Analogies: Two-dimensional v. three-dimensional vector fields
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Two-dimensional vector fields	Three-dimensional vector fields
Let $\mathbf{F}(x, y) = \langle P, Q \rangle$ be a vector field defined on an open, connected region D in the plane. The functions $P(x, y)$ and $Q(x, y)$ should have continuous first partial derivatives on D.	Let $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ be a vector field defined on an open, connected region E in space. The functions $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ should have continuous first partial derivatives on E.
Theorem: If $\mathbf{F}$ is conservative, then $$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$	**Theorem:** If $\mathbf{F}$ is conservative, then $$\text{curl}(\mathbf{F}) = \mathbf{0}$$
Theorem: Suppose the region D is *simply connected* (i.e. no "holes"). If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, then $\mathbf{F}$ is conservative.	**Theorem:** Suppose the region E is *simply connected* (i.e. no "holes"). If $\text{curl}(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F}$ is conservative.
Now let C be a simple, closed, positively-oriented, piecewise-smooth curve. Let D be the region of the plane enclosed by the curve C.	Let C be a simple, closed, positively-oriented, piecewise-smooth curve. Let S be any surface in space whose "edge" or "boundary" is the curve C. (For example, the unit circle is the boundary curve for the upper half of the unit sphere.)
Green's Theorem $$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$	**Stoke's Theorem** $$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS$$

NOTE: It is not hard to see that Stoke's Theorem is really just a three-dimensional version of Green's Theorem. We can think of the two-dimensional vector field $\mathbf{F} = \langle P, Q \rangle$ as a three-dimensional vector field $\mathbf{F} = \langle P, Q, 0 \rangle$. In this case, $\text{curl}(\mathbf{F}) = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$, and the vector $\mathbf{k}$ is always normal to the region D. Thus, the expression $\text{curl}(\mathbf{F}) \cdot \mathbf{n}$ appearing in Stoke's Theorem becomes

$$\text{curl}(\mathbf{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

which is the integrand in Green's Theorem.