The Geometry of Curves, Part I

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From Murray State University's Calculus III, Fall 2001

NOTE: This material supplements Sections 13.3 and 13.4 of the text Calculus with Early Transcendentals, 4th ed., by James Stewart.

I. Introduction

We're going to study the geometry of curves (e.g. lines, circles, and more exotic things like *helices*) using calculus techniques. But before talking about the calculus techniques that we will need, we should first understand what we mean by "geometry." The following definition of geometry was offered up by a former student of mine. The way she put it, *geometry is the study of shapes.*

Now, in Calculus when we study an object (like a line or a circle), we place the object in a coordinate system. So, for example, we usually draw our circles centered at the origin. But the "geometric properties" of a circle shouldn't depend on where we draw it. For instance, we would say that two circles that have the same radius are *congruent* (no matter where you decide to draw them in the plane). Thus, the radius of a circle is a geometric property.

More precisely, we'll say that geometry is the study of properties (of curves) that don't change when you *shift*, *spin*, or *flip* the curves. Said in the parlance of advanced mathematics: geometry is the study of properties (of curves) that are invariant under *translations*, *rotations*, and *reflections*.

Exercise 1. List several geometric properties of ellipses, rectangles, and triangles.

In Calculus I you studied graphs of functions in the plane by using the first derivative (to figure out where the curve is increasing or decreasing) and the second derivative (to figure out the concavity). This Calculus I approach had some limitations, though. For instance, we could analyze graphs of functions, but this left out the graphs of things that aren't functions—like circles. (Eventually we took care of this problem by introducing parametric equations for curves.) At any rate, our analysis of curves in Calculus I wasn't really "geometric" because we were very concerned with how and where the graph of a function sits in the plane. And ultimately, the biggest drawback is that it's not clear at this point how to use Calculus I methods to analyze space curves.

Our main goal is to develop a method for analyzing the geometry of space curves that will also work for curves in the plane. If Calculus I is any indication, it seems pretty clear that the derivative $\mathbf{r}'(t)$ (what we have been calling the "velocity") and the second derivative $\mathbf{r}''(t)$ (the "acceleration") should figure prominently into our analysis of curves. But this time everything will be couched in the language of *vectors*. We've already seen one simple advantage of vectors: the vector form for the equation of a line in space— $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$ —is exactly the same as the vector form for the equation of a line in the plane.

The following exercise is an important warm-up for what's coming up.

Exercise 2. An *orthonormal basis* for $\mathbb{R}^3$ is a set of three vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ when $i \neq j$, and such that $\mathbf{e}_i \cdot \mathbf{e}_i = 1$. Notice, then, that each vector is a unit vector (*why?*), and that the vectors are pairwise orthogonal, i.e. each vector is orthogonal to the other two.

(a) Produce two different orthonormal bases for $\mathbb{R}^3$.

(b) Let $\mathbf{v} = \langle x, y, z \rangle$ be any vector in $\mathbb{R}^3$. How can you compute the numbers a, b, and c such that

$$\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3?$$

(HINT: Consider $\mathbf{v} \cdot \mathbf{e}_i$.)

(c) Now let $\mathbf{v}$ and $\mathbf{w}$ be any two vectors in $\mathbb{R}^3$. Suppose that $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$ and that $\mathbf{w} = r\mathbf{e}_1 + s\mathbf{e}_2 + t\mathbf{e}_3$. Prove that $\mathbf{v} = \mathbf{w}$ if and only if $a = r$, $b = s$, and $c = t$.

II. Unit Speed Curves

The most convenient place to start is with *unit speed curves* defined over closed intervals. This is simply a parameterized curve $\mathbf{r}(t)$, defined for all t in a closed interval $[a, b]$, and having the property that $\|\mathbf{r}'(t)\| = 1$ for all t. There are two immediate consequences of this simplifying assumption that suggest that unit speed curves should be easier to study than arbitrary speed curves.

Exercise 1.

(a) If $\mathbf{r}(t)$ is a unit speed curve (where $a \leq t \leq b$), then what is the arc length of the curve $\mathbf{r}(t)$?

(b) Show that $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are orthogonal.

(c) Your text says that the curve $\mathbf{r}(t)$ is *arc-length parameterized* if the arc length of the curve $\mathbf{r}(t)$ over the interval $[a, t]$ is simply $t - a$, for all t in the interval $[a, b]$. In other words,

$$\int_a^t \|\mathbf{r}'(u)\| du = t - a.$$

Prove that $\mathbf{r}(t)$ is arc-length parameterized if and only $\mathbf{r}(t)$ is unit speed.

Remark. It seems that in insisting on only studying unit speed curves, we might risk eliminating from consideration a lot of non-unit speed curves that are perfectly fine otherwise. For example, even the simplest possible "twisted cubic" $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ is NOT unit speed. Even though it isn't immediately obvious, it turns out that in reality we really don't lose much by making the unit speed assumption: *If* $\mathbf{r}(t)$ *is a curve such that* $\mathbf{r}'(t)$ *is never zero, then there exists a parameterization* $\mathbf{u}(s)$ *for the same curve that is unit speed.* (See comments from Section 13.3, p. 850 in your text.)

Exercise 2.

(a) Verify that

$$\mathbf{r}(t) = \frac{\sqrt{2}}{2}\sin t\,\mathbf{i} + \frac{1}{2}(t - \cos t)\mathbf{j} + \frac{1}{2}(t + \cos t)\mathbf{k}$$

is a unit speed curve.

(b) *The circular helix.* The circular helix can be defined by the following function:

$$\mathbf{r}(t) = (a\cos\frac{t}{c})\mathbf{i} + (a\sin\frac{t}{c})\mathbf{j} + (\frac{bt}{c})\mathbf{k}.$$

It is convenient to assume that a and c are positive numbers. What other assumptions do we need to make about a, b, and c in order to be sure that $\mathbf{r}(t)$ is unit speed?

Exercise 3.

(a) Find a unit speed parameterization $\mathbf{u}(s)$ for a circle of radius r centered at the origin, i.e. for $\mathbf{r}(t) = \langle r\cos(t), r\sin(t)\rangle$. Check that $L = \mathit{Length}(\mathbf{u}) = 2\pi r$.

(b) Compute $\|\mathbf{u}''(s)\|$ (with $\mathbf{u}(s)$ as in part (a)). What do you notice?

(c) If $\mathbf{r}(t)$ is a unit speed parameterization for some space curve, then $\|\mathbf{r}''(t)\|$ is called the *curvature* of $\mathbf{r}(t)$. (We'll give a more complete characterization of curvature below.) In part (b), you found that the curvature of a circle depends on its radius, as expected.

Investigate the curvature of lines as follows. Let a line have equation

$$\mathbf{r}(t) = (at + x_1)\mathbf{i} + (bt + y_1)\mathbf{j} + (ct + z_1)\mathbf{k}.$$

First, find a unit speed parameterization $\mathbf{u}(s)$ for this line. Then determine the curvature of the line by computing $\|\mathbf{u}''(s)\|$.

III. The Frenet Frame

Let $\mathbf{r}(t)$ be a unit speed curve, with $a \le t \le b$. So just how exactly should we use the derivatives $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ in our analysis? We are going to associate three vector-valued functions to $\mathbf{r}(t)$, and each function will be defined in terms of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

Definition. The *Frenet frame* associated to $\mathbf{r}(t)$ is a collection of three vector-valued functions $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ defined as follows:

$$\begin{array}{rrcll}
\textit{The unit tangent vector} & \mathbf{T}(t) & = & \mathbf{r}'(t) & \\
\textit{The unit normal vector} & \mathbf{N}(t) & = & \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} & = \dfrac{\mathbf{r}''(t)}{\|\mathbf{r}''(t)\|} \\
\textit{The binormal unit vector} & \mathbf{B}(t) & = & \mathbf{T}(t) \times \mathbf{N}(t) &
\end{array}$$

Exercise 1. Confirm that $\mathbf{N}$ and $\mathbf{T}$ are orthogonal for all t.

You should notice that for a fixed number t_0, there are three vectors $\mathbf{T}(t_0)$, $\mathbf{N}(t_0)$, and $\mathbf{B}(t_0)$ associated to the point $\mathbf{r}(t_0)$ on the curve. But if you pick another number t_1, then you'll get three different vectors $\mathbf{T}(t_1)$, $\mathbf{N}(t_1)$, and $\mathbf{B}(t_1)$ associated to the new point $\mathbf{r}(t_1)$ on the curve. So, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are functions of t.

Exercise 2. Let $\mathbf{T}(t), \mathbf{N}(t)$, and $\mathbf{B}(t)$ be the Frenet frame for a *unit speed* curve $\mathbf{r}(t)$.

(a) Confirm that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are orthonormal in the sense of Exercise I.2.

(b) Prove that $\mathbf{T}'$ is orthogonal to $\mathbf{T}$ and $\mathbf{B}$. HINT: Differentiate both sides of $\mathbf{T}(t)\cdot\mathbf{T}(t) = 1$.

(c) Prove that $\mathbf{N}'$ is orthogonal to $\mathbf{N}$. HINT: Differentiate both sides of $\mathbf{N}(t)\cdot\mathbf{N}(t) = 1$.

(d) Prove that $\mathbf{B}'$ is orthogonal to $\mathbf{T}$ and $\mathbf{B}$. HINT: Differentiate both sides of $\mathbf{B}(t)\cdot\mathbf{B}(t) = 1$.

(e) Prove that $\mathbf{T}'\cdot\mathbf{N} = -\mathbf{N}'\cdot\mathbf{T}$, and that $\mathbf{B}'\cdot\mathbf{N} = -\mathbf{N}'\cdot\mathbf{B}$. HINT: Differentiate both sides of $\mathbf{T}(t)\cdot\mathbf{N}(t) = 0$ and $\mathbf{B}(t)\cdot\mathbf{N}(t) = 0$.

(f) Now, since the Frenet frame vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are orthonormal, we can compute $\mathbf{T}'$, $\mathbf{N}'$, and $\mathbf{B}'$ in terms of $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ (cf. Exercise I.2). Let $\kappa(t) = \mathbf{T}'(t)\cdot\mathbf{N}(t)$ and let $\tau(t) = \mathbf{N}'(t)\cdot\mathbf{B}(t)$. (We often abbreviate this by simply writing $\kappa = \mathbf{T}'\cdot\mathbf{N}$ and $\tau = \mathbf{N}'\cdot\mathbf{B}$, but don't forget that these are all functions of t, and the differentiation is with respect to t.) Use the previous parts of this exercise to show that:

$$\begin{aligned} \mathbf{T}' &= && \kappa\mathbf{N} && \\ \mathbf{N}' &= -\kappa\mathbf{T} && && +\tau\mathbf{B} \\ \mathbf{B}' &= && -\tau\mathbf{N} && \end{aligned}$$

These are the celebrated *Frenet formulas.*

Exercise 3. Compute $\mathbf{T}, \mathbf{N}$, $\mathbf{B}$, κ, and τ for a unit-speed parameterization of a circle of radius r (refer to Exercise II.3), and verify the results of the previous exercise for this Frenet frame.

Exercise 4. Repeat Exercise 3 above for the circular helix of Exercise II.2.(b).

Exercise 5. Let $\mathbf{r}(t)$ be a unit speed curve, with $a \le t \le b$, and define $\kappa(t)$ as in Exercise 2, above. Show that $\kappa(t) = \|\mathbf{r}''(t)\|$. Show that if $\mathbf{r}''(t) \neq \mathbf{0}$ for all t, then $\kappa(t) > 0$ for all t. Show that $\kappa(t) = 0$ for all time t if and only if $\mathbf{r}(t)$ is part of a line.

The question we now turn to is: Is there a geometric interpretation of these functions κ and τ, and if so, how will this help us analyze curves?

IV. Curvature and Torsion

What do the words *curvature* and *torsion* mean to you? Both words are suggestive of shape, but to me they have slightly different connotations. The *curvature* of an object is the amount of "bend" that it has; or, maybe more specifically, curvature is a measure of how circular the object is. (Obviously, a line would have no curvature.) While *torsion* also seems to speak of the amount of bend in an object, to me this word also has connotations of "twisting" or "distorting."

Now suppose someone walks up to me on the street and gives me a curve and asks me to say *just two* things about what the curve is like near a specific point. What should I say? First, I might tell her which circular arc most closely approximates the curve near the given point. And then second, I might try to give some indication of how close the curve is to lying in a plane.

The technical names for these two ideas are (as you might have guessed): curvature and torsion. Once more, curvature should measure the circularity of the curve at a point, and torsion should measure the extent to which the curve lies in a plane.

Let $\mathbf{r}(t)$ be a parameterized curve, with $a \leq t \leq b$. While it isn't at all obvious at this point, the functions $\kappa : [a, b] \to \mathbb{R}$ and $\tau : [a, b] \to \mathbb{R}$ that we encountered in our exploration of Frenet frames above are exactly what we need to measure curvature and torsion. In fact, from now on we will refer to the function κ as the *curvature* of $\mathbf{r}(t)$, and we will refer to the function τ as the *torsion* of $\mathbf{r}(t)$.

Let's think about curvature first. In Exercise III.5, we showed that $\mathbf{r}(t)$ is part of a line if and only if $\kappa(t) = 0$ for all t. In Exercise II.3.(b) we saw that for a circle of radius r, we have $\kappa(t) = 1/r$. Thus, the larger the radius of the circle, the smaller the curvature κ. Both of these examples are consistent with our intuition of what the curvature should be in each case. So it makes sense to call $\kappa(t)$ the "curvature" for these special space curves.

Exercise 1. Why does it make sense to call $\kappa(t)$ the "curvature" if $\mathbf{r}(t)$ is some randomly chosen unit speed curve (and not one of the easy curves of the previous paragraph)?

Next we ask: why should $\tau(t)$ have anything to do with "torsion"? Well, the best evidence that this is indeed the right interpretation for τ is in the following exercise:

Exercise 2. Let $\mathbf{r}(t)$ be a unit speed curve defined on the closed interval $[a, b]$. Prove that $\mathbf{r}(t)$ is contained entirely within a plane *if and only if* $\tau(t) = 0$ for all t. Use the following steps as a guide:

(a) First, let's show that if $\mathbf{r}(t)$ sits in a plane, then its torsion $\tau(t)$ is zero for all t.

1. Let $\mathbf{a}$ be a *unit* normal vector for this plane, and let $\mathbf{b}$ be a point on the plane. Then of course $(\mathbf{r}(t) - \mathbf{b}) \cdot \mathbf{a} = 0$ for all t. Prove that $\mathbf{r}'(t) \cdot \mathbf{a} = 0$, and hence that $\mathbf{T}(t) \cdot \mathbf{a} = 0$.
2. Use this now to show that $\mathbf{r}''(t) \cdot \mathbf{a} = 0$, and conclude that $\mathbf{N}(t) \cdot \mathbf{a} = 0$.
3. Put these two parts together to show that $\mathbf{B}(t) = \pm\mathbf{a}$ for all time t.
4. Next, use the previous part to conclude that $\mathbf{B}'(t) = \mathbf{0}$ for all t.
5. Finally, use the third Frenet formula (Exercise III.2.(f)) to conclude that $\tau(t) = 0$ for all t.

(b) Second, let's suppose that $\tau(t) = 0$ for all t, and see if we can prove that $\mathbf{r}(t)$ sits in a plane.

1. Use the third Frenet formula to see that $\mathbf{B}'(t) = \mathbf{0}$ for all t.
2. Now prove that $\mathbf{r}(t) \cdot \mathbf{B}$ is a constant. (Hint: Show that $\frac{d}{dt}\Big(\mathbf{r}(t) \cdot \mathbf{B}\Big) = 0$.)
3. Conclude from this that $\mathbf{r}(t) \cdot \mathbf{B} = \mathbf{r}(a) \cdot \mathbf{B}$ for any t. (Here, a is the left endpoint of the interval $[a, b]$.)
4. From this, we can see that $\Big(\mathbf{r}(t) - \mathbf{r}(a)\Big) \cdot \mathbf{B} = 0$ for all t. Why can we now conclude that $\mathbf{r}(t)$ is on a plane?

There are some simpler formulas for $\kappa(t)$ and $\tau(t)$:

$$\begin{aligned} \kappa(t) &= \mathbf{T}' \cdot \mathbf{N} && \text{(By definition)} \\ &= \|\mathbf{r}''(t)\| && \text{(Prove this!)} \end{aligned}$$

$$\begin{aligned} \tau(t) &= \mathbf{N}' \cdot \mathbf{B} && \text{(By definition)} \\ &= \Big(\mathbf{r}'(t) \times \mathbf{r}''(t)\Big) \cdot \mathbf{r}'''(t) && \text{(Extra credit!)} \end{aligned}$$

Exercise 3. Compute $\kappa(t)$ and $\tau(t)$ for the unit speed curve $\mathbf{r}(t)$ from Exercise II.2.(a).