf{r}(t)$ be a curve in the plane. Fix a point $\mathbf{r}(t_0) = \langle x_0, y_0\rangle$ on the curve. The *circle of curvature at* $\langle x_0, y_0\rangle$ is the circle that best approximates the shape of $\mathbf{r}(t)$ near the point $\langle x_0, y_0\rangle$. If the curvature $\kappa(t_0)$ of $\mathbf{r}$ is positive at this point, then the radius of the circle of curvature is defined to be $r = 1/\kappa(t_0)$. We also want the normal $\mathbf{N}(t_0)$ to point toward the center of the circle of curvature. Thus the center of the circle of curvature is the point $\mathbf{c} = \langle x_0, y_0\rangle + r\mathbf{N}(t_0)$.

(a) Find an equation for the circle of curvature for the parabola $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2\rangle$ when $t_0 = 0$ and when $t_0 = 2$. Sketch these and see if you believe that the circle of curvature works as an approximation to the curve near these points.

(b) Repeat the previous exercise for the cubic $\mathbf{r}(t) = \langle t, \frac{1}{3}t^3\rangle$ and $t_0 = 1$.

(c) Find an equation for the circle of curvature for the circle of radius $r = 5$ centered at the point $\langle 2, 3\rangle$.