Convergent and divergent numbers games for certain collections of edge-weighted graphs

Robert G. Donnelly
Department of Mathematics and Statistics, Murray State University, Murray, KY 42071

Abstract

The numbers game is a one-player game played on a finite simple graph with certain “amplitudes” assigned to its edges and with an initial assignment of real numbers to its nodes. The moves of the game successively transform the numbers at the nodes using the amplitudes in a certain way. Here, the edge amplitudes will be negative integers. Combinatorial methods are used to investigate the convergence and divergence of numbers games played on certain such graphs. The results obtained here provide support for results in a companion paper.

Keywords: numbers game, generalized Cartan matrix, Dynkin diagram

Contents

1. Introduction, definitions, and preliminary results
2. Convergent numbers games on Dynkin diagrams of finite type
3. Divergent games for some families of graphs

1. Introduction, definitions, and preliminary results

The numbers game is a one-player game played on a finite simple graph with weights (which we call “amplitudes”) on its edges and with an initial assignment of real numbers to its nodes. Here, each of the two edge amplitudes (one for each direction) will be negative integers. The move a player can make is to “fire” one of the nodes with a positive number. This move transforms the number at the fired node by changing its sign, and it also transforms the number at each adjacent node in a certain way using an amplitude along the incident edge. The player fires the nodes in some sequence of the player’s choosing, continuing until no node has a positive number.

The numbers game as formulated by Mozes [Moz] has also been studied by Proctor [Pro1], [Pro2], Björner [Björ], Eriksson [Erik1], [Erik2], [Erik3], [Erik4], [Erik5], [Erik6], [DE], Wildberger [Wil1], [Wil2], [Wil3] and Donnelly [Don2]. Wildberger studies a dual version which he calls the “mutation game.” See Alon et al [AKP] for a brief and readable treatment of the numbers game on “unweighted” cyclic graphs. Much of the numbers game discussion in §4.3 of the book [BB] by Björner and Brenti can be found in [Erik2] and [Erik5]. See these references for discussions of how the numbers game is a combinatorial encoding of information for geometric representations of Weyl groups (and more generally Coxeter groups) and has uses for computing orbits, finding reduced decompositions of Weyl group elements, solving the word problem, and obtaining combinatorial models for Weyl groups. Proctor developed this process in [Pro1] to compute Weyl group orbits of weights with respect to the fundamental weight basis. Here we use his perspective of firing nodes with positive, as opposed to negative, numbers. Mozes studied numbers games on graphs for which the matrix $M$ of integer amplitudes is “symmetrizable” (i.e. there is a nonsingular diagonal matrix $D$ such that $D^{-1}M$ is symmetric); in [Moz] he obtained “strong convergence” results and a geometric characterization of the initial positions for which the game terminates. There will be no symmetrizable assumption here.
Such graphs-with-amplitudes will henceforth be called “GCM graphs” for reasons explained below. Given any such graph, an initial “position” is an assignment of numbers to the nodes. The position is is “nonzero” if at least one of the numbers is nonzero. A numbers game played from some initial position is “convergent” if it terminates after a finite number of node firings; otherwise we say the game is “divergent.”

Here we investigate convergence and divergence of numbers games played on certain GCM graphs. The purpose is to provide supporting details for the proof of a result of [DE]. In particular, we aim to give straightforward combinatorial proofs of Propositions 2.3 and 3.1. These results are used in the proof of the first main result of [DE]: A connected GCM graph has a convergent numbers game played from a nonzero initial position with nonnegative numbers if and only if the graph is one of the “Dynkin diagrams” of Figure 1.1, in which case all numbers games played from a given initial position will converge to the same terminal position in the same number of steps. Proposition 2.3 asserts that for the GCM graphs in Figure 1.1, all numbers games are convergent. Applying results of Eriksson, we will then see that two numbers games played from the same initial position on one of these graphs converge to the same terminal position in the same number of steps. Proposition 3.1 asserts that for the GCM graphs of Figure 3.1, a numbers game is divergent from any nonzero initial position with nonnegative numbers. Two key results needed for proofs of both propositions are Eriksson’s Strong Convergence and Comparison Theorems (see Theorems 1.1 and 1.3 below). Proofs of both propositions also involve case analysis arguments that are fairly routine, can be checked by hand, and are often easily expedited using a computer algebra system to automate some of the computations. Complete details are provided here.

Fix a positive integer \( n \) and a totally ordered set \( I_n \) with \( n \) elements (usually \( I_n := \{1 < \ldots < n\} \)). A generalized Cartan matrix (or GCM) is an \( n \times n \) matrix \( M = (M_{ij})_{i,j \in I_n} \) with integer entries satisfying the requirements that each main diagonal matrix entry is 2, that all other matrix entries are nonpositive, and that if a matrix entry \( M_{ij} \) is nonzero then its transpose entry \( M_{ji} \) is also nonzero. Generalized Cartan matrices are the starting point for the study of Kac–Moody algebras: beginning with a GCM, one can write down a list of the defining relations for a Kac–Moody algebra as well as the associated Weyl group (see [Kac] or [Kum]). To an \( n \times n \) generalized Cartan matrix \( M = (M_{ij})_{i,j \in I_n} \) we associate a finite graph \( \Gamma \) (which has undirected edges, no loops, and no multiple edges) as follows: The nodes \( (\gamma_i)_{i \in I_n} \) of \( \Gamma \) are indexed by the set \( I_n \), and an edge is placed between nodes \( \gamma_i \) and \( \gamma_j \) if and only if \( i \neq j \) and the matrix entries \( M_{ij} \) and \( M_{ji} \) are nonzero.

We call the pair \((\Gamma, M)\) a GCM graph. We consider two GCM graphs \((\Gamma, M = (M_{ij})_{i,j \in I_n})\) and \((\Gamma', M' = (M'_{pq})_{p,q \in I_n'})\) to be the same if under some bijection \( \sigma : I_n \to I_n' \) we have nodes \( \gamma_i \) and \( \gamma_j \) in \( \Gamma \) adjacent if and only if \( \gamma'_{\sigma(i)} \) and \( \gamma'_{\sigma(j)} \) are adjacent in \( \Gamma' \) with \( M_{ij} = M'_{\sigma(i), \sigma(j)} \). With \( p = -M_{12} \) and \( q = -M_{21} \), we depict a generic connected two-node GCM graph as follows:

\[
\begin{array}{c}
\gamma_1 & \quad p \quad & \quad \gamma_2 \\
\end{array}
\]

We use special names and notation to refer to two-node GCM graphs which have \( p = 1 \) and \( q = 1 \), 2, or 3 respectively:

- \( A_2 \)
- \( B_2 \)
- \( G_2 \)

When \( p = 1 \) and \( q = 1 \) it is convenient to use the graph \( \gamma_1 - \gamma_2 \) to represent the GCM graph \( A_2 \). A GCM graph \((\Gamma, M)\) is a Dynkin diagram of finite type if each connected component
A position $\lambda = (\lambda_i)_{i \in I_n}$ is an assignment of real numbers to the nodes of the GCM graph $(\Gamma, M)$. The position $\lambda$ is dominant (respectively, strongly dominant) if $\lambda_i \geq 0$ (resp. $\lambda_i > 0$) for all $i \in I_n$; $\lambda$ is nonzero if at least one $\lambda_i \neq 0$. For $i \in I_n$, the fundamental position $\omega_i$ is the assignment of the number 1 at node $\gamma_i$ and the number 0 at all other nodes. Given a position $\lambda$ on a GCM graph $(\Gamma, M)$, to fire a node $\gamma_i$ is to change the number at each node $\gamma_j$ of $\Gamma$ by the transformation

$$\lambda_j \mapsto \lambda_j - M_{ij}\lambda_i,$$

provided the number at node $\gamma_i$ is positive; otherwise node $\gamma_i$ is not allowed to be fired. Since the generalized Cartan matrix $M$ assigns a pair of amplitudes ($M_{ij}$ and $M_{ji}$) to each edge of the graph $\Gamma$, we sometimes refer to GCMs as amplitude matrices. The numbers game is the one-player game on a GCM graph $(\Gamma, M)$ in which the player (1) Assigns an initial position to the nodes of $\Gamma$; (2) Chooses a node with a positive number and fires the node to obtain a new position; and (3) Repeats step (2) for the new position if there is at least one node with a positive number. Consider now the GCM graph $B_2$. As we can see in Figure 1.2, the numbers game terminates in a finite number of steps for any initial position and any legal sequence of node firings, if it is understood that the player will continue to fire as long as there is at least one node with a positive number. In general, given a position $\lambda$, a game sequence for $\lambda$ is the (possibly empty, possibly infinite) sequence...
The following preliminary results are needed for the proofs of Propositions 2.3 and 3.1. These results also appear in [DE] and [Don2] for use in proofs of key theorems of those papers. Proofs or references for these results are also given here. Following [Erik2] and [Erik6], we say the numbers game on a GCM graph \((\Gamma, M)\) is strongly convergent if given any initial position, any two game sequences either both diverge or both converge to the same terminal position in the same number of steps. The next result follows from Theorem 3.1 of [Erik6] (or see Theorem 3.6 of [Erik2]).

**Theorem 1.1 (Eriksson’s Strong Convergence Theorem)** The numbers game on a connected GCM graph is strongly convergent.

The following weaker result also applies when the GCM graph is not connected:
Lemma 1.2  For any GCM graph, if a game sequence for an initial position $\lambda$ diverges, then all game sequences for $\lambda$ diverge.

The next result is an immediate consequence of Theorem 4.3 of [Erik2] or Theorem 4.5 of [Erik5]. Eriksson’s proof of this result in [Erik2] uses only combinatorial and linear algebraic methods.

**Theorem 1.3 (Eriksson’s Comparison Theorem)**  Given a GCM graph, suppose that a game sequence for an initial position $\lambda = (\lambda_i)_{i \in I_n}$ converges. Suppose that a position $\lambda' = (\lambda'_i)_{i \in I_n}$ has the property that $\lambda'_i \leq \lambda_i$ for all $i \in I_n$. Then some game sequence for the initial position $\lambda'$ also converges.

Let $r$ be a positive real number. Observe that if $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ is a convergent game sequence for an initial position $\lambda = (\lambda_i)_{i \in I_n}$, then $(r\gamma_{i_1}, \ldots, r\gamma_{i_l})$ is a convergent game sequence for the initial position $r\lambda := (r\lambda_i)_{i \in I_n}$. This observation and Theorem 1.3 imply the following result:

**Lemma 1.4**  Let $\lambda = (\lambda_i)_{i \in I_n}$ be a dominant initial position such that $\lambda_j > 0$ for some $j \in I_n$. Suppose that a game sequence for $\lambda$ converges. Then some game sequence for the fundamental position $\omega_j$ also converges.

The following is an immediate consequence of Lemmas 1.2 and 1.4:

**Lemma 1.5**  A GCM graph is not admissible if for each fundamental position there is a divergent game sequence.

2. **Convergent numbers games on Dynkin diagrams of finite type**

Eriksson’s Strong Convergence and Comparison Theorems are key steps in our proof of Proposition 2.3. The remaining step, which accounts for most of the length of this section, is to provide convergent game sequences for numbers games played from strongly dominant positions on connected Dynkin diagrams of finite type. Finding convergent game sequences for numbers games played on Dynkin diagrams of finite type may seem like a difficult task at first, but in view of Proposition 2.3, there is no way to go wrong: any two numbers games played from the same initial position will terminate in the same finite number of steps.

A general theory connecting the numbers game and Coxeter/Weyl group actions was developed by Eriksson in [Erik2] and [Erik5]. From this theory it follows that for a numbers game played from a strongly dominant position on a Dynkin diagram of finite type, any game sequence corresponds to a reduced expression for the longest element of the corresponding Weyl group and conversely any reduced expression for the longest Weyl group element corresponds to a game sequence (see Propositions 4.1 and 4.2 of [Erik5] or Theorem 4.3.1 part (iv) of [BB]). Moreover, the length of the game sequence is the length of any such reduced expression and is also equal to the number of positive roots in the associated root system. For further discussion of this phenomenon, see [Don2].

For the four infinite families of connected Dynkin diagrams of finite type, the next results are proved by induction on $n$, the number of nodes. This is effected by observing natural “GCM subgraph” inclusions $A_{n-1} \hookrightarrow A_n$ ($n \geq 2$), $B_{n-1} \hookrightarrow B_n$ ($n \geq 3$), $C_{n-1} \hookrightarrow C_n$ ($n \geq 4$), and $D_{n-1} \hookrightarrow D_n$ ($n \geq 5$). If $I'_m$ is a subset of the node set $I_n$ of a GCM graph $(\Gamma, M)$, then let $\Gamma'$ be the subgraph of $\Gamma$ with node set $I'_m$ and the induced set of edges, and let $M'$ be the corresponding submatrix of the amplitude matrix $M$; we call $(\Gamma', M')$ a GCM subgraph of $(\Gamma, M)$.

**Lemma 2.1**

A.  For $n \geq 2$ and for any strongly dominant position $(a_1, \ldots, a_n)$ on $A_n$, one can obtain the
position \((a_1 + \cdots + a_n, -a_n, \ldots, -a_3, -a_2)\) by the sequence \((s_n, s_{n-1}, \ldots, s_2)\) of legal node firings where \(s_i\) is the subsequence \((\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-1}, \gamma_n)\) for \(2 \leq i \leq n\).

**B.** A similar statement holds for \(B_n\) with \(n \geq 2\): The initial strongly dominant position is \((a_1, \ldots, a_n)\), and the position \((a_1 + 2a_2 + \cdots + 2a_{n-1} + a_n, -a_2, -a_3, \ldots, -a_n)\) is obtained by the sequence \((s_n, s_{n-1}, \ldots, s_2)\) of legal node firings where \(s_i\) is the subsequence \((\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+1}, \gamma_i)\) for \(2 \leq i \leq n - 1\) and \(s_n = (\gamma_n)\).

**C.** A similar statement holds for \(C_n\) with \(n \geq 3\): The initial strongly dominant position is \((a_1, \ldots, a_n)\), and the position \((a_1 + 2a_2 + \cdots + 2a_{n-1} + 2a_n, -a_2, -a_3, \ldots, -a_n)\) is obtained by the sequence \((s_n, s_{n-1}, \ldots, s_2)\) of legal node firings where \(s_i\) is the subsequence \((\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+1}, \gamma_i)\) for \(2 \leq i \leq n - 1\) and \(s_n = (\gamma_n)\).

**D.** A similar statement holds for \(D_n\) with \(n \geq 4\): The initial strongly dominant position is \((a_1, \ldots, a_n)\). Let \(b_{n-1} := a_{n-1}\) and \(b_n := a_n\) when \(n\) is odd and where \(b_{n-1} := a_n\) and \(b_n := a_{n-1}\) when \(n\) is even. The position \((a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1} + a_n, -a_2, -a_3, \ldots, -a_{n-2}, -b_{n-1}, -b_n)\) is obtained by the sequence \((s_{n-1}, s_{n-2}, \ldots, s_2)\) of legal node firings where \(s_i\) is the subsequence \((\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-2}, \gamma_{n-1}, \gamma_n, \gamma_{n-2}, \ldots, \gamma_{i+1}, \gamma_i)\) for \(2 \leq i \leq n - 2\) and \(s_n = (\gamma_{n-1}, \gamma_n)\).

**Proof.** In case A, the result clearly holds for the two-node graph. As our induction hypothesis, assume the lemma statement holds for all type A Dynkin diagrams with fewer than \(n\) nodes. Given \(A_n\), the GCM subgraph determined by the \(n - 1\) rightmost nodes is an \(A_{n-1}\) Dynkin diagram. Applying the induction hypothesis, the legal firing sequence \((s_3, \ldots, s_3)\) from the strongly dominant position \((a_1, a_2, \ldots, a_n)\) results in the position \((a_1 + a_2 + \cdots + a_n, -a_n, \ldots, -a_3)\). To this position we now apply the sequence \(s_2 = (\gamma_2, \ldots, \gamma_{n-1}, \gamma_n)\). Once \(\gamma_2\) is fired, then for \(3 \leq i \leq n\) it is easily seen that just before \(\gamma_i\) is fired in the sequence \(s_2\) the position is \((a_1 + a_2 + \cdots + a_n, -a_n, -a_{n-1}, \ldots, -a_{n+5-i}, -a_2 - a_3 - \cdots - a_{n+3-i}, a_2 + \cdots + a_{n+2-i}, -a_{n+2-i}, -a_{n+1-i}, \ldots, -a_4, -a_3)\). Then each firing in the sequence \(s_2\) is legal, and the resulting position is \((a_1 + a_2 + \cdots + a_n, -a_n, -a_{n-1}, \ldots, -a_3, -a_2)\).

In case B, the result clearly holds for the two-node graph. As our induction hypothesis, assume the lemma statement holds for all type B Dynkin diagrams with fewer than \(n\) nodes. Given \(B_n\), the GCM subgraph determined by the \(n - 1\) rightmost nodes is a \(B_{n-1}\) Dynkin diagram. Applying the induction hypothesis, the legal firing sequence \((s_n, \ldots, s_3)\) from the strongly dominant position \((a_1, a_2, \ldots, a_n)\) results in the position \((a_1 + 2a_2 + \cdots + 2a_{n-1} + a_n, -a_2, -a_3, \ldots, -a_n)\). To this position we now apply the sequence \(s_2 = (\gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_2)\). Once \(\gamma_2\) is fired, then for \(3 \leq i \leq n - 1\) it is easily seen that just before \(\gamma_i\) is fired for the first time in the sequence \(s_2\) the position is \((a_1 + a_2 + 2a_3 + \cdots + 2a_{n-1} + a_n, -a_3, -a_4, \ldots, -a_{i-1}, -a_2 - a_3 - \cdots - a_{i-1} - 2a_i - \cdots - 2a_{n-1} - a_n, a_2 + a_3 + \cdots + a_{i-1} + a_1 + 2a_{i+1} + \cdots + 2a_{n-1} + a_n, -a_{i+1}, -a_{i+2}, \ldots, -a_n)\). One now sees that just before \(\gamma_n\) is fired the position is \((a_1 + a_2 + 2a_3 + \cdots + 2a_{n-1} + a_n, -a_3, -a_4, \ldots, -a_{n-1}, -a_2 - a_3 - \cdots - a_{n-1} - a_n, 2a_2 + 2a_3 + \cdots + 2a_{n-1} + a_n)\). Now for \(3 \leq i \leq n - 1\), it is easily seen that just before \(\gamma_i\) is fired for the second time in the sequence \(s_2\) the position is \((a_1 + a_2 + 2a_3 + \cdots + 2a_{n-1} + a_n, -a_3, -a_4, \ldots, -a_i, a_2 + a_3 + \cdots + a_{i-1} + a_1, -a_2 - a_3 - \cdots - a_i - a_{i+1}, -a_{i+2}, -a_{i+3}, \ldots, -a_n)\). Then each firing in the sequence \(s_2\) is legal, and the resulting position is \((a_1 + a_2 + 2a_3 + \cdots + 2a_{n-1} + a_n, -a_2, -a_3, \ldots, -a_n)\).
In case C, it is easy to confirm that the specified sequence of four legal node firings played from a strongly dominant position on the three-node graph yields the stated resulting position. As our induction hypothesis, assume the lemma statement holds for all type C Dynkin diagrams with fewer than n nodes. Given Cn, the GCM subgraph determined by the n − 1 rightmost nodes is a Cn−1 Dynkin diagram. Applying the induction hypothesis, the legal firing sequence (s1, . . . , s3) from the strongly dominant position (a1, a2, . . . , an) results in the position (a1, a2 + 2a3 + ⋯ + 2an, −a3, . . . , −an). To this position we now apply the sequence s2 = (γ2, . . . , γn−1, γn, γn−1, . . . , γ2). Once γ2 is fired, then for 3 ≤ i ≤ n − 1 it is easily seen that just before γi is fired for the first time in the sequence s2 the position is (a1 + a2 + 2a3 + ⋯ + 2an, −a3, −a4, . . . , −ai−1, −a2 − a3 − ⋯ − ai−1 − 2ai − ⋯ − 2an, a2 + a3 + ⋯ + ai−1 + ai + 2ai+1 + ⋯ + 2an, −ai+1, −ai+2, . . . , −an). One now sees that just before γn is fired the position is (a1 + a2 + 2a3 + ⋯ + 2an, −a3, −a4, . . . , −an−1, −a2 − a3 − ⋯ − an−1 − 2an, a2 + a3 + ⋯ + an). Now for 3 ≤ i ≤ n − 1, it is easily seen that just before γi is fired for the second time in the sequence s2 the position is (a1 + a2 + 2a3 + ⋯ + 2an, −a3, −a4, . . . , −ai, a2 + a3 + ⋯ + ai−1 + ai, −a2 − a3 − ⋯ − ai − ai+1, −ai+2, −ai+3, . . . , −an). Finally, fire γ2 from the position (a1 + a2 + 2a3 + ⋯ + 2an, −a2, −a3, −a4, . . . , −an). Then each firing in the sequence s2 is legal, and the resulting position is (a1 + 2a2 + ⋯ + 2an, −a2, −a3, . . . , −an).

In case D, it is easy to confirm that the specified sequence of six legal node firings played from a strongly dominant position on the four-node graph yields the stated resulting position. As our induction hypothesis, assume the lemma statement holds for all type D Dynkin diagrams with fewer than n nodes. Given Dn, the GCM subgraph determined by the n − 1 rightmost nodes is a Dn−1 Dynkin diagram. Assume for the moment that n is even, so n − 1 is odd. Applying the induction hypothesis, the legal firing sequence (s1−1, . . . , s3) from the strongly dominant position (a1, a2, . . . , an) results in the position (a1, a2 + 2a3 + ⋯ + 2an−2 + an−1 + an, −a3, . . . , −an−2, −an−1, −an). To this position we now apply the sequence s2 = (γ2, . . . , γn−2, γn−1, γn, γn−2, . . . , γ2). Once γ2 is fired, then for 3 ≤ i ≤ n − 3 it is easily seen that just before γi is fired for the first time in the sequence s2 the position is (a1 + a2 + 2a3 + ⋯ + 2an−2 + an−1 + an, −a3, −a4, . . . , −ai−1, −a2 − a3 − ⋯ − ai−1 − 2ai − ⋯ − 2an−2 + an−1 − an, a2 + a3 + ⋯ + ai−1 + ai + 2ai+1 + ⋯ + 2an−2 + an−1 + an, −ai+1, −ai+2, . . . , −an−2, −an−1, −an). One now sees that just before γn−2 is fired for the first time in the sequence the position is (a1 + a2 + 2a3 + ⋯ + 2an−2 + an−1 + an, −a3, −a4, . . . , −an−3, −a2 − a3 − ⋯ − an−3 − 2an−2 − an−1 − an, a2 + a3 + ⋯ + an−2 + an−1 + an, −an−1, −an). Then just before γn−1 is fired the position is (a1 + a2 + 2a3 + ⋯ + 2an−2 + an−1 + an, −a3, −a4, . . . , −an−3, −a2 − a3 − ⋯ − an−2 − an−1 − an, a2 + a3 + ⋯ + an−2 + an−1 + an, −an−1, −an). Just before γn−1 is fired the position is (a1 + a2 + 2a3 + ⋯ + 2an−2 + an−1 + an, −a3, −a4, . . . , −an−3, −a2 − a3 − ⋯ − an−2 − an−1 − an, a2 + a3 + ⋯ + an−2 + an−1 + an, −an−1, −an). So just before γ2 is fired for the second time in the sequence s2 the position is (a1 + a2 + 2a3 + ⋯ + 2an−2 + an−1 + an, −a3, −a4, . . . , −an−3, −a2 − a3 − ⋯ − an−2 − an−1 − an, a2 + a3 + ⋯ + an−2 + an−1 + an). The last time γ2 is fired, then for 3 ≤ i ≤ n it is easily seen that just before γi is fired for the second time in the sequence s2 the position is (a1 + a2 + 2a3 + ⋯ + 2an−2 + an−1 + an, −a3, −a4, . . . , −an−3, −a2 − a3 − ⋯ − an−2 − an−1 − an, a2 + a3 + ⋯ + an−2 + an−1 + an, −an−1, −an). When n is odd, the argument is entirely similar.
From this we immediately obtain the following:

Lemma 2.2
A. For any positive integer $n$ and for any strongly dominant position $(a_1,\ldots,a_n)$ on $A_n$, one can obtain the position $(-a_n,\ldots,-a_2,-a_1)$ by the sequence $(s_n, s_{n-1}, \ldots, s_1)$ of legal node firings where $s_i$ is the subsequence $(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n, \gamma_n)$ for $1 \leq i \leq n$.

B. A similar statement holds for $B_n$ with $n \geq 2$: The initial strongly dominant position is $(a_1,\ldots,a_n)$, and the position $(-a_1,-a_2,\ldots,-a_n)$ is obtained by the sequence $(s_n, s_{n-1}, \ldots, s_1)$ of legal node firings where $s_i$ is the subsequence $(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n, \gamma_n)$ for $1 \leq i \leq n-1$ and $s_n = (\gamma_n)$.

C. A similar statement holds for $C_n$ with $n \geq 3$: The initial strongly dominant position is $(a_1,\ldots,a_n)$, and the position $(-a_1,-a_2,\ldots,-a_n)$ is obtained by the sequence $(s_n, s_{n-1}, \ldots, s_1)$ of legal node firings where $s_i$ is the subsequence $(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n, \gamma_n)$ for $1 \leq i \leq n-1$ and $s_n = (\gamma_n)$.

D. A similar statement holds for $D_n$ with $n \geq 4$: The initial strongly dominant position is $(a_1,\ldots,a_n)$. Let $b_{n-1} := a_{n-1}$ and $b_n := a_n$ when $n$ is even and where $b_{n-1} := a_n$ and $b_n := a_{n-1}$ when $n$ is odd. The position $(a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1} + a_n, -a_2, -a_3, \ldots, -a_{n-2}, -b_{n-1}, -b_n)$ is obtained by the sequence $(s_n, s_{n-2}, \ldots, s_1)$ of legal node firings where $s_i$ is the subsequence $(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n, \gamma_n)$ for $1 \leq i \leq n-2$ and $s_{n-1} = (\gamma_n)$.

Proof. View $A_n$ (respectively $B_n$, $C_n$, $D_n$) as a GCM subgraph of $A_{n+1}$ (respectively $B_{n+1}$, $C_{n+1}$, $D_{n+1}$) by adding a node “to the left” of $\gamma_1$. Conclude by applying Lemma 2.1.

Proposition 2.3 A connected Dynkin diagram $(\Gamma, M)$ of finite type is admissible. Moreover, for any initial position on $(\Gamma, M)$, all game sequences converge to the same terminal position in the same finite number of steps.

Proof. The Strong Convergence Theorem shows that if a game sequence for some initial position $\lambda$ on $(\Gamma, M)$ converges, then all game sequences from $\lambda$ converge to the same terminal position in the same finite number of steps. Then in light of The Comparison Theorem, it suffices to show that for any strongly dominant initial position on $(\Gamma, M)$, there is a convergent game sequence.

For the four infinite families ($A_n$, $B_n$, $C_n$, and $D_n$), this is handled by Lemma 2.2. For the exceptional graphs ($E_6$, $E_7$, $E_8$, $F_4$, and $G_2$) this can be checked by hand (requiring 36, 63, 120, 24, and 6 firings respectively). For $G_2$, begin with strongly dominant position $\lambda = (a, b)$ with positive number $a$ on node $\gamma_1$ and positive number $b$ on $\gamma_2$. The following sequence of six node firings is easily seen to be legal: $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$. The resulting position is $(-a, -b)$. For $F_4$, begin with strongly dominant position $(a,b,c,d)$. Play the numbers game from this initial position to see that the following sequence of 24 node firings is legal: $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_10, \gamma_11, \gamma_12, \gamma_13, \gamma_14, \gamma_15, \gamma_16, \gamma_17, \gamma_18, \gamma_19, \gamma_20, \gamma_21, \gamma_22, \gamma_23, \gamma_24)$. For $E_6$, begin with strongly dominant position $(a,b,c,d,e,f)$. Play the numbers game from this initial position to see that the following sequence of 36 node firings is legal: $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_10, \gamma_11, \gamma_12, \gamma_13, \gamma_14, \gamma_15, \gamma_16, \gamma_17, \gamma_18, \gamma_19, \gamma_20, \gamma_21, \gamma_22, \gamma_23, \gamma_24, \gamma_25, \gamma_26, \gamma_27, \gamma_28, \gamma_29, \gamma_30, \gamma_31, \gamma_32, \gamma_33, \gamma_34, \gamma_35, \gamma_36)$. For $E_7$, begin with strongly dominant position $(a,b,c,d,e,f,g)$. Play the numbers game from this initial position to see that the following sequence of 63 node firings is legal (the first 36 of these node firings are exactly the previous game sequence played on the $E_6$ subgraph): $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_10, \gamma_11, \gamma_12, \gamma_13, \gamma_14, \gamma_15, \gamma_16, \gamma_17, \gamma_18, \gamma_19, \gamma_20, \gamma_21, \gamma_22, \gamma_23, \gamma_24, \gamma_25, \gamma_26, \gamma_27, \gamma_28, \gamma_29, \gamma_30, \gamma_31, \gamma_32, \gamma_33, \gamma_34, \gamma_35, \gamma_36, \gamma_37, \gamma_38, \gamma_39, \gamma_40, \gamma_41, \gamma_42, \gamma_43, \gamma_44, \gamma_45, \gamma_46, \gamma_47, \gamma_48, \gamma_49, \gamma_50, \gamma_51, \gamma_52, \gamma_53, \gamma_54, \gamma_55, \gamma_56, \gamma_57, \gamma_58, \gamma_59, \gamma_60, \gamma_61, \gamma_62, \gamma_63)$. For $E_8$, begin with strongly dominant position $(a,b,c,d,e,f,g,h,i)$.
The infinite game sequences for these cases, but rather to show that such game sequences can be found and handled using a common line of reasoning: A sequence of legal node firings is applied to a position \( \gamma \). It is then observed that the numbers for the resulting position are linear expressions of the same form with respect to the variable \( k + 1 \) and that the firing sequence can be repeated. The \( \tilde{G} \) cases and the families of small cycles are handled using a variation of this kind of argument: A sequence of legal node firings is applied to a generic position satisfying certain inequalities, and it is shown that the resulting position also satisfies these inequalities so that the firing sequence can be repeated. Each paragraph in what follows demonstrates inadmissibility for some graph in our list. Our case-analysis argument is lengthy in part because we have tried to make each paragraph reasonably self-contained.

The \( \tilde{A} \) family

The infinite \( \tilde{A} \) family of GCM graphs of Figure 3.1 is the family of cycles with amplitude products of unity on all edges. Such cycles were in fact the graphs that motivated Mozes’ study of the numbers game in [Moz]. The argument we give here demonstrating inadmissibility for each graph in this family is a special case of the proof of Lemma 3.1 of [Don2]. For an \( n \)-node graph in the \( \tilde{A} \) family (we take \( n \geq 3 \)), number the top node \( \gamma_1 \) and the remaining nodes \( \gamma_2, \ldots, \gamma_n \) in succession in the clockwise order around the cycle. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. By symmetry, it suffices to do so for the fundamental position \( \omega_1 = (1, 0, \ldots, 0) \). This is the \( k = 0 \)
The $\tilde{A}$ family

The $\tilde{B}$ family

The $\tilde{C}$ family

The $\tilde{D}$ family

The $\tilde{E}$ family

version of the position $(2k + 1, -k, 0, \ldots, 0, -k)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)$. This sequence results in the position $(2(k + 1) + 1, -(k + 1), 0, \ldots, 0, -(k + 1))$. This gives the desired divergent game sequence. We conclude that any such GCM graph is inadmissible.

First, we show why $\tilde{B}$ is not admissible. Label the leftmost nodes as $\gamma_1$ and $\gamma_2$, the middle node as $\gamma_3$, and the rightmost node as $\gamma_4$. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. From the fundamental position $\omega_1 = (1, 0, 0, 0)$, play the (legal) sequence $(\gamma_1, \gamma_3, \gamma_2, \gamma_4, \gamma_3)$ to obtain the position $(2, 1, -2, 2)$. This is the $k = 0$ version of the position $(k + 2, k + 1, -2(k + 1), 2(k + 1))$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1, \gamma_2, \gamma_4, \gamma_3, \gamma_1, \gamma_2, \gamma_4, \gamma_3)$. This sequence results in the position $((k + 1) + 2, (k + 1) + 1, -2[(k + 1) + 1], 2[(k + 1) + 1])$. By symmetry, we also obtain a divergent game sequence from the fundamental position $\omega_2$. The fundamental position $\omega_2 = (0, 0, 1, 0)$ is the $k = 0$ version of the position $(-2k, -2k, 2k + 1, 0)$. From any such position with $k \geq 0$, the following
sequence of node firings is easily seen to be legal: \((\gamma_3, \gamma_4, \gamma_3, \gamma_2, \gamma_1)\). This sequence results in the position \((-2(k+1), -2(k+1), 2(k+1) + 1, 0)\). The fundamental position \(\omega_4 = (0, 0, 0, 1)\) is the \(k = 0\) version of the position \((0, 0, -k, 2k + 1)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: \((\gamma_4, \gamma_3, \gamma_2, \gamma_1, \gamma_3)\). This sequence results in the position \((0, 0, -(k+1), 2(k+1) + 1)\).

Next, we show why \(\gamma_3, \gamma_4, \gamma_3, \gamma_2, \gamma_1\) is not admissible when the graph has \(n \geq 5\) nodes. Label the leftmost nodes as \(\gamma_1\) and \(\gamma_2\), and label the remaining nodes in succession from left to right as \(\gamma_3, \ldots, \gamma_{n-1}, \gamma_n\). For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \(\omega_1 = (1, 0, \ldots, 0)\) is the \(k = 0\) version of the position \((2k+1, -2k, 0, \ldots, 0)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_3, \gamma_4, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \gamma_4, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)\). This sequence results in the position \((2(k+1) + 1, -2(k+1), 0, \ldots, 0)\). By symmetry, we also obtain a divergent game sequence from the fundamental position \(\omega_2\). The fundamental position \(\omega_3 = (0, 0, 1, 0, \ldots, 0)\) is the \(k = 0\) version of the position \((-2k, -2k, 2k + 1, 0, \ldots, 0)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: \((\gamma_3, \gamma_4, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2, \gamma_1)\). This sequence results in the position \((-2(k+1), -2(k+1), 2(k+1) + 1, 0, \ldots, 0)\). For \(4 \leq i \leq n-1\), any fundamental position \(\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)\) is the \(k = 0\) version of the position \((0, \ldots, 0, -2k, 2k + 1, 0, \ldots, 0)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: \((\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+1}, \gamma_i, \gamma_{i-1}, \ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \ldots, \gamma_{i-1})\). This sequence results in the position \((0, \ldots, 0, -2(k+1), 2(k+1) + 1, 0, \ldots, 0)\). The fundamental position \(\omega_n = (0, \ldots, 0, 1)\) is the \(k = 0\) version of the position \((0, \ldots, 0, -k, 2k + 1)\). From any
such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \ldots, \gamma_{n-1})\). This sequence results in the position \((0, \ldots, 0, -(k+1), 2(k+1)+1)\).

Next, we show why \( \cdots \) is not admissible. Label the nodes as \( \gamma_1, \gamma_2 \), and \( \gamma_3 \) from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( \omega_1 = (1, 0, 0) \) is the \( k = 0 \) version of the position \((2k + 1, -k, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_2, \gamma_3, \gamma_2)\). This sequence results in the position \((2(k+1)+1, -(k+1), 0)\). By symmetry, we also obtain a divergent game sequence from the fundamental position \( \omega_3 \). The fundamental position \( \omega_2 = (0, 1, 0) \) is the \( k = 0 \) version of the position \((-4k, 2k + 1, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_2, \gamma_3, \gamma_2, \gamma_1)\). This sequence results in the position \((-4(k+1), 2(k+1)+1, 0)\).

We finish the \( \tilde{B} \) family by showing why \( \cdots \) is not admissible when the graph has \( n \geq 4 \) nodes. Label the nodes as \( \gamma_1, \gamma_2, \ldots, \gamma_{n-1} \), and \( \gamma_n \) from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( \omega_1 = (1, 0, 0, \ldots, 0) \) is the \( k = 0 \) version of the position \((2k + 1, -k, 0, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)\). This sequence results in the position \((2(k+1)+1, -2(k+1), 0, \ldots, 0)\). By symmetry, we also obtain a divergent game sequence from the fundamental position \( \omega_n \). For \( 2 \leq i \leq n - 1 \), any fundamental position \( \omega_i = (0, \ldots, 0, 1, 0, 0, \ldots, 0) \) is the \( k = 0 \) version of the position \((0, \ldots, 0, 2k+1, -2k, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_i, \gamma_{i-1}, \ldots, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_i+2, \gamma_{i+1})\). This sequence results in the position \((0, \ldots, 0, 2(k+1)+1, -2(k+1), 0, \ldots, 0)\).

The \( \tilde{C} \) family First, we show why \( \cdots \) is not admissible when the graph has \( n \geq 3 \) nodes. (Since firing the middle node in the \( n = 3 \) case is comparable to firing either \( \gamma_2 \) or \( \gamma_{n-1} \) in the \( n \geq 4 \) cases, then the \( n = 3 \) case does not need to be considered separately here.) Label the nodes as \( \gamma_1, \gamma_2, \ldots, \gamma_{n-1} \), and \( \gamma_n \) from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( \omega_1 = (1, 0, 0, \ldots, 0) \) is the \( k = 0 \) version of the position \((2k + 1, -2k, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)\). This sequence results in the position \((2(k+1)+1, -2(k+1), 0, \ldots, 0)\). By symmetry, we also obtain a divergent game sequence from the fundamental position \( \omega_n \). For \( 2 \leq i \leq n - 1 \), any fundamental position \( \omega_i = (0, \ldots, 0, 1, 0, 0, \ldots, 0) \) is the \( k = 0 \) version of the position \((0, \ldots, 0, 2k+1, -2k, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_i, \gamma_{i-1}, \ldots, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+2}, \gamma_{i+1})\). This sequence results in the position \((0, \ldots, 0, 2(k+1)+1, -2(k+1), 0, \ldots, 0)\).

Next, we show why \( \cdots \) is not admissible when the graph has \( n \geq 3 \) nodes. (Since firing the middle node in the \( n = 3 \) case is comparable to firing \( \gamma_2 \) in the \( n \geq 4 \) cases, then the \( n = 3 \) case does not need to be considered separately here.) Label the nodes as \( \gamma_1, \gamma_2, \ldots, \gamma_{n-1} \), and \( \gamma_n \) from left to right. For each fundamental position, we
exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( \omega_1 = (1,0,\ldots,0) \) is the \( k = 0 \) version of the position \((2k + 1, -k, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_2, \ldots, \gamma_{n-2}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)\). This sequence results in the position \((2(k + 1) + 1, -(k + 1), 0, \ldots, 0)\). For \( 2 \leq i \leq n - 1 \), any fundamental position \( \omega_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) is the \( k = 0 \) version of the position \((0, \ldots, 0, 2k + 1, -2k, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_i, \gamma_{i-1}, \ldots, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_i + 2, \gamma_i + 1)\). This sequence results in the position \((0, \ldots, 0, 2(k + 1) + 1, -2(k + 1), 0, \ldots, 0)\). The fundamental position \( \omega_n = (0, \ldots, 0, 1) \) is the \( k = 0 \) version of the position \((0, \ldots, 0, -2k; 2k + 1)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_n, \gamma_{n-1}, \ldots, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_n - 2, \gamma_n - 1)\). This sequence results in the position \((0, \ldots, 0, -2(k + 1), 2(k + 1) + 1)\).

We finish the \( \tilde{C} \) family by showing why \[
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is not admissible when the graph has \( n \geq 4 \) nodes. (Since firing the middle node in the \( n = 4 \) case is comparable to firing \( \gamma_3 \) in the \( n \geq 5 \) cases, then the \( n = 4 \) case does not need to be considered separately here.) Label the leftmost nodes as \( \gamma_1 \) and \( \gamma_2 \), and label the remaining nodes in succession from left to right as \( \gamma_3, \ldots, \gamma_{n-1}, \gamma_n \). For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( \omega_1 = (1,0,\ldots,0) \) is the \( k = 0 \) version of the position \((2k + 1, -2k, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_3, \gamma_4, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \gamma_4, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)\). This sequence results in the position \((2(k + 1) + 1, -(k + 1), 0, \ldots, 0)\). By symmetry, we also obtain a divergent game sequence from the fundamental position \( \omega_2 \). The fundamental position \( \omega_3 = (0,0,1,0,\ldots,0) \) is the \( k = 0 \) version of the position \((-2k, -2k, 2k + 1, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_3, \gamma_4, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2, \gamma_1)\). This sequence results in the position \((-2(k + 1), -2(k + 1), 2(k + 1) + 1, 0, \ldots, 0)\). For \( 4 \leq i \leq n \), any fundamental position \( \omega_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) is the \( k = 0 \) version of the position \((0, \ldots, 0, -2k, 2k + 1, 0, \ldots, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+1}, \gamma_i, \gamma_{i-1}, \ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \ldots, \gamma_{i-1})\). This sequence results in the position \((0, \ldots, 0, -2(k + 1), 2(k + 1) + 1, 0, \ldots, 0)\).

The \( \tilde{D} \) family First, we show why the five-node graph

is not admissible. Label the leftmost nodes as \( \gamma_1 \) and \( \gamma_2 \), label the middle node as \( \gamma_3 \), and label the rightmost nodes as \( \gamma_4 \) and \( \gamma_5 \). For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. From the fundamental position \( \omega_1 = (1,0,0,0,0,0) \), play the (legal) sequence \((\gamma_1, \gamma_3, \gamma_2, \gamma_4, \gamma_5, \gamma_3)\) to obtain the position \((2,1, -2, 1, 1)\). This is the \( k = 0 \) version of the position \((k + 2, k + 1, -2(k + 1), k + 1, k + 1)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_2, \gamma_4, \gamma_5, \gamma_3, \gamma_1, \gamma_2, \gamma_4, \gamma_5, \gamma_3)\). This sequence results in the position \(((k + 1) + 2, (k + 1) + 1, -2[(k + 1) + 1], (k + 1) + 1, (k + 1) + 1)\). By symmetry, we also obtain divergent game sequences from the fundamental positions \( \omega_2, \omega_4, \) and \( \omega_5 \). The fundamental position \( \omega_3 = (0,0,1,0,0) \) is the \( k = 0 \) version of the position \((-k, -k, 2k + 1, -k, -k)\). From any such position with \( k \geq 0 \), the following sequence of
node firings is easily seen to be legal: \((\gamma_3, \gamma_1, \gamma_2, \gamma_4, \gamma_5)\). This sequence results in the position 
\((-k+1), -(k+1), 2(k+1) + 1, -(k+1), -(k+1)\).

We finish the \(\tilde{D}\) family by showing why \(\cdots \gamma_1 \gamma_4 \gamma_2 \cdots \) is not admissible when the graph has \(n \geq 6\) nodes. Label the leftmost nodes as \(\gamma_1\) and \(\gamma_2\), label the “isthmus” nodes in succession from left to right as \(\gamma_3, \ldots, \gamma_{n-2}\), and label the rightmost nodes as \(\gamma_{n-1}\) and \(\gamma_n\). For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \(\omega_1 = (1, 0, \ldots, 0)\) is the \(k = 0\) version of the position \((2k + 1, -2k, 0, \ldots, 0)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: 
\((\gamma_1, \gamma_3, \gamma_4, \ldots, \gamma_{n-2}, \gamma_{n-1}, \gamma_n, \gamma_n - 2, \ldots, \gamma_4, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \ldots, \gamma_{n-2}, \gamma_{n-1}, \gamma_n, \gamma_{n-2}, \ldots, \gamma_4, \gamma_3, \gamma_2)\). This sequence results in the position \((2k + 1, 1, -2(k + 1), 0, \ldots, 0)\). By symmetry, we also obtain divergent game sequences from the fundamental positions \(\omega_2, \omega_{n-1}\), and \(\omega_n\). The fundamental position \(\omega_3 = (0, 0, 1, 0, \ldots, 0)\) is the \(k = 0\) version of the position \((-2k, -2k, 2k + 1, 0, \ldots, 0)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: 
\((\gamma_3, \gamma_4, \ldots, \gamma_{n-2}, \gamma_{n-1}, \gamma_n, \gamma_n - 2, \ldots, \gamma_4, \gamma_3, \gamma_2, \gamma_1)\). This sequence results in the position \((-2(k + 1), -2(k + 1), 2(k + 1) + 1, 0, \ldots, 0)\). By symmetry, we also obtain a divergent game sequence from the fundamental position \(\omega_{n-2}\). For \(4 \leq i \leq n - 2\), any fundamental position \(\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)\) is the \(k = 0\) version of the position \((0, \ldots, 0, -k, 2k + 1, -k, 0, \ldots, 0)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: 
\((\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-2}, \gamma_{n-1}, \gamma_n, \gamma_n - 2, \ldots, \gamma_{i+1}, \gamma_{i-1}, \ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \ldots, \gamma_{i-1})\). This sequence results in the position \((0, \ldots, 0, -(k + 1), 2(k + 1) + 1, -(k + 1), 0, \ldots, 0)\).

The \(\tilde{E}\) family

First, we show why

\[ \gamma_1 \gamma_4 \gamma_3 \gamma_6 \gamma_7 \]

is not admissible. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \(\omega_1 = (1, 0, 0, 0, 0, 0, 0)\) is the \(k = 0\) version of the position \((2k + 1, 0, 0, -k, 0, 0, 0)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: 
\((\gamma_1, \gamma_4, \gamma_5, \gamma_3, \gamma_2, \gamma_6, \gamma_7, \gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_6, \gamma_7, \gamma_6, \gamma_5, \gamma_3, \gamma_2, \gamma_4, \gamma_5, \gamma_3, \gamma_3, \gamma_6, \gamma_5, \gamma_4)\). This results in the position \((2k + 1, 1, 0, 0, -k + 1, 0, 0, 0)\). By symmetry, we also obtain divergent game sequences from the fundamental positions \(\omega_2\) and \(\omega_7\). The fundamental position \(\omega_4 = (0, 0, 0, 1, 0, 0, 0)\) is the \(k = 0\) version of the position \((-4k, 0, 0, 2k + 1, 0, 0, 0)\). From any such position with \(k \geq 0\), the following sequence of node firings is easily seen to be legal: 
\((\gamma_4, \gamma_5, \gamma_3, \gamma_2, \gamma_6, \gamma_5, \gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_6, \gamma_7, \gamma_6, \gamma_5, \gamma_3, \gamma_2, \gamma_4, \gamma_5, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_1)\). This results in the position \((-4k + 1, 0, 0, 2(k + 1) + 1, 0, 0, 0)\). By symmetry, we also obtain divergent game sequences from the fundamental positions \(\omega_3\) and \(\omega_6\). The fundamental position \(\omega_5 = (0, 0, 0, 0, 1, 0, 0, 0)\) is the \(k = 0\) version of the position \((-6k, 0, 0, -6k, 6k + 1, 0, 0, 0)\). From any such position with \(k \geq 0\), play the game to see that the following sequence of node firings is legal: 
\((\gamma_5, \gamma_3, \gamma_2, \gamma_4, \gamma_5, \gamma_3, \gamma_6, \gamma_5, \gamma_3, \gamma_2, \gamma_4, \gamma_5, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_1)\). This results in the position \((-6(k + 1), 0, 0, -6(k + 1), 6(k + 1) + 1, 0, 0, 0)\).
Next, we show why

\[ γ \]

is not admissible. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( ω_1 = (1, 0, 0, 0, 0, 0, 0) \) is the \( k = 0 \) version of the position \((2k + 1, 0, -k, 0, 0, 0, 0)\). From any such position with \( k \geq 0 \), play the game to see that the following sequence of node firings is legal:

\[ (γ_1, γ_3, γ_4, γ_5, γ_6, γ_7, γ_8). \]

This results in the position \((2(k + 1) + 1, 0, -(k + 1), 0, 0, 0, 0)\). By symmetry, we also obtain a divergent game sequence from the fundamental position \( ω_8 \). From the fundamental position \( ω_2 = (0, 1, 0, 0, 0, 0, 0) \), play the (legal) sequence \((γ_2, γ_3, γ_4, γ_5, γ_6, γ_7, γ_8) \)

is not admissible. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( ω_1 = (1, 0, 0, 0, 0, 0, 0) \) is the \( k = 0 \) version of the position \((2k + 1, -k, 0, 0, 0, 0, 0)\). From any
such position with \( k \geq 0 \), play the game to see that the following sequence of node firings is legal:

\[
(\gamma_1, \gamma_3, \gamma_4, \gamma_2, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_7, \gamma_6, \gamma_5, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6).
\]

This results in the position \( (2(k+1)+1, -(k+1), 0, 0, 0, 0, 0, 0, 0, 0) \). The fundamental position \( \omega_2 = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0) \) is the \( k = 0 \) version of the position \( (0, 3k + 1, 0, -k, 0, 0, -k, 0, 0, 0) \). From any such position with \( k \geq 0 \), play the game to see that the following sequence \( s \) of node firings is legal when played twice in a row: \( s := (\gamma_2, \gamma_4, \gamma_3, \gamma_1, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_7) \). Playing \( s \) twice in a row results in the position \( (0, 3(k+1)+1, 0, -(k+1), 0, 0, -(k+1), 0, 0) \). From the fundamental position \( \omega_3 = (0, 0, 1, 0, 0, 0, 0, 0, 0) \), play the (legal) sequence \( (\gamma_3, \gamma_1, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_6) \) to obtain the position \( (0, 2, 0, 0, 0, 0, 0, -1, 0, 0) \). This is the \( k = 0 \) version of the position \( (0, 6k + 2, 0, -3k, 0, 0, -3k, -1, 0) \). From any such position with \( k \geq 0 \), play the game to see that the following sequence \( s \) of node firings is legal when played three times in a row: \( s := (\gamma_2, \gamma_4, \gamma_3, \gamma_1, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_7, \gamma_2) \). Playing \( s \) three times in a row results in the position \( (0, 6(k+1)+2, 0, -3(k+1), 0, 0, -3(k+1), -1, 0, 0) \). The fundamental position \( \omega_4 \) is the \( k = 0 \) version of the position \( (0, -3k, 0, 2k + 1, 0, 0, -k, 0, 0) \). From any such position with \( k \geq 0 \), play the game to see that the following sequence \( s \) of node firings is legal when played six times in a row: \( s := (\gamma_2, \gamma_4, \gamma_3, \gamma_1, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_7) \). Playing \( s \) six times in a row results in the position \( (0, 15(k+1)+3, 0, -5(k+1), 0, -1, -5(k+1), 0, 0) \). From the fundamental position \( \omega_5 = (0, 0, 0, 0, 1, 0, 0, 0, 0, 0) \), play the (legal) sequence \( (\gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_9, \gamma_8, \gamma_7, \gamma_2) \) to obtain the position \( (0, 3, 0, 0, 0, -1, 0, 0, 0) \). This is the \( k = 0 \) version of the position \( (0, 12k + 3, 0, -4k, -1, 0, -4k, 0, 0) \). From any such position with \( k \geq 0 \), play the game to see that the following sequence \( s \) of node firings is legal when played six times in a row: \( s := (\gamma_2, \gamma_4, \gamma_3, \gamma_1, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_7, \gamma_2) \). Playing \( s \) six times in a row results in the position \( (0, 12(k+1)+3, 0, -4(k+1), -1, 0, -4(k+1), 0, 0) \). From the fundamental position \( \omega_7 = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0) \), play the (legal) sequence \( (\gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_7, \gamma_6, \gamma_5, \gamma_9, \gamma_8, \gamma_7, \gamma_2) \) to obtain the position \( (0, 3, 0, -1, 0, 0, 0, 0, 0) \). This is the \( k = 0 \) version of the position \( (0, 3k + 3, 0, -k - 1, 0, 0, -k, 0, 0) \). From any such position with \( k \geq 0 \), play the game to see that the following sequence \( s \) of node firings is legal when played twice in a row: \( s := (\gamma_2, \gamma_4, \gamma_3, \gamma_1, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_7) \). Playing \( s \) twice in a row results in the position \( (0, 3(k+1)+3, 0, -k(k+1), 1, 0, -k(k+1), 0, 0) \). From the fundamental position \( \omega_8 = (0, 0, 0, 0, 0, 0, 1, 0, 0, 0) \), play the (legal) sequence \( (\gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1, \gamma_9, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_2, \gamma_4, \gamma_3, \gamma_1, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_2) \) to obtain the position \( (0, 4, 0, -1, 0, -1, 0, 0, 0, 0) \). This is the \( k = 0 \) version of the position \( (0, 6k + 4, 0, -2k - 1, 0, -1, -2k, 0, 0) \). From any such position with \( k \geq 0 \), play the game to see that the following sequence \( s \) of node firings is legal when played six times in a row: \( s := (\gamma_2, \gamma_4, \gamma_3, \gamma_1, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_8, \gamma_7) \). Playing \( s \) six times in a row results in the position \( (0, 6(k+1)+4, 0, -2(k+1) - 1, 0, -1, -2(k+1), 0, 0) \). The fundamental position \( \omega_9 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1) \) is the \( k = 0 \) version of the position \( (0, 0, 0, 0, 0, 0, -k, 2k + 1) \).
From any such position with \( k \geq 0 \), play the game to see that the following sequence of node firings is legal: \((\gamma_9, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_3, \gamma_1, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_2, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8)\). This results in the position \((0, 0, 0, 0, 0, 0, 0, -(k + 1), 2(k + 1) + 1)\).

The \( \tilde{F} \) family

First, we show why \( \cdots \) is not admissible. Label the nodes as \( \gamma_1, \gamma_2, \gamma_3, \gamma_4, \) and \( \gamma_5 \) from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( \omega_1 = (1, 0, 0, 0, 0) \) is the \( k = 0 \) version of the position \((2k + 1, -k, 0, 0, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_2, \gamma_3, \gamma_2, \gamma_4, \gamma_3, \gamma_2, \gamma_5, \gamma_3, \gamma_2)\). This results in the position \((2(k + 1) + 1, -(k + 1), 0, 0, 0)\). The fundamental position \( \omega_2 = (0, 1, 0, 0, 0) \) is the \( k = 0 \) version of the position \((-2k, 4k + 1, -2k, -2k, -2k)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_1)\). This results in the position \((-2(k + 1), 4(k + 1) + 1, -2(k + 1), -2(k + 1))\). The fundamental position \( \omega_3 = (0, 0, 1, 0, 0) \) is the \( k = 0 \) version of the position \((-k, -k, 3k + 1, -k, -k)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_3, \gamma_4, \gamma_5, \gamma_2, \gamma_1, \gamma_3, \gamma_4, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_2, \gamma_1)\). This results in the position \((-k + 1), -(k + 1), 3(k + 1) + 1, -(k + 1), -(k + 1))\). The fundamental position \( \omega_4 = (0, 0, 0, 1, 0) \) is the \( k = 0 \) version of the position \((-k, -k, 2k + 1, -k)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_4, \gamma_5, \gamma_3, \gamma_2, \gamma_1, \gamma_4, \gamma_5, \gamma_3, \gamma_2, \gamma_1)\). This results in the position \((-k + 1), -(k + 1), k + 1, 2(k + 1) + 1, -(k + 1))\). The fundamental position \( \omega_5 = (0, 0, 0, 0, 1) \) is the \( k = 0 \) version of the position \((0, 0, 0, -k, 2k + 1)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_5, \gamma_4, \gamma_3, \gamma_2, \gamma_3, \gamma_4, \gamma_2, \gamma_3, \gamma_4, \gamma_2, \gamma_3, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_2)\). This results in the position \((0, 0, 0, -(k + 1), 2(k + 1) + 1)\).

To finish our analysis of the \( \tilde{F} \) family, we show why \( \cdots \) is not admissible. Label the nodes as \( \gamma_1, \gamma_2, \gamma_3, \gamma_4, \) and \( \gamma_5 \) from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position \( \omega_1 = (1, 0, 0, 0, 0) \) is the \( k = 0 \) version of the position \((2k + 1, -k, 0, 0, 0)\). From any such position with \( k \geq 0 \), the following sequence of node firings is easily seen to be legal: \((\gamma_1, \gamma_2, \gamma_3, \gamma_2, \gamma_4, \gamma_3, \gamma_2, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8)\). This results in the position \((0, 0, 0, 0, 0, 0, 0, -(k + 1), 2(k + 1) + 1)\).
sequence of node firings is easily seen to be legal: $(\gamma_5, \gamma_4, \gamma_3, \gamma_2, \gamma_3, \gamma_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$. This results in the position $(0, 0, 0, -(k + 1), 2(k + 1) + 1)$.

The $G$ family First, we show why is not admissible. Label the nodes as $\gamma_1, \gamma_2,$ and $\gamma_3$ from left to right. A position $(a, b, c)$ meets condition $(\ast)$ if $a \leq 0$, $b \leq 0$, and $a + 2b + c > 0$. The following inequalities are immediate: (1) $c > 0$, (2) $b + c > 0$, (3) $a + 3b + 3c > 0$, (4) $a + 2b + 2c > 0$, (5) $2a + 3b + 3c > 0$, (6) $a + b + c > 0$, and (7) $2a + 4b + 3c > 0$. From (1) through (6) it now follows that all node firings of the sequence $s := (\gamma_3, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ are legal: The left-hand side of each inequality is the number at the respective node of the sequence when that node is fired. The resulting position $(a_1, b_1, c_1)$ has $a_1 = a$, $b_1 = -(a + b + c)$, and $c_1 = 2a + 4b + 3c$. Clearly $a_1 \leq 0$. By inequality (6), it follows that $b_1 < 0$. From inequality (7) we get $c_1 > 0$. Finally, $a_1 + 2b_1 + c_1 = a + 2b + c > 0$, so $(a_1, b_1, c_1)$ meets condition $(\ast)$. So from any position which meets condition $(\ast)$, the firing sequence $s$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(\ast)$ using a sequence of legal node firings. The fundamental position $\omega_3 = (0, 0, 1)$ meets condition $(\ast)$. Now take fundamental position $\omega_2 = (0, 1, 0)$ and apply the legal firing sequence $(\gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ to get the resulting position $(0, -1, 4)$. The latter meets condition $(\ast)$. For the fundamental position $\omega_1 = (1, 0, 0)$, apply the legal firing sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1)$ to get the resulting position $(-1, 0, 2)$. The latter meets condition $(\ast)$.

Next, we show why is not admissible. Our argument is similar to the previous case. Label the nodes as $\gamma_1, \gamma_2,$ and $\gamma_3$ from left to right. A position $(a, b, c)$ meets condition $(\ast)$ if $b \leq 0$, $c \leq 0$, $a + 3b > 0$, and $a + b + c > 0$. The following inequalities are easy to see: (1) $a > 0$, (2) $a + b > 0$, (3) $2a + 3b > 0$, (4) $a + 2b > 0$, (5) $a + 3b > 0$, (6) $2a + 3b + c > 0$, (7) $4a + 7b + 2c > 0$, (8) $2a + 4b + c > 0$, and (9) $11a + 18b + 6c > 0$. From (1) through (8) it now follows that all node firings of the sequence $s := (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_3)$ are legal: The left-hand side of each inequality is the number at the respective node of the sequence when that node is fired. The resulting position $(a_1, b_1, c_1)$ has $a_1 = 11a + 18b + c$, $b_1 = b$, and $c_1 = -(2a + 4b + c)$. Clearly $b_1 \leq 0$. Inequality (8) gives $c_1 < 0$. From (9) we get $a_1 > 0$. Note that $a_1 + 3b_1 = 11a + 21b + 6c = 6(a + b + c) + 5(a + 3b) > 0$. Finally, $a_1 + b_1 + c_1 = 9a + 15b + 5c = 5(a + b + c) + 2(a + 2b) + 2(a + 3b) > 0$, so $(a_1, b_1, c_1)$ meets condition $(\ast)$. So from any position which meets condition $(\ast)$, the firing sequence $s$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(\ast)$ using a sequence of legal node firings. The fundamental position $\omega_2 = (0, 1, 0)$ meets condition $(\ast)$. For the fundamental position $\omega_2 = (0, 1, 0)$, apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_2)$ to get the resulting position $(6, -1, 0)$. The latter meets condition $(\ast)$. For the fundamental position $\omega_3 = (0, 0, 1)$, apply the legal firing sequence $(\gamma_3, \gamma_2, \gamma_3)$ to get the resulting position $(6, 0, -1)$, which meets condition $(\ast)$.

Next, we show why is not admissible. Our argument is entirely similar to the previous case. Label the nodes as $\gamma_1, \gamma_2,$ and $\gamma_3$ from left to right. A position $(a, b, c)$ meets condition $(\ast)$ if $b \leq 0$, $c \leq 0$, $a + 3b > 0$, and $a + b + c > 0$. The following inequalities are easy to see: (1) $a > 0$, (2) $a + b > 0$, (3) $2a + 3b > 0$, (4) $a + 2b > 0$, (5) $a + 3b > 0$, (6) $2a + 3b + c > 0$, (7) $6a + 10b + 3c > 0$, (8) $4a + 7b + 2c > 0$, (9) $6a + 11b + 3c > 0$, (10) $2a + 4b + c > 0$, and (11) $35a + 60b + 18c > 0$. From (1) through (10) it now follows that all node firings of the sequence $s := (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_3, \gamma_2, \gamma_3)$ are legal: The left-hand side of each inequality is the number
at the respective node of the sequence when that node is fired. The resulting position \((a_1, b_1, c_1)\) has 
\[ a_1 = 35a + 60b + 18c, \quad b_1 = b, \quad \text{and} \quad c_1 = -(2a + 4b + c). \] 
Clearly \(b_1 \leq 0\). Inequality (10) gives \(c_1 < 0\). From (11) we get \(a_1 > 0\). Note that 
\[ a_1 + 3b_1 = 35a + 63b + 18c = 18(a + b + c) + 15(a + 3b) + 2a > 0. \] 
Finally, \(a_1 + b_1 + c_1 = 33a + 57b + 17c = 17(a + b + c) + 8(a + 2b) + 8(a + 3b) > 0\), so \((a_1, b_1, c_1)\) meets
condition \((*)\). So from any position which meets condition \((*)\), the firing sequence \(s\) can be legally
applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each
fundamental position we can reach a position which meets condition \((*)\) using a sequence of legal
node firings. The fundamental position \(\omega_1 = (1, 0, 0)\) meets condition \((*)\). For the fundamental
position \(\omega_2 = (0, 1, 0)\), apply the legal firing sequence \((\gamma_2, \gamma_3, \gamma_2, \gamma_3, \gamma_2)\) to get the resulting position
\((12, -1, 0)\). The latter meets condition \((*)\). For the fundamental position \(\omega_3 = (0, 0, 1)\), apply the legal
firing sequence \((\gamma_3, \gamma_2, \gamma_3, \gamma_2, \gamma_3)\) to get the resulting position \((18, 0, -1)\), which meets
condition \((*)\).

Next, we show why \(\ldots\) is not admissible. Our argument is entirely similar to the
previous case. Label the nodes as \(\gamma_1, \gamma_2, \text{and} \gamma_3\) from left to right. A position \((a, b, c)\) meets
condition \((*)\) if \(b \leq 0, \quad a + 3b > 0, \quad \text{and} \quad 3a + 6b + c > 0\). The following inequalities are easy to see:
\(\ldots\) From (11) we get \(a_1 > 0\). Note that 
\[ a_1 + 3b_1 = 35a + 63b + 6c = 6(3a + 6b + c) + 9(a + 3b) + 8a > 0. \] 
Finally, \(3a_1 + 6b_1 + c_1 = 99a + 174b + 17c = 17(3a + 6b + c) + 24(2a + 3b) > 0\), so \((a_1, b_1, c_1)\) meets
condition \((*)\). So from any position which meets condition \((*)\), the firing sequence \(s\) can be legally
applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each
fundamental position we can reach a position which meets condition \((*)\) using a sequence of legal
node firings. The fundamental position $\omega_1 = (1, 0, 0)$ meets condition (•). For the fundamental position $\omega_2 = (0, 1, 0)$, apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_2)$ to get the resulting position $(6, -1, 0)$. The latter meets condition (•). For the fundamental position $\omega_3 = (0, 0, 1)$, apply the legal firing sequence $(\gamma_3, \gamma_2, \gamma_3)$ to get the resulting position $(3, 0, -1)$, which meets condition (•).

To finish our analysis of the $\bar{G}$ family, we show why $\bar{G}$ is not admissible. Our argument is similar to the previous case. Label the nodes as $\gamma_1, \gamma_2, \text{and } \gamma_3$ from left to right. A position $(a, b, c)$ meets condition (•) if $a \leq 0$, $b \leq 0$, and $3a + 2b + c > 0$. The following inequalities are easy to see: (1) $c > 0$, (2) $b + c > 0$, (3) $a + b + c > 0$, (4) $3a + 2b + 2c > 0$, (5) $2a + b + c > 0$, (6) $3a + b + c > 0$, and (7) $6a + 4b + 3c > 0$. From (1) through (6) it now follows that all node firings of the sequence $s := (\gamma_3, \gamma_2, \gamma_1, \gamma_2, \gamma_3)$ are easy to see: (1) $3a + b + c > 0$, (2) $3a + 2b + c > 0$, (3) $3a + b + c > 0$. Since the fundamental positions $(a_1, b_1, c_1)$ have $a_1 = a$, $b_1 = -(3a + b + c)$, and $c_1 = 6a + 4b + 3c$. Clearly $a_1 \leq 0$. Inequality (6) gives $b_1 < 0$. Note that $3a_1 + 2b_1 + c_1 = 3a + 2b + c > 0$, so $(a_1, b_1, c_1)$ meets condition (•). So from any position which meets condition (•), the firing sequence $s$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition (•) using a sequence of legal node firings. The fundamental position $\omega_3 = (0, 0, 1)$ meets condition (•). For the fundamental position $\omega_2 = (0, 1, 0)$, apply the legal firing sequence $(\gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ to get the resulting position $(0, -1, 4)$, which meets condition (•). For the fundamental position $\omega_1 = (1, 0, 0)$, apply the legal firing sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1)$ to get the resulting position $(-1, 0, 6)$, which meets condition (•).

**Families of small cycles** First, we show why GCM graphs of the form are not admissible.

Assign numbers $a$, $b$, and $c$ as follows:

Assume for now that $a \geq 0$, $b \geq 0$, $c \leq 0$, and $\kappa > 0$; when these inequalities hold we will say the position $(a, b, c)$ meets condition (•). Under condition (•) notice that $a$ and $b$ cannot both be zero. Begin by firing only at the two rightmost nodes. When this is no longer possible, fire at the leftmost node. The resulting corresponding numbers are $a_1 = q_1(\kappa + \frac{1}{q_2}a)$, $b_1 = q_2(\kappa + \frac{1}{q_1}b)$, and $c_1 = -\kappa - \frac{1}{q_2}a - \frac{1}{q_1}b$. In particular, $a_1 > 0$, $b_1 > 0$, and $c_1 < 0$. Next we check that $\kappa_1 := (p_1 + p_2 - \frac{1}{q_2})a_1 + (p_1 + p_2 - \frac{1}{q_1})b_1 + c_1$ is also positive. Now

$$\kappa_1 = Q\kappa + Q_1a + Q_2b,$$

where $Q = q_1(p_2 - \frac{1}{q_2}) + q_2(p_1 - \frac{1}{q_1}) + (p_1q_1 + p_2q_2 - 1)$, $Q_1 = \frac{1}{q_2}[q_1(p_2 - \frac{1}{q_2}) + (p_1q_1 - 1)]$, and $Q_2 = \frac{1}{q_1}[q_2(p_1 - \frac{1}{q_1}) + (p_2q_2 - 1)]$. Since each parenthesized quantity in our expression for $Q$ is nonnegative and the last of these is positive, then $Q > 0$. Similar reasoning shows that each bracketed quantity in our expressions for $Q_1$ and $Q_2$ is nonnegative, hence $Q_1 \geq 0$ and $Q_2 \geq 0$. Since $\kappa > 0$ by hypothesis, it now follows that $\kappa_1 > 0$. Then $(a_1, b_1, c_1)$ meets condition (•), so we can legally repeat the above firing sequence from position $(a_1, b_1, c_1)$ to obtain another position $(a_2, b_2, c_2)$ that meets condition (•), etc. Since the fundamental positions $(a, b, c) = (1, 0, 0)$ and...
(a, b, c) = (0, 1, 0) meet condition (\(*\)), then we see that the indicated legal firing sequence can be repeated indefinitely from these positions. For the fundamental position (a, b, c) = (0, 0, 1), begin by firing at the leftmost node to obtain the position (q_1, q_2, -1). This latter position meets condition (\(*\)) with \( \kappa = Q \), and so the legal firing sequence indicated above can be repeated indefinitely from this position.

Next, we show why GCM graphs of the form are not admissible. We assume that the amplitude products \( p_1 q_1 \) and \( p_2 q_2 \) are at least two. The argument is entirely similar to the previous case. Assign numbers \( a, b, \) and \( c \) as follows: Set \( \kappa := (2p_1 + 2p_2 - \frac{1}{q_1})a + (p_1 + 2p_2 - \frac{1}{q_2})b + c. \)

Assume for now that \( a \geq 0, b \geq 0, c \leq 0, \) and \( \kappa > 0; \) when these inequalities hold we will say the position \( (a, b, c) \) meets condition (\(*\)). Using the same firing sequence as before, the resulting corresponding numbers are \( a_1 = q_1(\kappa + \frac{1}{q_2}b), \) \( b_1 = q_2(\kappa + \frac{1}{q_1}a), \) and \( c_1 = -\kappa - \frac{1}{q_1}a - \frac{1}{q_2}b. \) In particular, \( a_1 > 0, b_1 > 0, \) and \( c_1 < 0. \) Next we check that \( \kappa_1 := (2p_1 + 2p_2 - \frac{1}{q_1})a_1 + (p_1 + 2p_2 - \frac{1}{q_2})b_1 + c_1 \) is also positive. Now

\[
\kappa_1 = Q \kappa + Q_1 a + Q_2 b,
\]

where \( Q = q_1(2p_2 - \frac{1}{q_1}) + q_2(p_1 - \frac{1}{q_2}) + (2p_1q_1 + 2p_2q_2 - 1), \) \( Q_1 = \frac{1}{q_1}[q_2(p_1 - \frac{1}{q_2}) + (2p_2q_2 - 1)], \) and \( Q_2 = \frac{1}{q_2}[q_1(2p_2 - \frac{1}{q_1}) + (2p_1q_1 - 1)]. \) Since each parenthesized quantity in our expression for \( Q \) is nonnegative and the last of these is positive, then \( Q > 0. \) Similar reasoning shows that each bracketed quantity in our expressions for \( Q_1 \) and \( Q_2 \) is nonnegative, hence \( Q_1 \geq 0 \) and \( Q_2 \geq 0. \) Since \( \kappa > 0 \) by hypothesis, it now follows that \( \kappa_1 > 0. \) Conclude as in the previous case.

Next, we show why GCM graphs of the form are not admissible. We assume that the amplitude products \( p_1 q_1 \) and \( p_2 q_2 \) are at least three. The argument is entirely similar to the previous two cases. Assign numbers \( a, b, \) and \( c \) as follows: Set \( \kappa := (4p_1 + 6p_2 - \frac{1}{q_1})a + (2p_1 + 4p_2 - \frac{1}{q_2})b + c. \)

Assume for now that \( a \geq 0, b \geq 0, c \leq 0, \) and \( \kappa > 0; \) when these inequalities hold we will say the position \( (a, b, c) \) meets condition (\(*\)). Using the same firing sequence as in the previous two cases, the resulting corresponding numbers are \( a_1 = q_1(\kappa + \frac{1}{q_2}b), \) \( b_1 = q_2(\kappa + \frac{1}{q_1}a), \) and \( c_1 = -\kappa - \frac{1}{q_1}a - \frac{1}{q_2}b. \) In particular, \( a_1 > 0, b_1 > 0, \) and \( c_1 < 0. \) Next we check that \( \kappa_1 := (4p_1 + 6p_2 - \frac{1}{q_1})a_1 + (2p_1 + 4p_2 - \frac{1}{q_2})b_1 + c_1 \) is also positive. Now

\[
\kappa_1 = Q \kappa + Q_1 a + Q_2 b,
\]

where \( Q = q_1(6p_2 - \frac{1}{q_1}) + q_2(2p_1 - \frac{1}{q_2}) + (4p_1q_1 + 4p_2q_2 - 1), \) \( Q_1 = \frac{1}{q_1}[q_2(2p_1 - \frac{1}{q_2}) + (4p_2q_2 - 1)], \) and \( Q_2 = \frac{1}{q_2}[q_1(6p_2 - \frac{1}{q_1}) + (4p_1q_1 - 1)]. \) Since each parenthesized quantity in our expression for
$Q$ is nonnegative and the last of these is positive, then $Q > 0$. Similar reasoning shows that each bracketed quantity in our expressions for $Q_1$ and $Q_2$ is nonnegative, hence $Q_1 \geq 0$ and $Q_2 \geq 0$. Since $\kappa > 0$ by hypothesis, it now follows that $\kappa_1 > 0$. Conclude as in the previous two cases.

Next, we show why \includegraphics{example.png} is not admissible. The argument is similar to the previous three cases, but simpler since the amplitudes are all known. Number the nodes with $\gamma_1$ as the North vertex, $\gamma_2$ as the East vertex, $\gamma_3$ as the South vertex, and $\gamma_4$ as the West vertex. We say an initial position $(a, b, c, d)$ meets condition $(\ast)$ if the following inequalities are satisfied: $b \geq 0$, $c \geq 0$, $d \leq 0$, and $a + d > 0$. The firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is easily seen to be legal from any such position. The resulting position is $(a_1, b_1, c_1, d_1)$ with $a_1 = 4a + 2b + c + d$, $b_1 = c$, $c_1 = a + d$, and $d_1 = -(3a + b + c + d)$. It is easy now to check that $(a_1, b_1, c_1, d_1)$ also meets condition $(\ast)$. (In fact, the inequalities $c_1 > 0$ and $d_1 < 0$ are now strict.) So from any position which meets condition $(\ast)$, the firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(\ast)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0, 0)$ meets condition $(\ast)$. Now take fundamental position $\omega_2 = (0, 1, 0, 0)$ and apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(2, 0, 0, -1)$. The latter meets condition $(\ast)$. Next take fundamental position $\omega_3 = (0, 0, 1, 0)$ and apply the legal firing sequence $(\gamma_3, \gamma_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(5, 0, 0, -3)$. The latter meets condition $(\ast)$. For fundamental position $\omega_4 = (0, 0, 0, 1)$, apply the legal firing sequence $(\gamma_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(4, 1, 0, -3)$. The latter meets condition $(\ast)$.

Next, we show why \includegraphics{example.png} is not admissible. The argument is entirely similar to the previous case. Number the nodes with $\gamma_1$ as the North vertex, $\gamma_2$ as the East vertex, $\gamma_3$ as the South vertex, and $\gamma_4$ as the West vertex. We say an initial position $(a, b, c, d)$ meets condition $(\ast)$ if the following inequalities are satisfied: $a > 0$, $b \geq 0$, $c \geq 0$, $d \leq 0$, and $3a + b + c + 2d > 0$. The firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is easily seen to be legal from any such position. The resulting position is $(a_1, b_1, c_1, d_1)$ with $a_1 = 4a + 2b + c + d$, $b_1 = c$, $c_1 = 4a + b + c + 2d$, and $d_1 = -(3a + b + c + d)$. It is easy now to check that $(a_1, b_1, c_1, d_1)$ also meets condition $(\ast)$. (In fact, the inequalities $c_1 > 0$ and $d_1 < 0$ are now strict.) So from any position which meets condition $(\ast)$, the firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(\ast)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0, 0)$ meets condition $(\ast)$. Now take fundamental position $\omega_2 = (0, 1, 0, 0)$ and apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(2, 0, 0, -1)$. The latter meets condition $(\ast)$. Next take fundamental position $\omega_3 = (0, 0, 1, 0)$ and apply the legal firing sequence $(\gamma_3, \gamma_4)$ to get the resulting position $(1, 1, 1, -1)$. The latter meets condition $(\ast)$. For fundamental position $\omega_4 = (0, 0, 0, 1)$, apply the legal firing sequence $(\gamma_4)$ to get the resulting position $(1, 0, 2, -1)$. The latter meets condition $(\ast)$. 

22
Next, we show why \( \bullet \bullet \) is not admissible. The argument is entirely similar to the previous case. Number the nodes with \( \gamma_1 \) as the North vertex, \( \gamma_2 \) as the East vertex, \( \gamma_3 \) as the South vertex, and \( \gamma_4 \) as the West vertex. We say an initial position \( (a, b, c, d) \) meets condition \((\ast)\) if the following inequalities are satisfied: \( a > 0, b \geq 0, c \geq 0, d \leq 0, \) and \( 3a + b + c + d > 0. \) The firing sequence \( (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \) is easily seen to be legal from any such position. The resulting position is \( (a_1, b_1, c_1, d_1) \) with \( a_1 = 6a + 3b + 2c + d, b_1 = c, c_1 = 3a + b + c + d, \) and \( d_1 = -(5a + 2b + 2c + d). \)

It is easy now to check that \( (a_1, b_1, c_1, d_1) \) also meets condition \((\ast)\). (In fact, the inequalities \( c_1 > 0 \) and \( d_1 < 0 \) are now strict.) So from any position which meets condition \((\ast)\), the firing sequence \( (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \) can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition \((\ast)\) using a sequence of legal node firings. The fundamental position \( \omega_1 = (1, 0, 0, 0) \) meets condition \((\ast)\). Now take fundamental position \( \omega_2 = (0, 1, 0, 0) \) and apply the legal firing sequence \( (\gamma_2, \gamma_3, \gamma_4) \) to get the resulting position \( (3, 0, 0, -2) \). The latter meets condition \((\ast)\). Next take fundamental position \( \omega_3 = (0, 0, 1, 0) \) and apply the legal firing sequence \( (\gamma_3, \gamma_4) \) to get the resulting position \( (2, 1, 1, -2) \). The latter meets condition \((\ast)\). For fundamental position \( \omega_4 = (0, 0, 0, 1) \), apply the legal firing sequence \( (\gamma_4) \) to get the resulting position \( (1, 0, 1, -1) \). The latter meets condition \((\ast)\).

To finish our analysis of families of small cycles, we show why \( \bullet \bullet \) is not admissible.

The argument is entirely similar to the previous case. Number the nodes with \( \gamma_1 \) as the North vertex and \( \gamma_2, \gamma_3, \gamma_4, \) and \( \gamma_5 \) in succession in the clockwise order around the cycle. We say an initial position \( (a, b, c, d, e) \) meets condition \((\ast)\) if the following inequalities are satisfied: \( b \geq 0, c \geq 0, d \geq 0, e \leq 0, \) and \( a + e > 0. \) The firing sequence \( (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \) is easily seen to be legal from any such position. The resulting position is \( (a_1, b_1, c_1, d_1, e_1) \) with \( a_1 = 4a + 2b + c + d + e, b_1 = c, c_1 = d, d_1 = a + e, \) and \( e_1 = -(3a + b + c + d + e). \) It is easy now to check that \( (a_1, b_1, c_1, d_1, e_1) \) also meets condition \((\ast)\). (In fact, the inequalities \( d_1 > 0 \) and \( e_1 < 0 \) are now strict.) So from any position which meets condition \((\ast)\), the firing sequence \( (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \) can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition \((\ast)\) using a sequence of legal node firings. The fundamental position \( \omega_1 = (1, 0, 0, 0, 0) \) meets condition \((\ast)\). Now take fundamental position \( \omega_2 = (0, 1, 0, 0, 0) \) and apply the legal firing sequence \( (\gamma_2, \gamma_3, \gamma_4, \gamma_5) \) to get the resulting position \( (2, 0, 0, 0, -1) \). The latter meets condition \((\ast)\). Next take fundamental position \( \omega_3 = (0, 0, 1, 0, 0) \) and apply the legal firing sequence \( (\gamma_3, \gamma_4, \gamma_5, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \) to get the resulting position \( (5, 0, 0, 0, -3) \). The latter meets condition \((\ast)\). Next take fundamental position \( \omega_4 = (0, 0, 0, 1, 0) \) and apply the legal firing sequence \( (\gamma_4, \gamma_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \) to get the resulting position \( (4, 1, 0, 0, -3) \). The latter meets condition \((\ast)\). For fundamental position \( \omega_4 = (0, 0, 0, 0, 1) \), apply the legal firing sequence \( (\gamma_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \) to get the resulting position \( (4, 0, 1, 0, -3) \). The latter meets condition \((\ast)\).

This completes the proof of Proposition 3.1.
References


