How do I love enumeration?

Let me count the ways.

Rob Donnelly

April 28, 2008

- Basics: The multiplication rule, factorials, etc
- Generating functions
- Recurrence relations
- q-analogs
- Algebraic methods

The multiplication rule

Suppose a task can be performed in two steps.

There are r choices for the first step.
There are s choices for the second step.

Number of choices for second step do not depend on choices for first step.

Then there are $r \cdot s$ ways to perform the task.

A task can be performed in a sequence of k steps.

At i^{th} step, r_i choices, number not dependent on choices from previous steps.

Then there are $r_1 \cdot r_2 \cdots r_k$ ways to perform the task.

Example 1

Q: n people in a room. How many ways to line them up?

A: $$\underset{\substack{\uparrow \\ \text{1st} \\ \text{position}}}{n} \cdot \underset{\substack{\uparrow \\ \text{2nd} \\ \text{position}}}{(n-1)} \cdot \underset{\substack{\uparrow \\ \text{3rd} \\ \text{position}}}{(n-2)} \cdots \underset{\substack{\uparrow \\ \text{Next-} \\ \text{to-last}}}{(2)} \underset{\substack{\uparrow \\ \text{Last}}}{(1)} \stackrel{\text{def}}{=}: n!$$

"n factorial"

NOTE: $0! =$ empty product $= 1$

Example 2

Q: n people in a room. How many ways to line up k of them?

A: $$n \cdot (n-1) \cdot (n-2) \cdots (n-(k-1))$$
$$= n(n-1)(n-2)\cdots(n-k+1) \stackrel{\text{def}}{=}: (n)_k$$

"Partial factorial"

NOTE: $(n)_0 =$ empty product $= 1$

Example 3

Q: n letters in an alphabet. How many k-letter words?

A:

$$\underset{\substack{\uparrow\\ \text{1st}\\ \text{letter}}}{n} \cdot \underset{\substack{\uparrow\\ \text{2nd}\\ \text{letter}}}{n} \cdot \underset{\substack{\uparrow\\ \text{3rd}\\ \text{letter}}}{n} \cdots \underset{\substack{\uparrow\\ \text{Last}\\ \text{letter}}}{n} = n^k$$

NOTE: $n^0 = \text{empty product} = 1$

Example 4

Q: How many subsets of an n-element set?

A: $\{1, 2, 3, \ldots, n-1, n\}$

$$\underset{\substack{\uparrow\\ \text{Is "1"}\\ \text{in or out?}}}{2} \cdot \underset{\substack{\uparrow\\ \text{Is "2"}\\ \text{in or out?}}}{2} \cdot 2 \cdots \underset{\substack{\uparrow\\ \text{Is "n-1"}\\ \text{in or out?}}}{2} \cdot \underset{\substack{\uparrow\\ \text{Is "n"}\\ \text{in or out?}}}{2} = 2^n$$

Generating functions

Given a sequence $a_0, a_1, a_2, \ldots$ (finite or infinite),

a generating function for the sequence is a function $f(q)$ such that

$$f(q) = \sum_{k \geq 0} a_k q^k$$

A polynomial or a power series

Example 5

Q: How many k-element subsets of an n-element set?

A: Denote by $\binom{n}{k}$ the number of k-element subsets of an n-element set.

"n choose k"

NOTE: $\binom{n}{0} = \binom{n}{n} = 1$

$\binom{n}{k} = 0$ if $k < 0$ or $k > n$.

Binomial Theorem $\quad (1+q)^n = \sum_{k=0}^{n} \binom{n}{k} q^k$

i.e. $(1+q)^n$ is a generating function for the $\binom{n}{k}$'s.

Proof: $(1+q)^n = (1+q)(1+q)(1+q) \cdots (1+q)$

To get a "q^k" from RHS, choose "q" from k of the factors and "1" from the remaining $n-k$ factors.

There are $\binom{n}{k}$ such choices. So in expanding $(1+q)^n$, the coefficient of q^k is $\binom{n}{k}$. //

So, $\binom{n}{k}$ is the coefficient of q^k in the expansion of $(1+q)^n$.

"Binomial coefficients"

For $n \geq 1$,

$$(1+q)^n = (1+q)(1+q)^{n-1}$$

$$= (1+q) \sum_{k=0}^{n-1} \binom{n-1}{k} q^k$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} q \quad + \quad \sum_{k=0}^{n-1} \binom{n-1}{k} q^{k+1}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} q^k + \sum_{k=1}^{n} \binom{n-1}{k-1} q^k$$

$$= \sum_{k=0}^{n} \binom{n-1}{k} q^k + \sum_{k=0}^{n} \binom{n-1}{k-1} q^k$$

$$= \sum_{k=0}^{n} \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) q^k$$

Since $(1+q)^n = \sum_{k=0}^{n} \binom{n}{k} q^k$, then by equating coefficients,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

"Recurrence relation"

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Symmetric,
unimodal

The nth row of Pascal's Δ gives coefficients for $(1+q)^n$.

NOTE Let $\boxed{C(n,k) := \dfrac{(n)_k}{k!}}$ for $0 \leq k \leq n$,

with $C(n,k) = 0$ if $k < 0$ or $k > n$.

Check that $\boxed{C(n,k) = C(n-1,k) + C(n-1,k-1)}$

with $C(n,0) = C(n,n) = 1$.

Then $\boxed{C(n,k) = \binom{n}{k}}$.

q-analogs

A function $f(q)$ is a q-analog of an integer N if

$$f(q)\big|_{q\to 1} = N$$

<u>Example 6</u> $(1+q)^n$ is a q-analog of 2^n

<u>Example 7</u> $\dfrac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}$ is a q-analog of n

Let $[n]_q := \dfrac{1-q^n}{1-q}$ ← "q-integers"

<u>NOTE:</u> $[1]_q = 1$

<u>Example 8</u> Let $\binom{n}{k}_q \stackrel{\text{def}}{:=} \dfrac{[n]_q\,[n-1]_q\cdots[n-k+1]_q}{[k]_q\,[k-1]_q\cdots[1]_q}$ for $0 \le k \le n$,

with $\binom{n}{k}_q = 0$ if $k<0$ or $k>n$.

Then $\binom{n}{k}_q$ is a q-analog of $\binom{n}{k}$.

"q-binomial coefficients"

Q: Is $\binom{n}{k}_q$ a polynomial?

A: Yes, since $\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$ //

In fact, $\deg \binom{n}{k}_q = k \cdot (n-k)$

and $\binom{n}{k}_q$ has positive integer coefficients.

Let $\binom{n}{k}_q = \sum_{i=0}^{k(n-k)} a_i q^i$, where the a_i's are these positive integer coefficients.

Then $\binom{n}{k}_q$ is a generating function for the a_i's.

Q: What do the a_i's count?

A:

Let $b_i = \#\left\{ k \left\{ \right. \right.$  $\left. \middle| \right.$ there are exactly i shaded boxes, shadings satisfy these properties $\}$

Property #1

Shaded boxes must be "left-justified" in any row

NO

Property #2

$(j+1)$st row can't have more shaded boxes than the jth row has

NO

Claim $a_i = b_i$

To see this, let $P(n,k,q) = \sum_{i=0}^{k(n-k)} b_i q^i$.

Then show that $P(n,k,q) = q^k P(n-1,k,q) + P(n-1,k-1,q)$.

Conclude that $\binom{n}{k}_q = P(n,k,q)$,

i.e. the a_i's and b_i's have the same generating function. //

Q: Recurrence relation for a_i's? Explicit formula for a_i's?

Yes

No?

All of the q-identities

$$\frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

$$(1+q)^n = \sum_{k=0}^{n} \binom{n}{k} q^n$$

$$\binom{n}{k}_q = \sum_{i=0}^{k(n-k)} a_i q^i$$

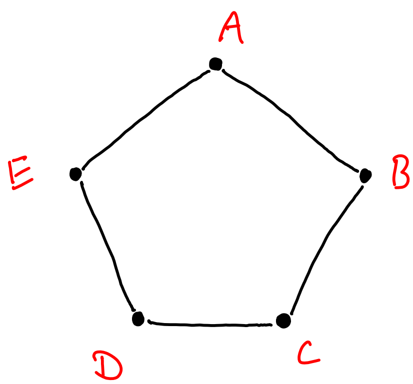
are closely related to the following q-analog of Weyl's dimension formula for irreducible finite-dimensional representations of finite-dimensional complex semisimple Lie algebras:

$$WDF(q) = \prod_{\alpha \in \Phi^+} \left(\frac{1 - q^{\langle \lambda + \rho, \alpha^\vee \rangle}}{1 - q^{\langle \rho, \alpha^\vee \rangle}} \right)$$

This is a polynomial in q with positive integer coefficients.

The sequence of coefficients is symmetric and unimodal.

Counting symmetries



Q: How many symmetries?

A: That depends ...

Rotations only : 5

Rotations & reflections : $\underset{\substack{\uparrow \\ \text{Where can A go?}}}{5} \cdot \underset{\substack{\uparrow \\ \text{How many symmetries keep A fixed?}}}{2} = 10$

Q: How many rotation/reflection symmetries for a regular n-gon with $n \geq 3$?

A: $\# \text{ of symmetries} = \underset{\substack{\uparrow \\ \text{Where can A go?}}}{n} \cdot \underset{\substack{\uparrow \\ \text{How many keep A fixed?}}}{2} = 2n$

The Orbit-Stablizer Theorem

$$\#\text{ of symmetries} = |O_x| \cdot |S_x|$$

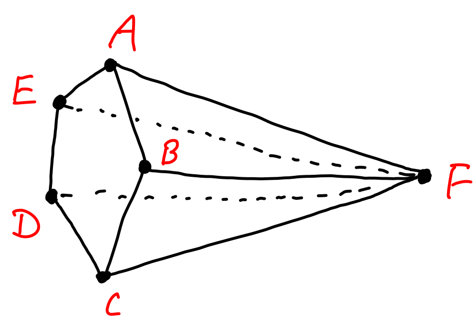
Proof MAT 421

$|O_x|$: Where can x go

$|S_x|$: How many keep x fixed

NOTE: To apply this theorem, the underlying shape does not have to be "regular"

Ex:



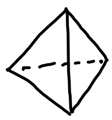
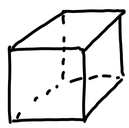
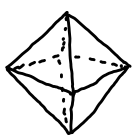
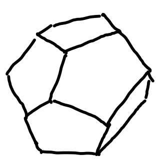
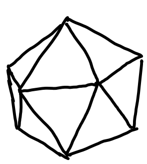
$$\#\text{ symmetries} = 5 \cdot 1 = 1 \cdot 5 = 1 \cdot 5$$

$5 \cdot 1$: Where can A go? · How many keep A fixed?

$1 \cdot 5$: Where can F go? · How many keep F fixed?

$1 \cdot 5$: Where can bottom face go? · How many keep bottom face fixed?

Example The platonic solids

	Name	# Symmetries Rotations only	 Rotations + Reflections
	Tetrahedron	$4 \cdot 3 = 12$	$4 \cdot 6 = 24$
	Cube	$8 \cdot 3 = 24$	$8 \cdot 6 = 48$
	Octahedron	$6 \cdot 4 = 24$	$6 \cdot 8 = 48$
	Dodecahedron	$20 \cdot 3 = 60$	$20 \cdot 6 = 120$
	Icosahedron	$12 \cdot 5 = 60$	$12 \cdot 10 = 120$