January 4, 2010

Root systems for asymmetric geometric representations of Coxeter groups

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Abstract

Results are obtained concerning root systems for asymmetric geometric representations of Coxeter groups. These representations were independently introduced by Vinberg and Eriksson, and generalize the standard geometric representation of a Coxeter group in such a way as to include all Kac–Moody Weyl groups. In particular, a characterization of when a non-trivial multiple of a root may also be a root is given in the general context. Characterizations of when the number of such multiples of a root is finite and when the number of positive roots sent to negative roots by a group element is finite are also given. These characterizations are stated in terms of combinatorial conditions on a graph closely related to the Coxeter graph for the group. Other finiteness results for the symmetric case which are connected to the Tits cone and to a natural partial order on positive roots are extended to this asymmetric setting.

Mathematics Subject Classification: 20F55 (05E99)
Keywords: Coxeter group, geometric representation, root system, Tits cone, Kac–Moody algebra, numbers game

§1 Introduction. A certain natural symmetric bilinear form is used to define the familiar geometric representation of a given Coxeter group, often called the "standard" geometric representation. See [Bour] Ch. 5, [Hum] Ch. 5, or [BB] §4.4. These representations are well understood and are useful for studying Coxeter groups and their applications in many different contexts. See for example [Gun] and references therein. Following work of Vinberg and Eriksson, when considering geometric representations of Coxeter groups in Chapter 4 of the book [BB], Björner and Brenti initially do not require that the bilinear form be symmetric. The purpose here is to further study the root systems associated to such representations. Much of what we record here generalizes the standard theory as presented for example in §5.3, 5.4, 5.6, and 5.13 of [Hum] and extends §4.3 of [BB]. Since the form is no longer required to be symmetric, all statements here may be applied to the sets of real roots of Kac–Moody algebras. This yields new proofs of standard Kac–Moody results (one direction of the first statement in Corollary 3.7, one direction of the second statement in Corollary 3.10).

These asymmetric geometric realizations of Coxeter groups were introduced by Vinberg in [Vin], for geometric reasons. A main focus of Vinberg's study is the behavior of the "fundamental chamber" (a convex polyhedral cone) under the group action. In a different context, Lusztig used such asymmetric forms when constructing certain irreducible representations of Hecke algebras [Lus]. Eriksson applied asymmetric geometric representations of Coxeter groups in [Erik1] (§4.3, §6.9, Ch. 8) and [Erik2] (§3, 4) in connection with the combinatorial numbers game of Mozes [Moz]. While the numbers game is of combinatorial interest in its own right, it is also helpful for facilitating computations with Coxeter groups and their geometric representations (e.g. computing orbits, solving the word problem, or finding reduced decompositions) and for obtaining combinatorial models of Coxeter groups. See for example §4.3 of [BB]. The results of this paper are needed for our further study of the numbers game in [Don]. There we further investigate connections between moves of the game and reduced decompositions for group elements, characterize "full commutativity" of group elements in terms of the game, characterize when all positive roots can be obtained

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from game play, and obtain a new Dynkin diagram classification theorem whose answer consists of versions of Coxeter graphs for finite Coxeter groups.

The possible asymmetry of the bilinear forms here leads to some curious differences with the standard case. In Exercise 4.9 of [BB], the authors point out that without symmetry of the bilinear forms, some important properties of root systems would not be true. However, we will see that these properties do not fail too badly, at least not all of the time. In particular, we determine precisely when non-trivial scalar multiples of roots can also be roots (Theorem 3.2), and we relate the finiteness of this set of root multiples to a combinatorial condition on a graph closely related to the Coxeter graph for the group (Theorem 3.6). Further, we determine when the number of positive roots sent to negative roots by a given group element is finite, and we say how this quantity is related to the length of the given group element (Theorem 3.9). An asymmetric version of Brink and Howlett's fundamental result on the finiteness of the set of "dominance-minimal" roots is obtained in Theorem 3.13. In Theorem 4.5, we show that finiteness of an irreducible Coxeter group is equivalent to certain conditions on the asymmetric version of the Tits cone.

The original version of this paper was written with only the numbers game motivations above in mind. Recently, for unrelated reasons Proctor decided to relate the treatment of Weyl groups in [Kac] and [Kum] to the study of asymmetric geometric representations of Coxeter groups in [BB]. This led to the definition of 'real Weyl groups' in [Pro] and his realization that our Theorem 3.2 would play a key role in those notes. Quoting from an earlier draft of [Pro]: "There are many statements concerning Weyl groups and the 'real' roots of Kac–Moody algebras which can at least be conjectured in the general context of real Weyl groups. If still true, it would seem that each of these statements should be provable without any reference to Lie brackets or to root spaces, if one could formulate suitable sufficient conditions for them in terms of real Weyl group concepts. One example of such a statement is "no 'non-trivial' real multiple of a real root is also a root". Within the general context, two successive restricting assumptions (which are both automatically satisfied by Weyl groups) guarantee [via our Theorem 3.2] that this example statement holds true in a context which is still much more general than that of Weyl groups or of Section 4.4 of [BB]."

At the end of Section 2 we observe that any Kac–Moody Weyl group arises as one of our representing groups $\sigma(W) \subset GL(V)$. Hence all of our results pertain to the special case consisting of arbitrary Kac–Moody Weyl groups. Our complete characterizations of the "no non-trivial multiple of a (real) root is also a root" (Corollary 3.7) and the "set of positive (real) roots sent to negative is finite" (Corollary 3.10) properties are given proofs which are naturally set in a general environment which encompasses both the standard geometric representations of Coxeter groups and Kac–Moody Weyl groups. Only combinatorial positivity arguments are used in these proofs; no references to Lie brackets or root spaces are needed.

§2 Definitions and preliminaries. In this section we present the main objects of interest for this paper. The crucial information identifying an asymmetric geometric representation of a Coxeter group is a certain real matrix analog of a generalized Cartan matrix. We take this matrix as our starting point. Fix a positive integer n and a totally ordered set I_n with n elements (usually

$I_n := \{1 < \ldots < n\}$). An *E-generalized Cartan matrix* (*E-GCM*)[2] is an $n \times n$ matrix $A = (a_{ij})_{i,j \in I_n}$ with real entries satisfying the requirements that each main diagonal matrix entry is 2, that all other matrix entries are nonpositive, that if a_{ij} is nonzero then a_{ji} is also nonzero, and that for $i \neq j$ either $a_{ij}a_{ji} \geq 4$ or $a_{ij}a_{ji} = 4\cos^2(\pi/k_{ij})$ for some integer $k_{ij} \geq 2$. The peculiar quantities $4\cos^2(\pi/k)$ appear in the developments of [Bour], [Hum] as the products of transpose entries of a symmetric matrix for the defining bilinear form of the standard geometric representation of a Coxeter group. To an $n \times n$ E-generalized Cartan matrix $A = (a_{ij})_{i,j \in I_n}$ we associate a finite graph Γ as follows: The nodes $(\gamma_i)_{i \in I_n}$ of Γ are indexed by the set I_n, and an edge is placed between nodes γ_i and γ_j if and only if $i \neq j$ and the matrix entries a_{ij} and a_{ji} are nonzero. We display this edge as $\gamma_i \bullet\!\xrightarrow{\ p\ }\!\xleftarrow{\ q\ }\!\bullet\, \gamma_j$, where $p = -a_{ij}$ and $q = -a_{ji}$. We call the pair (Γ, A) an *E-GCM graph*. See Figure 3.1 for a six-node example.

Define the associated Coxeter group $W(\Gamma, A)$ to be the Coxeter group with identity ε, generators $\{s_i\}_{i \in I_n}$, and defining relations $s_i^2 = \varepsilon$ for $i \in I_n$ and $(s_i s_j)^{m_{ij}} = \varepsilon$ for all $i \neq j$, where the m_{ij} are determined by:

$$m_{ij} = \begin{cases} k_{ij} & \text{if } a_{ij}a_{ji} = 4\cos^2(\pi/k_{ij}) \text{ for some integer } k_{ij} \geq 2 \\ \infty & \text{if } a_{ij}a_{ji} \geq 4 \end{cases}$$

(Conventionally, $m_{ij} = \infty$ means there is no relation between generators s_i and s_j.) When A is a generalized Cartan matrix or GCM (i.e. an E-GCM with integer entries), then $W(\Gamma, A)$ is a *Weyl group*. In this case, m_{ij} is finite only for the pairs $\{-a_{ij}, -a_{ji}\} = \{0,0\}, \{1,1\}, \{1,2\}, \{1,3\}$; the corresponding values of such m_{ij} are $2, 3, 4, 6$. One can think of the E-GCM graph as a refinement of the information from the Coxeter graph for the associated Coxeter group. Observe that any Coxeter group on a finite set of generators is isomorphic to $W(\Gamma, A)$ for some E-GCM graph (Γ, A). We let ℓ denote the length function for $W = W(\Gamma, A)$. An expression $s_{i_1} s_{i_2} \cdots s_{i_p}$ for an element of W is *reduced* if $\ell(s_{i_1} s_{i_2} \cdots s_{i_p}) = p$. For $J \subseteq I_n$, let W_J be the subgroup generated by $\{s_i\}_{i \in J}$, a *parabolic* subgroup, and $W^J := \{w \in W \,|\, \ell(ws_j) > \ell(w) \text{ for all } j \in J\}$ is the set of *minimal coset representatives*. If $J = \{i, j\}$, then W_J is a dihedral group of order $2m_{ij}$.

From here on, fix an arbitrary E-GCM graph (Γ, A) with index set I_n and associated Coxeter group $W = W(\Gamma, A)$. We now define the representations of W which are of interest to us here, cf. §4.2 of [BB]. To fix notation that will help set up some subsequent arguments, we present some of the details here. Let V be a real n-dimensional vector space freely generated by $(\alpha_i)_{i \in I_n}$. (Elements of this ordered basis are *simple roots*.) Equip V with a possibly asymmetric bilinear form $B : V \times V \to \mathbb{R}$ defined on the basis $(\alpha_i)_{i \in I_n}$ by $B(\alpha_i, \alpha_j) := \frac{1}{2}a_{ij}$. For each $i \in I_n$ define an operator $S_i : V \to V$ by the rule $S_i(v) := v - 2B(\alpha_i, v)\alpha_i$ for each $v \in V$. One can check that S_i^2 is the identity transformation, so $S_i \in GL(V)$. Fix $i \neq j$ and set $V_{i,j} := \mathrm{span}_{\mathbb{R}}\{\alpha_i, \alpha_j\}$. Observe that $S_k(V_{i,j}) \subseteq V_{i,j}$ for $k = i, j$. Let $\mathfrak{B}$ be the ordered basis (α_i, α_j) for $V_{i,j}$, and for any linear mapping

[2]Motivation for terminology: E-GCM's with integer entries are generalizations of 'generalized' Cartan matrices (GCM's), which are the starting point for the study of Kac-Moody algebras. Here we use the modifier "E" because of the relationship between these matrices and the combinatorics of Eriksson's E-games [Erik1], [Erik2].

$T : V_{i,j} \to V_{i,j}$ let $[T]_{\mathfrak{B}}$ be the matrix for T relative to $\mathfrak{B}$. Then

$$[S_i|_{V_{i,j}}]_{\mathfrak{B}} = \begin{pmatrix} -1 & -a_{ij} \\ 0 & 1 \end{pmatrix}, [S_j|_{V_{i,j}}]_{\mathfrak{B}} = \begin{pmatrix} 1 & 0 \\ -a_{ji} & -1 \end{pmatrix}, [S_iS_j|_{V_{i,j}}]_{\mathfrak{B}} = \begin{pmatrix} a_{ij}a_{ji}-1 & a_{ij} \\ -a_{ji} & -1 \end{pmatrix}$$

Analysis of the eigenvalues for $X_{i,j} := [S_iS_j|_{V_{i,j}}]_{\mathfrak{B}}$ as in the proofs of Proposition 3.13 of [Kac] and Proposition 1.3.21 of [Kum] shows that $X_{i,j}$ has infinite order when $a_{ij}a_{ji} \geq 4$, and hence S_iS_j has infinite order as an element of $GL(V)$. When $0 < a_{ij}a_{ji} < 4$, write $a_{ij}a_{ji} = 4\cos^2\theta$ for $\theta := \pi/m_{ij}$. In this case check that $X_{i,j}$ has two distinct complex eigenvalues ($e^{2i\theta}$ and $e^{-2i\theta}$). It follows that $X_{i,j}$ has finite order m_{ij}. When $a_{ij}a_{ji} = 0$, then $X_{i,j} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which clearly has order $m_{ij} = 2$. Now assume $0 \leq a_{ij}a_{ji} < 4$, and set $V'_{i,j} := \{v \in V \mid B(\alpha_i, v) = 0 = B(\alpha_j, v)\}$. One can easily check that $V_{i,j} \cap V'_{i,j} = \{0\}$. The facts that $\dim V_{i,j} = 2$, $\dim V'_{i,j} \geq n-2$, and $V_{i,j} \cap V'_{i,j} = \{0\}$ together imply that $\dim V'_{i,j} = n-2$ and $V = V_{i,j} \oplus V'_{i,j}$. Since S_iS_j acts as the identity on $V'_{i,j}$, it follows that S_iS_j has order m_{ij} as an element of $GL(V)$.

Then there is a unique homomorphism $\sigma_A : W \to GL(V)$ for which $\sigma_A(s_i) = S_i$. With the dependence on A understood, we set $\sigma := \sigma_A$. We now have W acting on V, and for all $w \in W$ and $v \in V$ we write $w.v$ for $\sigma(w)(v)$. We call σ a *geometric representation* of W. If A is symmetric such that $a_{kl}a_{lk} \geq 4 \Rightarrow a_{kl} = a_{lk} = -2$ for all $k \neq l$, then σ is the *standard* geometric representation. The *root system* for σ is $\Phi := \Phi_A := \{w.\alpha_i\}_{i \in I_n, w \in W}$. For each $w \in W$, $\sigma(w)$ permutes Φ, so σ induces an action of W on Φ. Evidently, $\Phi = -\Phi$. Elements of Φ are *roots* and are necessarily nonzero. If $\alpha = \sum c_i\alpha_i$ is a root with all c_i nonnegative (respectively nonpositive), then say α is a *positive* (resp. *negative*) root. Let Φ^+ and Φ^- denote the collections of positive and negative roots respectively. Clearly $\Phi^+ \cap \Phi^- = \emptyset$. The next statement is Proposition 4.2.5 of [BB] and appears in a somewhat different form as Corollary 4.3 in [Erik1].

Proposition 2.1 *Let $w \in W$ and $i \in I_n$. If $\ell(ws_i) > \ell(w)$, then $w.\alpha_i \in \Phi^+$. If $\ell(ws_i) < \ell(w)$, then $w.\alpha_i \in \Phi^-$.*

This result analogizes Theorem 5.4 of [Hum], which handles the standard case. As with Corollary 5.4 of [Hum], it is a consequence of Proposition 2.1 that the representation σ is faithful. (See [BB] Theorem 4.2.7.) It also follows that $\Phi = \Phi^+ \cup \Phi^-$. This is Equation 4.24 of [BB], which actually could have been derived at the end of Section 4.2 of that text.

Kac–Moody Weyl groups are subsumed into this paper as follows: Let A be a generalized Cartan matrix. We identify our simple roots $\{\alpha_1, \ldots, \alpha_n\}$ with the simple roots in $\mathfrak{h}^*_{\mathbb{R}}$ of [Kac], which is the dual of a real vector space $\mathfrak{h}_{\mathbb{R}}$ of dimension $n+l$, where $l = \text{nullity}(A)$. The simple "coroots" of [Kac] are a linearly independent set $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}_{\mathbb{R}}$ for which $\alpha_j(\alpha_i^\vee) = a_{ij}$. Now for $1 \leq i \leq n$, a mapping $R_i : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ is defined in [Kac] by $R_i(v) = v - v(\alpha_i^\vee)\alpha_i$. The associated Kac–Moody Weyl group is the subgroup of $GL(\mathfrak{h}^*_{\mathbb{R}})$ generated by $\{R_i\}_{i=1}^n$. If we identify our V with $\text{span}_{\mathbb{R}}\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{h}^*_{\mathbb{R}}$ and restrict each R_i to V, then the homomorphism $W \to GL(V)$ determined by $s_i \mapsto R_i|_V$ is the representation σ. The real roots of Kac–Moody theory are the roots $\Phi \subset V$ obtained here from this geometric representation of W.

§3 Root system results. Asymmetry of the bilinear form leads to crucial differences with the symmetric case. Most notably, $\sigma(W)$ preserves the form B if and only if A is symmetric. From this fact for symmetric A it readily follows that if $K\alpha_x \in \Phi$ for some $x \in I_n$ and real number K, then $K = \pm 1$. (See equation 4.27 of [BB].) However, when A is asymmetric sometimes $K\alpha_x$ is a root for $K \neq \pm 1$, as can be seen in Exercise 4.9 of [BB] and Example 3.12 below.[3] To understand how such a W-action can generate scalar multiples of roots in Φ, we first analyze how s_i and s_j act in tandem on $V_{i,j}$. Our next result strengthens Lemma 4.2.4 of [BB] and provides a different proof. It also answers Exercise 4.6 of [BB].

Lemma 3.1 *Fix $i \neq j$ in I_n, and let k be a positive integer. If $m_{ij} = \infty$, then $(s_is_j)^k.\alpha_i = a\alpha_i + b\alpha_j$ and $s_j(s_is_j)^k.\alpha_i = c\alpha_i + d\alpha_j$ for positive coefficients a, b, c, and d. Now suppose $m_{ij} < \infty$. If $2k < m_{ij}$, then $(s_is_j)^k.\alpha_i = a\alpha_i + b\alpha_j$ with $a \geq 0$ and $b > 0$. In this case, $a = 0$ if and only if m_{ij} is odd and $k = (m_{ij}-1)/2$, and consequently $(s_is_j)^k.\alpha_i = \frac{-a_{ji}}{2\cos(\pi/m_{ij})}\alpha_j$. Similarly, if $2k < m_{ij} - 1$, then $s_j(s_is_j)^k.\alpha_i = c\alpha_i + d\alpha_j$ with $c > 0$ and $d \geq 0$. In this case, $d = 0$ if and only if m_{ij} is even and $k = (m_{ij}-2)/2$, and consequently $s_j(s_is_j)^k.\alpha_i = \alpha_i$.*

Proof. Let $\mathfrak{B}$ and $X_{i,j}$ be as above, and set $X_i := [S_i|_{V_{i,j}}]_{\mathfrak{B}}$ and $X_j := [S_j|_{V_{i,j}}]_{\mathfrak{B}}$. To understand $(s_is_j)^k.\alpha_i$ and $s_j(s_is_j)^k.\alpha_i$ we compute $X_{i,j}^k$ and $X_jX_{i,j}^k$. Set $p := -a_{ij}$ and $q := -a_{ji}$.

For $m_{ij} = \infty$, first take $pq = 4$. We can write $X_{i,j} = PYP^{-1}$ for nonsingular P and upper triangular Y as follows:

$$X_{i,j} = \frac{1}{p}\begin{pmatrix} p & p \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & p \\ 2 & -p \end{pmatrix}.$$

Then for any positive integer k we obtain $X_{i,j}^k = \begin{pmatrix} 2k+1 & -kp \\ kq & -2k+1 \end{pmatrix}$. It follows that $(s_is_j)^k.\alpha_i = (2k+1)\alpha_i + kq\alpha_j$, with both coefficients of the linear combination positive. From the first column of the matrix $X_jX_{i,j}^k$ we see that $s_j(s_is_j)^k.\alpha_i = (2k+1)\alpha_i + (k+1)q\alpha_j$, with both coefficients of the linear combination positive. Next take $pq > 4$. In this case we get distinct eigenvalues $\lambda = \frac{1}{2}(pq - 2 + \sqrt{pq(pq-4)}) > 1$ and $\mu = \frac{1}{2}(pq - 2 - \sqrt{pq(pq-4)}) < 1$ for $X_{i,j}$ (here we have $\lambda\mu = 1$). Similar to the above, we may write $X_{i,j} = PDP^{-1}$ for the diagonal matrix $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and a nonsingular matrix P, from which we obtain

$$X_{i,j}^k = \frac{1}{p(\lambda-\mu)}\begin{pmatrix} p & p \\ \mu' & \lambda' \end{pmatrix}\begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix}\begin{pmatrix} \lambda' & -p \\ -\mu' & p \end{pmatrix},$$

for any positive integer k, where $\lambda' := \lambda + 1$ and $\mu' := \mu + 1$. This (eventually) simplifies to

$$X_{i,j}^k = \frac{1}{\lambda-\mu}\begin{pmatrix} \lambda'\lambda^k - \mu'\mu^k & -p(\lambda^k - \mu^k) \\ q(\lambda^k - \mu^k) & \lambda'\mu^k - \mu'\lambda^k \end{pmatrix}.$$

From this we also get

$$X_jX_{i,j}^k = \frac{1}{\lambda-\mu}\begin{pmatrix} \lambda'\lambda^k - \mu'\mu^k & -p(\lambda^k - \mu^k) \\ q(\lambda^{k+1} - \mu^{k+1}) & \mu'\mu^k - \lambda'\lambda^k \end{pmatrix}.$$

[3] In Proposition 6.9 of [Erik1] and in [Erik2] just prior to Proposition 4.4, it is asserted that $s_x(\Phi^+ \setminus \{\alpha_x\}) = \Phi^+ \setminus \{\alpha_x\}$ for all $x \in I_n$. However, this will not be the case if $K\alpha_x$ is a root for some $K \neq \pm 1$. Only Theorem 6.9 of [Erik1] and Proposition 4.4 of [Erik2] are affected by this misstatement. (See Lemma 3.8 below.)

The factor $\frac{1}{\lambda-\mu}$ is positive, and for both matrices $X_{i,j}^k$ and $X_jX_{i,j}^k$, the first column entries are positive. So, $(s_is_j)^k.\alpha_i = a\alpha_i + b\alpha_j$ with both a and b positive, and $s_j(s_is_j)^k.\alpha_i = c\alpha_i + d\alpha_j$ with c and d both positive.

For the $m_{ij} < \infty$ case, set $\theta := \pi/m_{ij}$. Note that the hypotheses of the lemma require that $m_{ij} > 2$, so in particular p and q are nonzero. Check that $X_{i,j}$ can be written as $X_{i,j} = PDP^{-1}$ for a nonsingular matrix P and diagonal matrix D in the following way:

$$\frac{1}{q(e^{2i\theta}-e^{-2i\theta})}\begin{pmatrix} e^{2i\theta}+1 & e^{-2i\theta}+1 \\ q & q \end{pmatrix}\begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix}\begin{pmatrix} q & -e^{-2i\theta}-1 \\ -q & e^{2i\theta}+1 \end{pmatrix}.$$

Then for any positive integer k we have

$$X_{i,j}^k = PD^kP^{-1} = \frac{1}{\sin(2\theta)}\begin{pmatrix} \sin(2(k+1)\theta)+\sin(2k\theta) & -p\sin(2k\theta) \\ q\sin(2k\theta) & -\sin(2k\theta)-\sin(2(k-1)\theta) \end{pmatrix}$$

and

$$X_jX_{i,j}^k = \frac{1}{\sin(2\theta)}\begin{pmatrix} \sin(2(k+1)\theta)+\sin(2k\theta) & -p\sin(2k\theta) \\ q\sin(2(k+1)\theta) & (1-pq)\sin(2k\theta)+\sin(2(k-1)\theta) \end{pmatrix}$$

Use the first column of $X_{i,j}^k$ and $X_jX_{i,j}^k$ to see that $(s_is_j)^k.\alpha_i = \frac{1}{\sin(2\theta)}[\sin(2(k+1)\theta)+\sin(2k\theta)]\alpha_i + \frac{q}{\sin(2\theta)}\sin(2k\theta)\alpha_j$ and that $s_j(s_is_j)^k.\alpha_i = \frac{1}{\sin(2\theta)}[\sin(2(k+1)\theta)+\sin(2k\theta)]\alpha_i + \frac{q}{\sin(2\theta)}\sin(2(k+1)\theta)\alpha_j$. As long as $2(k+1) < m_{ij}$, then all the coefficients of these linear combinations will be positive. So now suppose $2(k+1) \geq m_{ij}$. First we consider $(s_is_j)^k = a\alpha_i + b\alpha_j$ for some positive k with $2k < m_{ij}$. There are two possibilities now: $2(k+1) = m_{ij}$ or $2(k+1) = m_{ij}+1$. In the former case both a and b are positive. In the latter case we have m_{ij} odd, $a = \frac{1}{\sin(2\theta)}[\sin(2(k+1)\theta)+\sin(2k\theta)] = 0$, and $b = \frac{q\sin\theta}{\sin(2\theta)} = \frac{q}{2\cos\theta}$. Second we consider $s_j(s_is_j)^k = c\alpha_i + d\alpha_j$ for some positive k with $2k < m_{ij}-1$. Now the fact that $2(k+1) \geq m_{ij}$ implies we have $2(k+1) = m_{ij}$. In particular, m_{ij} is even. With $k = (m_{ij}-2)/2$ now, one can check that $d = 0$ and $c = 1$. □

Distinct nodes γ_i and γ_j in (Γ, A) are *odd-neighborly* if m_{ij} is odd. If in addition we have $a_{ij} \neq a_{ji}$, then the adjacent nodes γ_i and γ_j form an *odd asymmetry*. For odd m_{ij}, let v_{ji} be the element $(s_is_j)^{(m_{ij}-1)/2}$ of W, and set $K_{ji} := \frac{-a_{ji}}{2\cos(\pi/m_{ij})} = \sqrt{\frac{a_{ji}}{a_{ij}}}$. In view of Lemma 3.1, $v_{ji}.\alpha_i = K_{ji}\alpha_j$. Observe that $K_{ij}K_{ji} = 1$ and that $v_{ij} = v_{ji}^{-1}$. We have $\ell(v_{ji}) = m_{ij}-1$. Say a sequence $\mathcal{P} := [\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_p}]$ of nodes from Γ is a *path of odd neighbors*, or *ON-path*, if consecutive nodes of $\mathcal{P}$ are odd neighbors. The ON-path $\mathcal{P}$ has length p, and we allow ON-paths to have length zero. We say γ_{i_0} and γ_{i_p} are the *start* and *end* nodes of the ON-path, respectively. Let $w_{\mathcal{P}} \in W$ be the Coxeter group element $v_{i_pi_{p-1}} \cdots v_{i_2i_1}v_{i_1i_0}$, and let $\Pi_{\mathcal{P}} := K_{i_pi_{p-1}} \cdots K_{i_2i_1}K_{i_1i_0}$, where $w_{\mathcal{P}} = \varepsilon$ with $\Pi_{\mathcal{P}} = 1$ when $\mathcal{P}$ has length zero. Then $w_{\mathcal{P}}.\alpha_{i_0} = \Pi_{\mathcal{P}}\alpha_{i_p}$. If ON-path $\mathcal{Q} = [\gamma_{j_0}, \gamma_{j_1}, \ldots, \gamma_{j_q}]$ has the same start node as the end node of $\mathcal{P}$, then their *concatenation* $\mathcal{P}\natural\mathcal{Q}$ is the ON-path $[\gamma_{i_0}, \gamma_{j_1}, \ldots, \gamma_{i_p} = \gamma_{j_0}, \ldots, \gamma_{j_q}]$. Note that $w_{\mathcal{P}\natural\mathcal{Q}} = w_{\mathcal{Q}}w_{\mathcal{P}}$.

Distinct nodes γ_i and γ_j in (Γ, A) are *even-related* if m_{ij} is even. For even m_{ij}, let v_{ji} be the element $s_j(s_is_j)^{(m_{ij}-2)/2}$ of W. Then $v_{ji}.\alpha_i = \alpha_i$ (for $m_{ij} \geq 4$ this is justified by Lemma 3.1), and $v_{ij} = v_{ji}^{-1}$. We have $\ell(v_{ji}) = m_{ij}-1$. Say a sequence $\mathcal{S} := [(\gamma_{i_0}, \gamma_{i_1}), (\gamma_{i_0}, \gamma_{i_2}), \ldots, (\gamma_{i_0}, \gamma_{i_p})]$

is a *sequence of even-related nodes*, or *ER-sequence*, *rooted at* γ_{i_0} if for each pair $(\gamma_{i_0}, \gamma_{i_k})$ of the sequence ($1 \leq k \leq p$) the nodes γ_{i_0} and γ_{i_k} are even-related. Say $\mathcal{S}$ has length p. We allow $\mathcal{S}$ to be the empty sequence, in which case it has length zero. Let $w_{\mathcal{S}} \in W$ be the Coxeter group element $v_{i_p i_0} \cdots v_{i_2 i_0} v_{i_1 i_0}$, with $w_{\mathcal{S}} = \varepsilon$ when $\mathcal{S}$ has length zero. Then $w_{\mathcal{S}}.\alpha_{i_0} = \alpha_{i_0}$. If ER-sequence $\mathcal{T} = [(\gamma_{i_0}, \gamma_{j_1}), \ldots, (\gamma_{i_0}, \gamma_{j_q})]$ is also rooted at γ_{i_0}, then the *concatenation* $\mathcal{S}\natural\mathcal{T}$ is the ER-sequence $[(\gamma_{i_0}, \gamma_{i_1}), \ldots, (\gamma_{i_0}, \gamma_{i_p}), (\gamma_{i_0}, \gamma_{j_1}), \ldots, (\gamma_{i_0}, \gamma_{j_q})]$ rooted at γ_{i_0}. Note that $w_{\mathcal{S}\natural\mathcal{T}} = w_{\mathcal{T}} w_{\mathcal{S}}$.

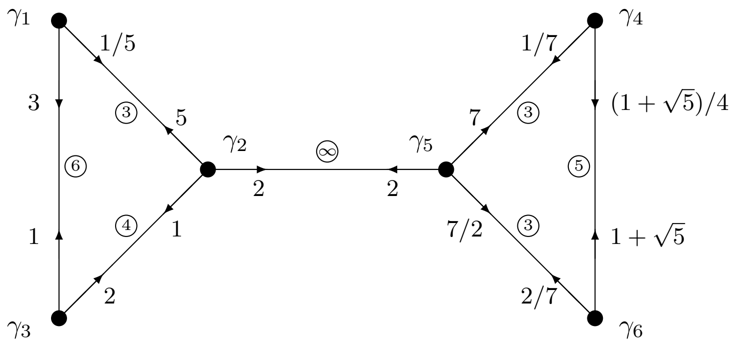
Theorem 3.2 *Let $w \in W$ with $w \neq \varepsilon$, and let $i \in I_n$. (1) Then $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and some $K > 0$ if and only if there is an ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$ and ER-sequences $\mathcal{S}_k$ rooted at γ_{i_k} ($0 \leq k \leq p$) such that $w = w_{\mathcal{S}_p} v_{i_p,i_{p-1}} w_{\mathcal{S}_{p-1}} v_{i_{p-1},i_{p-2}} \cdots v_{i_2,i_1} w_{\mathcal{S}_1} v_{i_1,i_0} w_{\mathcal{S}_0}$. In this case, $w.\alpha_i = w_{\mathcal{P}}.\alpha_i = \Pi_{\mathcal{P}}\alpha_x$. (2) Similarly $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and some $K < 0$ if and only if there is an ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$ and ER-sequences $\mathcal{S}_k$ rooted at γ_{i_k} ($0 \leq k \leq p$) such that $w = w_{\mathcal{S}_p} v_{i_p,i_{p-1}} w_{\mathcal{S}_{p-1}} v_{i_{p-1},i_{p-2}} \cdots v_{i_2,i_1} w_{\mathcal{S}_1} v_{i_1,i_0} w_{\mathcal{S}_0} s_i$. In this case, $w.\alpha_i = (w_{\mathcal{P}} s_i).\alpha_i = -\Pi_{\mathcal{P}}\alpha_x$.*

Proof. Note that (2) follows from (1). For (1), the "if" direction is handled by the two definitions paragraphs preceding the theorem statement. For the "only if" direction, we induct on $\ell(w)$. If $\ell(w) = 1$, then it is clear that $w = s_j$ for some $j \neq i$ in I_n and with $m_{ij} = 2$. So, $v_{ji} = s_j$. Taking $\mathcal{S}_0 = [(\gamma_i, \gamma_j)]$, $\mathcal{P} = [\gamma_i]$, and $\mathcal{S}_1$ the empty sequence, then w has the desired form. Now suppose $\ell(w) > 1$. Take any $j \in I_n$ for which $\ell(ws_j) = \ell(w) - 1$. Since $\ell(ws_i) > \ell(w)$, then $i \neq j$. Let $J := \{i, j\}$, and let v^J be the unique element in W^J and v_J the unique element in W_J for which $w = v^J v_J$. Then $\ell(w) = \ell(v^J) + \ell(v_J)$ by Proposition 2.4.4 of [BB]. Write $v_J.\alpha_i = a\alpha_i + b\alpha_j$. Since $\ell(ws_i) > \ell(w)$, then $\ell(v_J s_i) > \ell(v_J)$, and hence $v_J.\alpha_i \in \Phi^+$ (Proposition 2.1). So $a \geq 0$ and $b \geq 0$. Suppose $a > 0$ and $b > 0$. Now $v^J \in W^J$ implies that $\ell(v^J s_i) > \ell(v^J)$ and $\ell(v^J s_j) > \ell(v^J)$, and hence $v^J.\alpha_i \in \Phi^+$ and $v^J.\alpha_j \in \Phi^+$ (Proposition 2.1). Write $v^J.\alpha_i = \sum_{y \in I_n} c_y \alpha_y$ ($c_y \geq 0$) and $v^J.\alpha_j = \sum_{y \in I_n} d_y \alpha_y$ ($d_y \geq 0$). Then $K\alpha_x = w.\alpha_i = v^J.(a\alpha_i + b\alpha_j) = \sum_{y \in I_n}(ac_y + bd_y)\alpha_y$ implies that for all $y \neq x$, $ac_y + bd_y = 0$ and hence $c_y = d_y = 0$. Then $v^J.\alpha_i$ and $v^J.\alpha_j$ are both multiples of α_x. But then $(v^J)^{-1}.\alpha_x$ is a scalar multiple of α_i and of α_j, which is absurd. So we must have $a = 0$ or $b = 0$. Then by Lemma 3.1, it follows that m_{ij} is finite and $v_J = v_{ji}$.

If m_{ij} is even, then $v_{ji}.\alpha_i = \alpha_i$. So $v^J.\alpha_i = K\alpha_x$. If $v^J = \varepsilon$, then take $\mathcal{S}_0 = [(\gamma_i, \gamma_j)]$, $\mathcal{P} = [\gamma_i]$, and $\mathcal{S}_1$ the empty sequence to see that $w = v_{ji}$ has the desired form. Otherwise, since $\ell(v^J) < \ell(w)$ we may apply the induction hypothesis to v^J to see that there is an ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$ and ER-sequences $\mathcal{S}_k$ rooted at γ_{i_k} ($0 \leq k \leq p$) such that $v^J = w_{\mathcal{S}_p} v_{i_p,i_{p-1}} w_{\mathcal{S}_{p-1}} v_{i_{p-1},i_{p-2}} \cdots v_{i_2,i_1} w_{\mathcal{S}_1} v_{i_1,i_0} w_{\mathcal{S}_0}$. Let $\mathcal{S}_0' := [(\gamma_{i=i_0}, \gamma_j)]\natural\mathcal{S}_0$. Then we get $w = w_{\mathcal{S}_p} v_{i_p,i_{p-1}} w_{\mathcal{S}_{p-1}} v_{i_{p-1},i_{p-2}} \cdots v_{i_2,i_1} w_{\mathcal{S}_1} v_{i_1,i_0} w_{\mathcal{S}_0'}$, which has the desired form. On the other hand, if m_{ij} is odd, then $v_{ji}.\alpha_i = K_{ji}\alpha_j$. So $v^J.\alpha_j = \frac{K}{K_{ji}}\alpha_x$. If $v^J = \varepsilon$, then take $\mathcal{P} := [\gamma_i, \gamma_j]$ with $\mathcal{S}_0$ and $\mathcal{S}_1$ empty ER-sequences to see that $w = v_{ji}$ has the desired form. Otherwise apply the induction hypothesis to v^J to see that there is an ON-path $\mathcal{P} = [\gamma_{i_1=j}, \gamma_{i_2}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$ and ER-sequences $\mathcal{S}_k$ rooted at γ_{i_k} ($1 \leq k \leq p$) such that $v^J = w_{\mathcal{S}_p} v_{i_p,i_{p-1}} w_{\mathcal{S}_{p-1}} v_{i_{p-1},i_{p-2}} \cdots v_{i_2,i_1} w_{\mathcal{S}_1}$. Take $i_0 = i$, $\mathcal{P}' = [\gamma_i, \gamma_j]\natural\mathcal{P}$ (an ON-path), and $\mathcal{S}_0$ an empty ER-sequence. Then we get $w = w_{\mathcal{S}_p} v_{i_p,i_{p-1}} w_{\mathcal{S}_{p-1}} v_{i_{p-1},i_{p-2}} \cdots v_{i_2,i_1} w_{\mathcal{S}_1} v_{i_1,i_0} w_{\mathcal{S}_0}$ as desired. Whether m_{ij} is even or odd, we now see that $w.\alpha_i = w_{\mathcal{P}}.\alpha_i = \Pi_{\mathcal{P}}\alpha_x$. $\square$

Figure 3.1: A unital ON-cyclic E-GCM graph for Example 3.12.

The notation ⓜ (resp. ⊗) on an edge $\overset{\bullet\!\!\longrightarrow\ \longleftarrow\!\!\bullet}{\ \ p\qquad q}$ indicates that $pq = 4\cos^2(\pi/m)$ (resp. $pq \geq 4$).



The induction argument of the preceding proof can be viewed as a constructive method for obtaining the expression of the theorem statement for the Coxeter group element w. A further consequence of the proof is the following result about the length of w. It says, in effect, that if we write w as a product of v_{ji}'s as prescribed in the theorem statement and then in turn write each such v_{ji} as a shortest product of generators, the resulting expression for w is reduced.

Corollary 3.3 *Suppose $w \in W$, $i, x \in I_n$, and $w.\alpha_i = K\alpha_x$ for some $K > 0$. Suppose $w = w_{\mathcal{S}_p} v_{i_p,i_{p-1}} w_{\mathcal{S}_{p-1}} v_{i_{p-1},i_{p-2}} \cdots v_{i_2,i_1} w_{\mathcal{S}_1} v_{i_1,i_0} w_{\mathcal{S}_0}$ for an ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$ and ER-sequences $\mathcal{S}_k$ rooted at γ_{i_k} $(0 \leq k \leq p)$ obtained by the method of the preceding proof. For all $j, l \in I_n$, let $c(j,l)$ count the total number of occurrences of (γ_j, γ_l) as consecutive nodes (in this order) in the ON-path $\mathcal{P}$ or as a pair in the ER-sequences $\mathcal{S}_k$ $(0 \leq k \leq p)$. Then $\ell(w) = \sum_{j,l \in I_n} c(j,l)(m_{jl} - 1)$.* □

For any $\alpha \in \Phi$, set $\mathfrak{S}(\alpha) := \mathfrak{S}_A(\alpha) := \{K\alpha\}_{K \in \mathbb{R}} \cap \Phi^+$. Our analysis of the sets $\mathfrak{S}(\alpha)$ requires some additional notation. For ON-paths $\mathcal{P}$ and $\mathcal{Q}$, write $\mathcal{P} \sim \mathcal{Q}$ and say $\mathcal{P}$ and $\mathcal{Q}$ are Π-*equivalent* if these ON-paths have the same start and end nodes and $\Pi_{\mathcal{P}} = \Pi_{\mathcal{Q}}$. This is an equivalence relation on the set of all ON-paths. An ON-path $\mathcal{P}$ is *simple* if it has no repeated nodes with the possible exception that the start and end nodes may coincide. Two ON-paths $\mathcal{P}$ and $\mathcal{Q}$ are *scalar-distinct* if $\Pi_{\mathcal{P}} \neq \Pi_{\mathcal{Q}}$. An ON-path $\mathcal{P} = [\gamma_{i_0}, \ldots, \gamma_{i_p}]$ is an *ON-cycle* if $\gamma_{i_p} = \gamma_{i_0}$. It is *unital* if $\Pi_{\mathcal{P}} = 1$, i.e. $a_{i_0,i_1} a_{i_1,i_2} \cdots a_{i_{p-1},i_0} = a_{i_0,i_{p-1}} \cdots a_{i_2,i_1} \cdots a_{i_1,i_0}$. We say (Γ, A) is *unital ON-cyclic* if and only if $\Pi_{\mathcal{C}} = 1$ for all ON-cycles $\mathcal{C}$. See Figure 3.1. From the definitions it follows that (Γ, A) is unital ON-cyclic if it has no odd asymmetries. So if A is a GCM, then (Γ, A) is unital ON-cyclic. If A is a symmetrizable E-GCM, then by applying Exercise 2.1 of [Kac] or Exercise 1.5.E.1 of [Kum] to the environment of E-GCM's, one sees that (Γ, A) is unital ON-cyclic. However, a unital ON-cyclic E-GCM graph need not have a symmetrizable matrix A, as Example 3.12 shows. To check if an E-GCM graph is unital ON-cyclic, it is enough to check that each simple ON-cycle is unital. An E-GCM graph is *ON-connected* if any two nodes can be joined by an ON-path. An *ON-connected component* of (Γ, A) is an E-GCM subgraph (Γ', A') whose nodes form a maximal collection of nodes in (Γ, A) which can be pairwise joined by ON-paths.

Lemma 3.4 *Let α and β be roots in Φ. Suppose an element of $\mathfrak{S}(\alpha)$ is in the same orbit as an element of $\mathfrak{S}(\beta)$ under the action of W on Φ. Then there is a one-to-one correspondence between* *the sets $\mathfrak{S}(\alpha)$ and $\mathfrak{S}(\beta)$. If γ_i and γ_j are nodes in the same ON-connected component of (Γ, A), then there is a one-to-one correspondence between the sets $\mathfrak{S}(\alpha_i)$ and $\mathfrak{S}(\alpha_j)$.*

Proof. Since $\mathfrak{S}(\alpha) = \mathfrak{S}(K\alpha)$ for all $K\alpha \in \mathfrak{S}(\alpha)$, it suffices to assume that α and β are in the same W-orbit, i.e. $\beta = w.\alpha$ for some $w \in W$. It is easy to see that the mapping $\mathfrak{S}(\alpha) \to \mathfrak{S}(\beta)$ given by $\sigma(w)|_{\mathfrak{S}(\alpha)}$ gives the desired one-to-one correspondence. If γ_i and γ_j are in the same ON-connected component, then by Theorem 3.2, some positive scalar multiple of α_j is in the W-orbit of α_i. Thus there is a a one-to-one correspondence between the sets $\mathfrak{S}(\alpha_i)$ and $\mathfrak{S}(\alpha_j)$. $\square$

The proof of the following lemma is a routine verification, so it is omitted.

Lemma 3.5 *Suppose (Γ, A) is unital ON-cyclic. Then for any ON-path $\mathcal{P}$ there is a simple ON-path which is Π-equivalent to $\mathcal{P}$.*

Although Theorem 3.6 and Corollary 3.7 ask readers to look at a subgraph (Γ', A') of (Γ, A), the conclusions pertain to the action of $W = W(\Gamma, A)$ on Φ.

Theorem 3.6 *Choose any ON-connected component (Γ', A') of (Γ, A), and let $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$. Then the following are equivalent:*

(1) (Γ', A') is unital ON-cyclic.

(2) $|\mathfrak{S}(w.\alpha_x)| < \infty$ for some $x \in J$ and $w \in W$.

(3) $|\mathfrak{S}(w.\alpha_x)| < \infty$ for all $x \in J$ and $w \in W$.

In these cases for all $x, y \in J$ and $w \in W$, $|\mathfrak{S}(\alpha_x)| = |\mathfrak{S}(w.\alpha_y)|$. This common quantity is equal to the largest number of pairwise scalar-distinct simple ON-paths in (Γ, A) with end node γ_x.

Proof. We show (2) $\Rightarrow$ (1) $\Rightarrow$ (3), the implication (3) $\Rightarrow$ (2) being obvious. For (1) $\Rightarrow$ (3), let $x \in J$. Observe that if $K\alpha_x \in \Phi^+$, then by Theorem 3.2 we must have $K = \Pi_{\mathcal{P}}$ for some ON-path $\mathcal{P}$ with end node γ_x. Therefore $\mathcal{P}$ is in (Γ', A'). By Lemma 3.5, we may take a simple ON-path $\mathcal{Q}$ Π-equivalent to $\mathcal{P}$ (all ON-paths Π-equivalent to $\mathcal{P}$ must be in (Γ', A')), so that $K = \Pi_{\mathcal{Q}}$. Since there can be at most a finite number of simple ON-paths, then there can be at most finitely many positive roots that are scalar multiples of a given α_x. That $|\mathfrak{S}(w.\alpha_x)| = |\mathfrak{S}(\alpha_x)|$ for all $w \in W$ follows from Lemma 3.4. For (2) $\Rightarrow$ (1), we show the contrapositive. Let $\mathcal{C} = [\gamma_x, \ldots, \gamma_x]$ be a non-unital ON-cycle with start/end node γ_x for an $x \in J$. So necessarily $\mathcal{C}$ has nonzero length. Note that $w_{\mathcal{C}}.\alpha_x = \Pi_{\mathcal{C}}\alpha_x$. Next, for $y \in J$ (and possibly $y = x$) take any ON-path $\mathcal{P}$ with start node γ_x and end node γ_y. Since $w_{\mathcal{P}}.\alpha_x = \Pi_{\mathcal{P}}\alpha_y$, it follows that $w_{\mathcal{P}}w_{\mathcal{C}}^k.\alpha_x = \Pi_{\mathcal{P}}\Pi_{\mathcal{C}}^k\alpha_y$ for any integer k. In particular, for all $y \in J$, we have $|\mathfrak{S}(\alpha_y)| = \infty$. So by Lemma 3.4 $|\mathfrak{S}(w.\alpha_y)| = \infty$ for all $y \in J$, $w \in W$. The next-to-last claim of the theorem statement follows from Lemma 3.4. The final claim follows from our proof above of the (1) $\Rightarrow$ (3) part of the theorem statement. $\square$

From Theorem 3.2 it follows that *if (Γ, A) has an odd asymmetry, then there exists a root which is a non-trivial multiple of a simple root.* The following corollary of Theorem 3.6 contains a more general statement that includes the converse. When A is an integer matrix, odd neighbors γ_i and γ_j must have $\{-a_{ij}, -a_{ji}\} = \{1, 1\}$. These are not asymmetric. Therefore the matrices A defining Weyl groups have no odd asymmetries. In this integer matrix setting, Kac ([Kac] Proposition 5.1.b) and Kumar ([Kum] Corollary 1.3.6.a) show that for a "real" root α and real number K, $K\alpha$ is also a root if and only if $K = \pm 1$. Their proofs use Lie brackets and root space reasoning. But alternatively, this result is also a very special case of the following:

Corollary 3.7 *We have $|\mathfrak{S}(\alpha)| = 1$ for all $\alpha \in \Phi$ if and only if (Γ, A) has no odd asymmetries. More generally, choose any ON-connected component (Γ', A') of (Γ, A), and let $J := \{x \in I_n\}_{\gamma_x \in \Gamma'}$. Then $|\mathfrak{S}(w.\alpha_x)| = 1$ for some $x \in J$ and $w \in W$ if and only if $|\mathfrak{S}(w.\alpha_x)| = 1$ for all $x \in J$ and $w \in W$ if and only if (Γ', A') has no odd asymmetries.*

Proof. Follows from Theorems 3.2 and 3.6. □

Analogizing [BB] and [HRT], for any $w \in W$ set $N(w) := N_A(w) := \{\alpha \in \Phi^+ \mid w.\alpha \in \Phi^-\}$. (For the matrices A considered in [Kum] Ch. 1, this set is notated $\Phi_{w^{-1}}$.)

Lemma 3.8 *For any $i \in I_n$, $s_i(\Phi^+ \setminus \mathfrak{S}(\alpha_i)) = \Phi^+ \setminus \mathfrak{S}(\alpha_i)$. Now let $w \in W$. If $w.\alpha_i \in \Phi^+$, then $N(ws_i) = s_i(N(w)) \,\dot\cup\, \mathfrak{S}(\alpha_i)$, a disjoint union. If $w.\alpha_i \in \Phi^-$, then $N(ws_i) = s_i(N(w) \setminus \mathfrak{S}(\alpha_i))$.*

Proof. Using Proposition 2.1, the proof of Proposition 5.6.(a) from [Hum] is easily adjusted to prove the first claim. Proofs for the remaining claims involve routine set inclusion arguments. □

When (Γ, A) is ON-connected and unital ON-cyclic, set $f_{\Gamma,A} := |\mathfrak{S}(\alpha_x)|$ for any given $x \in I_n$. At this point, Proposition 2.1, Theorem 3.2, Lemma 3.8, and Theorem 3.6 allow us to modify the proof of Proposition 5.6 of [Hum] to obtain the result that for all $w \in W$, $|N(w)| = f_{\Gamma,A}\, \ell(w)$. Theorem 3.9 below generalizes this statement for arbitrary E-GCM graphs. For $J \subseteq I_n$, let $\mathfrak{C}(J)$ denote the set of all ON-connected components of (Γ, A) containing some node from the set $\{\gamma_x\}_{x \in J}$.

Theorem 3.9 *Let $w \in W$ with $p = \ell(w) > 0$. (1) Then $N(w)$ is finite if and only if w has a reduced expression $s_{i_1} \cdots s_{i_p}$ for which $\mathfrak{S}(\alpha_{i_q})$ is finite for all $1 \le q \le p$ if and only if every reduced expression $s_{i_1} \cdots s_{i_p}$ for w has $\mathfrak{S}(\alpha_{i_q})$ finite for all $1 \le q \le p$. (2) Now suppose $w = s_{i_1} \cdots s_{i_p}$ and $N(w)$ is finite. Let $J := \{i_1, \ldots, i_p\}$. In view of (1), let f_1 be the min and f_2 the max of all integers in the set $\{f_{\Gamma',A'} \mid (\Gamma', A') \in \mathfrak{C}(J)\}$. Then $f_1\, \ell(w) \le |N(w)| \le f_2\, \ell(w)$.*

Proof. (1) follows from Lemma 3.8. For (2), induct on $\ell(w)$. Take $w' := s_{i_1} \cdots s_{i_{p-1}}$ with $w = w' s_{i_p}$. Now γ_{i_p} is in an ON-connected component (Γ', A') of (Γ, A). Then by Lemma 3.8, $|N(w)| = |N(w')| + f_{\Gamma',A'}$. Since $f_1\, \ell(w') \le |N(w')| \le f_2\, \ell(w')$, the result follows. □

Apply Theorems 3.6 and 3.9 to get:

Corollary 3.10 *We have $N(w)$ finite for all $w \in W$ if and only if (Γ, A) is unital ON-cyclic. Moreover $|N(w)| = \ell(w)$ for all $w \in W$ if and only if (Γ, A) has no odd asymmetries.* □

When W is infinite, the length function must take arbitrarily large values. Then by Theorem 3.9, Φ is infinite as well. If W is finite, then Φ is finite as well, so $|\mathfrak{S}(\alpha_x)| < \infty$ for all $x \in I_n$. In this case let w_0 be the longest element of W (cf. Exercise 5.6.2 of [Hum]). It is easily seen that if $w_0 = s_{i_1} \cdots s_{i_l}$ is reduced, then $\{i_1, \ldots, i_l\} = I_n$.

Corollary 3.11 *Suppose W is finite. Let Φ_{std} denote the root system for the standard geometric representation. Then $f_1 |\Phi^+_{\mathrm{std}}| \le |\Phi^+| \le f_2 |\Phi^+_{\mathrm{std}}|$, where f_1 is the min and f_2 is the max of all integers in the set $\{f_{\Gamma',A'} \mid (\Gamma', A') \in \mathfrak{C}(I_n)\}$.*

Proof. Apply Proposition 2.1 to see that $N(w_0) = \Phi^+$. By Theorem 3.9, $f_1 \ell(w_0) \le |\Phi^+| \le f_2 \ell(w_0)$. To see that $\ell(w_0) = |\Phi^+_{\mathrm{std}}|$, apply the previous reasoning in the standard case. □

Example 3.12 In Figure 3.1 is depicted a connected, unital ON-cyclic E-GCM graph (Γ, A) with three ON-connected components: (Γ_1, A_1) is the E-GCM subgraph with nodes γ_1 and γ_2; (Γ_2, A_2) has nodes γ_4, γ_5, and γ_6; and (Γ_3, A_3) has only the node γ_3. The matrix A is not symmetrizable by Exercise 2.1 of [Kac] or Exercise 1.5.E.1 of [Kum]. Pertaining to the pair (γ_4, γ_6), we have $4\cos^2(\pi/5) = \frac{3+\sqrt{5}}{2}$ and $2\cos(\pi/5) = \frac{1+\sqrt{5}}{2}$. Since $a_{46} = -\frac{1+\sqrt{5}}{4}$ and $a_{64} = -(1+\sqrt{5})$, then $K_{46} = \frac{-a_{46}}{2\cos(\pi/5)} = \frac{1}{2}$ and $K_{64} = \frac{-a_{64}}{2\cos(\pi/5)} = 2$. For all other odd neighbors (γ_i, γ_j) in this graph, $m_{ij} = 3$, so $K_{ij} = -a_{ij}$ and $K_{ji} = -a_{ji}$. By the last statement of Theorem 3.6, $f_{\Gamma_1, A_1} = 2$ and $f_{\Gamma_2, A_2} = 3$. For example, $\mathfrak{S}(\alpha_2) = \{\alpha_2, \frac{1}{5}\alpha_2\} = N(s_2)$ and $\mathfrak{S}(\alpha_5) = \{\alpha_5, \frac{1}{7}\alpha_5, \frac{2}{7}\alpha_5\} = N(s_5)$. By Theorem 3.9, we can see that $f_{\Gamma_1, A_1}\ell(s_5 s_2) = 4 \leq |N(s_5 s_2)| \leq 6 = f_{\Gamma_2, A_2}\ell(s_5 s_2)$. More precisely, by Lemma 3.8 we get $N(s_5 s_2) = s_2(N(s_5)) \,\dot\cup\, \mathfrak{S}(\alpha_2)$, whence $|N(s_5 s_2)| = 5$. □

We now apply Theorem 3.2 to extend a finiteness result of Brink and Howlett concerning a natural partial order on positive roots, cf. Theorem 2.8 of [BH]. From here on, Φ_{std} denotes the root system for the standard geometric representation, and $\{\alpha_i^{\text{std}}\}_{i \in I_n}$ are its simple roots. Following [BH] and §4.7 of [BB], for roots $\alpha, \beta \in \Phi_{\text{std}}^+$, we say α *dominates* β and write α dom β if for all $w \in W$ we have $w.\beta \in \Phi_{\text{std}}^-$ whenever $w.\alpha \in \Phi_{\text{std}}^-$. It is known that the relation "dom" on Φ_{std}^+ is a partial order on Φ_{std}^+ ([BH], §4.7 of [BB]). Roots in Φ_{std}^+ that are minimal with respect to this partial order are *dominance-minimal*. Observe that simple roots in Φ_{std}^+ are dominance-minimal. The fact, due to Brink and Howlett in [BH], that the set of dominance-minimal elements is finite is viewed by some to be a fundamental result (see [Cas], §4.7 of [BB]). Notable consequences of this finiteness result are the so-called Parallel Wall Theorem (discussed in [BH], see also [Cap]) for the associated Davis complex, as well as the fact that Coxeter groups are automatic [BH]. However, the above definition of dominance does not extend nicely in the obvious way to the asymmetric setting: If adjacent nodes γ_i and γ_j in (Γ, A) form an odd asymmetry, then by Theorem 3.2, α_i and $K\alpha_i$ are both positive roots for some positive $K \neq 1$. Then, each root would dominate the other, so dominance would not be an anti-symmetric relation on Φ^+.

In what follows, we address this issue. In the general setting, let $\Psi^+ := \{\mathfrak{S}(\alpha)\}_{\alpha \in \Phi^+}$. For $\alpha, \beta \in \Phi^+$, say $\mathfrak{S}(\alpha)$ *dominates* $\mathfrak{S}(\beta)$ and write $\mathfrak{S}(\alpha)$ dom $\mathfrak{S}(\beta)$ if for all $w \in W$ we have $w.\beta \in \Phi^-$ whenever $w.\alpha \in \Phi^-$. It is easy to see that this definition is independent of the choice of representatives from each of $\mathfrak{S}(\alpha)$ and $\mathfrak{S}(\beta)$, so dominance is a well-defined relation on Ψ^+.

Theorem 3.13 *Define a function $\rho : \Psi^+ \longrightarrow \Phi_{\text{std}}^+$ by the rule: $\rho(\mathfrak{S}(\alpha)) := w.\alpha_i^{\text{std}}$ if $\alpha = w.\alpha_i \in \Phi^+$ for some $w \in W$ and $i \in I_n$. Then ρ is a well-defined bijection, and moreover ρ and ρ^{-1} both preserve dominance. In particular, "dom" is a partial order on Ψ^+ and the set of dominance-minimal elements of Ψ^+ is finite.*

Proof. Obviously ρ is surjective if it is well-defined. To see that ρ is well-defined and injective, we show that for any $w_1, w_2 \in W$ and $i, x \in I_n$, we have $w_1.\alpha_i = K w_2.\alpha_x \in \Phi^+$ for some $K > 0$ if and only if $w_1.\alpha_i^{\text{std}} = w_2.\alpha_x^{\text{std}} \in \Phi_{\text{std}}^+$. Now if $w_1.\alpha_i = K w_2.\alpha_x \in \Phi^+$, then $w.\alpha_i = K\alpha_x$ for $w := (w_2)^{-1} w_1$. By Theorem 3.2, there is an ON-path $\mathcal{P} = [\gamma_{i_0=i}, \gamma_{i_1}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p=x}]$ and ER-sequences $\mathcal{S}_k$ rooted at γ_{i_k} $(0 \leq k \leq p)$ such that $w = w_{\mathcal{S}_p} v_{i_p, i_{p-1}} w_{\mathcal{S}_{p-1}} v_{i_{p-1}, i_{p-2}} \cdots v_{i_2, i_1} w_{\mathcal{S}_1} v_{i_1, i_0} w_{\mathcal{S}_0}$. Using this expression for w, we can calculate that $w.\alpha_i^{\text{std}} = \alpha_x^{\text{std}}$. It follows that $w_1.\alpha_i^{\text{std}} = w_2.\alpha_x^{\text{std}}$, which is in Φ_{std}^+ by Proposition 2.1. The converse is entirely similar. Using Proposition 2.1, we have that $w.\alpha_i \in \Phi^+$ (respectively, Φ^-) if and only if $w.\alpha_i^{\mathrm{std}} \in \Phi^+_{\mathrm{std}}$ (resp. Φ^-_{std}). It now follows from the definitions that ρ and ρ^{-1} preserve dominance. So, "dom" is a partial order on Ψ^+. That the set of dominance-minimal elements of Ψ^+ is finite now follows from Theorem 2.8 of [BH]. $\square$

§4 An application concerning the Tits cone. We close with results which relate the size of a Coxeter group W and the behavior of a "fundamental domain" for the "contragredient" W-action. The main result of this section (Theorem 4.5) is derived in two ways as an application of Corollary 3.10/Proposition 4.3: first using the perspective of the numbers game, and second borrowing some results from [Vin]. We continue to consider $\sigma : W \to GL(V)$. We have the natural pairing $\langle \lambda, v\rangle := \lambda(v)$ for elements λ in the dual space V^* and vectors v in V. The contragredient representation $\sigma^* := \sigma^*_A : W \to GL(V^*)$ is determined by $\langle \sigma^*(w)(\lambda), v\rangle = \langle \lambda, \sigma(w^{-1})(v)\rangle$. When $w \in W$ and $\lambda \in V^*$, we write $w.\lambda$ for $\sigma^*(w)(\lambda)$. Let $D := \{\lambda \in V^* \,|\, \langle \lambda, \alpha_i\rangle \geq 0 \text{ for all } i \in I_n\}$. Following [Vin], [Erik1], [Erik2], the *Tits cone* is $U := U_A := \cup_{w\in W} wD$. This generalizes the standard case of [Hum]. In view of Proposition 2.1, the results of [Hum] §5.13 hold here. So D is the aforementioned fundamental domain, and U is a convex cone. Let $\overline{U}$ denote the closure of U. See the lecture notes of Howlett [How] for further discussion of properties of the Tits cone for the standard geometric representation σ, and in particular an investigation of phenomena in $\overline{U} \setminus U$. If T is any convex cone, let T_0 denote the maximal subspace contained in T. It is not hard to see that $T_0 = T \cap (-T)$.

Our Tits cone results below concern U_0. These results both use/produce consequences from/for the numbers game. Elements of V^* will be referred to as *positions* for (Γ, A). We define a process of acting on positions in V^* with certain sequences of Coxeter group generators that is equivalent to Eriksson's numbers game as presented in §4.3 of [BB]. For a positive integer p we say a sequence $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ from (Γ, A) is *legal* from a given position λ if $\langle s_{i_{q-1}} \cdots s_{i_1}.\lambda, \alpha_{i_q}\rangle > 0$ for all $1 \leq q \leq p$. Repeated application of Proposition 2.1 implies that in this case, $s_{i_p} \cdots s_{i_1}$ is reduced. Call this the Reduced Word Result. Next, say a position λ is *good* if $\lambda \in -D$ or there exists a legal sequence $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ from λ such that $s_{i_p} \cdots s_{i_1}.\lambda \in -D$. In the latter case say $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ is a *terminated* legal sequence. Think of a good position as a position from which there is a (possibly empty) terminated legal sequence. Eriksson's Strong Convergence Theorem (see Theorem 2.2 of [Erik2]) shows that all legal sequences of maximal length from a good λ terminate at the same "terminal position" in the same finite number of steps. Lemma 5.13 of [Hum] is the basis for an argument in §4 of [Erik2] showing that if $\lambda = w.\mu$ for $\mu \in -D$, then μ can be reached from λ by a legal sequence. Then we get the following characterization of the set of good positions:

Proposition 4.1 (Eriksson) *The set of good positions for* (Γ, A) *is precisely* $-U$.

Our next result generalizes Remark 4.4 of [Deo] to our current setting. This is needed for Proposition 4.3. For $J \subseteq I_n$, let $\Phi^J := \{\alpha \in \Phi^+ \,|\, \alpha \notin \mathrm{span}_{\mathbb{R}}\{\alpha_j\}_{j\in J}\}$.

Lemma 4.2 *If* (Γ, A) *is connected,* Φ *is infinite, and* $J \subset I_n$ *(proper), then* Φ^J *is infinite.*

Proof: In the "$(ix) \Rightarrow (ii)$" part of the proof of Proposition 4.2 in [Deo], assume $|\Phi^J| < \infty$ and begin reading at line -8 of page 620. $\square$

Proposition 3.2 of [HRT] states that if (Γ, A) is connected, σ is standard, and W is infinite, then $U_0 = \{0\}$. In view of Corollary 3.10 and Lemma 4.2, we can use the proof of Proposition 3.2 of

[HRT] verbatim to get the generalization of that result stated as Proposition 4.3 below. One can see that that proof will work if it is known that all $N(w)$ are finite; by Corollary 3.10 this is guaranteed by our hypothesis in the statement of Proposition 4.3 requiring that (Γ, A) is unital ON-cyclic.

Proposition 4.3 *Suppose (Γ, A) is connected and unital ON-cyclic and W is infinite. Then $U_0 = \{0\}$, i.e. U is a "strictly convex" cone.* □

In contrast, for finite W the overlap $U_0 = U \cap (-U)$ is <u>all</u> of V^*. This is a consequence of the following result due to Vinberg (see §7 of [Vin]). The proof below uses numbers game reasoning.

Proposition 4.4 *If W is finite, then $U = V^* = -U$, so $U_0 = V^*$.*

Proof. Since W is finite, then by the Reduced Word Result it follows that the set of good positions is all of V^*. Proposition 4.1 now implies that $V^* = -U$, hence $U = V^*$ also. □

When (Γ, A) is connected and unital ON-cyclic, if a nonzero $\lambda \in D$ is good, then by Propositions 4.1 and 4.3, W must be finite. This observation, together with the classification of finite Coxeter groups and reasoning based on the numbers game, is used in §6 of [Don] to prove the following result, which we refer to in Remark 4.7 below as result ($*$): *If (Γ, A) is connected, then $D \cap (-U) \neq \{0\}$ implies that W is finite.* (We know of three proofs of statement ($*$): See Theorem 6.1 of [Don]; see Remark 4.7 below for a proof that uses Proposition 4.3, results borrowed from [Vin], and a classification result due to H. S. M. Coxeter; or see §4 of [DE] for a proof that does not require Proposition 4.3 or the classification of finite Coxeter groups.) Now, it follows from the definitions that $U_0 \neq \{0\}$ if and only if $D \cap (-U) \neq \{0\}$. In view of Proposition 4.4, we thus obtain the following addition to the list of equivalences from Propositions 4.1 and 4.2 of [Deo] for an irreducible Coxeter group to be finite:

Theorem 4.5 *Let (Γ, A) be connected, so the Coxeter group W is irreducible. Then W is finite if and only if $U_0 \neq \{0\}$ if and only if $U_0 = V^*$.* □

See §2 of [Kra] for a proof of this result in the special case that the bilinear form B for the representing space V is symmetric.

Remark 4.6 A Tits cone is similarly defined in the context of Kac–Moody theory e.g. [Kac] §3.12, [Kum] §1.4. Let A be a GCM. Here we follow Kac [Kac] and the end of §2 above. The Kac–Moody Tits cone is the set $C := C_A := \{w.\lambda \,|\, w \in W, \lambda \in \mathfrak{h}_{\mathbb{R}}$ such that $\alpha_i(\lambda) \geq 0$ for $1 \leq i \leq n\} \subseteq \mathfrak{h}_{\mathbb{R}}$. When A is nondegenerate (nullity$(A) = 0$), then $V = \mathfrak{h}_{\mathbb{R}}^*$ and hence C and U coincide. Now suppose (Γ, A) is connected and W is infinite. We have that the GCM graph (Γ, A) is unital ON-cyclic. Thus if A is nondegenerate, the result $C_0 = \{0\}$ holds by Proposition 4.3. Allowing nullity$(A) \geq 0$, Kumar (personal communication) has supplied the following description of C_0: $C_0 = \{v \in \mathfrak{h}_{\mathbb{R}} \,|\, \alpha_i(v) = 0$ for $1 \leq i \leq n\}$. Here $\dim(C_0) = $ nullity(A). He notes that this statement may be deduced from Part (c) of Proposition 3.12 of [Kac]. Note that the topological interior of the Kac–Moody Tits cone can never intersect its negative. This follows from [Kac] Exercise 3.15 (see also [Kum] Exercise 1.4.E.1). □

Remark 4.7 In this remark we use Proposition 4.3 and results from [Vin] to prove the following version of result ($*$) above: *If (Γ, A) is connected, then W infinite implies that $U_0 = \{0\}$.* (Then, Theorem 6.1 of [Don] can be obtained as an easy consequence.) To prove this, we interpret our set-up here in terms of [Vin]. Assume throughout this remark that (Γ, A) is connected. Our V^*

plays the role of Vinberg's V, our D plays the role of his K, our α_i's play the role of his α_i's. Let us take elements $h_i \in V^*$ for which $h_i(\alpha_j) = a_{ij}$ for all $i, j \in I_n$. Each $R_i := \sigma^*(s_i) \in GL(V^*)$ is then a reflection in the sense of §1 of [Vin]. From [Hum] §5.13, we know that $w\,\overset{\circ}{D} \cap \overset{\circ}{D} = \emptyset$ for all $w \neq \varepsilon$ in W, where $\overset{\circ}{D}$ denotes the interior of D. Therefore our $\sigma^*(W) \subset GL(V^*)$ is, in Vinberg's language, a "linear Coxeter group."

With respect to some total ordering of I_n, think of an n-tuple $\mathbf{v} = (v_i)_{i \in I_n}$ as a column vector. Denote by $\mathbf{0}$ the zero vector. For column vectors $\mathbf{u}, \mathbf{v}$, say $\mathbf{u} > \mathbf{v}$ (respectively $\mathbf{u} \geq \mathbf{v}$) if $u_i > v_i$ (resp. $u_i \geq v_i$) for each $i \in I_n$. From §4 of [Vin] or Chapter 4 of [Kac], we have that exactly one of the following three statements (+), (0), or (–) is true: (+) $\det A \neq 0$; there exists $\mathbf{v} > \mathbf{0}$ such that $A\mathbf{v} > \mathbf{0}$; $A\mathbf{u} \geq \mathbf{0}$ implies $\mathbf{u} > \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$; (0) nullity$(A) = 1$; there exists $\mathbf{v} > \mathbf{0}$ such that $A\mathbf{v} = \mathbf{0}$; $A\mathbf{u} = \mathbf{0}$ implies that $\mathbf{u} \geq \mathbf{0}$; (–) There exists $\mathbf{v} > \mathbf{0}$ such that $A\mathbf{v} < \mathbf{0}$; $A\mathbf{u} \geq \mathbf{0}$, $\mathbf{u} \geq \mathbf{0}$ imply that $\mathbf{u} = \mathbf{0}$. Write $A = A^+$, $A = A^0$, or $A = A^-$ accordingly. By Lemma 15 and Proposition 25 of [Vin], we see that $A = A^+ \Rightarrow U = V^*$, $A = A^0 \Rightarrow (\overline{U})_0 = \mathrm{span}(h_i)_{i \in I_n}$, and $A = A^- \Rightarrow (\overline{U})_0 = \{0\}$. (Note that the set "Ann$[\alpha]$" in [Vin] is $\{0\}$ here, since it is just $\{\lambda \in V^* \,|\, \lambda(v) = 0 \text{ for all } v \in V\}$.)

Now we prove the version of result ($*$) stated at the beginning of this remark. We consider the three cases (+), (0), and (–). First, suppose $A = A^+$. Then by Proposition 22 of [Vin], W must be finite, contrary to our hypothesis. Second, suppose that $A = A^0$. Then by Proposition 23 of [Vin], W is an irreducible "parabolic" Coxeter group, also called an irreducible Euclidean reflection group, see e.g. [Dav]. The well-known classification of such groups is due to H. S. M. Coxeter [Cox]. For our purposes, it is enough to observe that any such (Γ, A) will possess a simple ON-cycle only in the case that (Γ, A) itself is a simple ON-cycle with $m_{ij} = 3$ for any adjacent γ_i and γ_j. By Proposition 23 of [Vin], it follows that this ON-cycle is unital. We conclude that whenever $A = A^0$, (Γ, A) is unital ON-cyclic. Since W is infinite (by, say, Proposition 22 of [Vin]), then by Proposition 4.3 above, we have $U_0 = \{0\}$. Finally, if $A = A^-$, then $(\overline{U})_0 = \{0\}$ implies that $U_0 = \{0\}$. (Note that W must infinite whenever $A = A^-$, by Proposition 22 of [Vin].) In any case, we see that when (Γ, A) is connected and W is infinite, then $U_0 = \{0\}$. $\square$

Example 4.8 For $(\Gamma, A) = \gamma_1 \bullet\!\xrightarrow{\;p\;}\!\!\xleftarrow{\;q\;}\!\bullet\, \gamma_2$ with $pq = 4$, we have that W is the infinite dihedral group. Since $A = A^0$ in the notation of Remark 4.7, then $(\overline{U})_0 = \mathrm{span}(h_i)_{i \in \{1,2\}}$. Relative to the basis $\{\omega_i\}_{i=1,2}$ for V^* dual to the simple root basis $\{\alpha_i\}_{i=1,2}$ for V, we have $h_1 = 2\omega_1 - p\omega_2$ and $h_2 = -q\omega_1 + 2\omega_2 = -\frac{q}{2}h_1$. Then, $(\overline{U})_0 = \mathrm{span}(h_i)_{i \in \{1,2\}} = \{x\omega_1 + y\omega_2 \,|\, y = -\frac{p}{2}x\}$. In fact, using the computational approach of the proof of Lemma 3.1 above, one can see that $U = \{x\omega_1 + y\omega_2 \,|\, y > -\frac{p}{2}x \text{ or } x = y = 0\}$, and hence that $U_0 = \{0\}$. $\square$

Acknowledgments We thank Kimmo Eriksson for providing us with a copy of his thesis and for many helpful conversations during the preparation of this paper. We thank Bob Proctor for his helpful feedback, which included the remarks concerning Weyl groups, and for sharing with us in advance some of the results of [Pro]. We thank Shrawan Kumar for helpful comments concerning Kac–Moody Tits cones.

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