Analogies: Calc I/II concepts in comparison with analogous Calc III concepts

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As any Calculus III student will know, mathematical structures build upon themselves, creating in mathematics a taut interdependence of ideas unlike many other disciplines. To help calculus students better understand this interdependence, I thought that it would be appropriate to compare and contrast fundamental ideas encountered in pre-calculus, Calculus I/II, and Calculus III.

Pre-calc/Calc I/Calc II	Calculus III
Line: $y = mx + b$ $A(x - x_0) + B(y - y_0) = 0$	Line: $\mathbf{r}(t) = t\mathbf{v} + \mathbf{r}_0$ Plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
Derivative for $y = f(x)$: $\dfrac{dy}{dx} = f'(x)$	Derivative for curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$: $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ Derivatives for a surface $z = f(x, y)$ $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$, and the directional derivative $D_{\mathbf{u}}(f)(x, y) = \nabla f(x, y) \cdot \mathbf{u}$
Tangent line: $y - y_0 = f'(x_0)(x - x_0)$	Tangent line for a curve $\mathbf{r}(t)$: $\mathbf{s}(t) = t\mathbf{r}'(t_0) + \mathbf{r}_0$ Tangent plane for a surface $z = f(x, y)$: $z - z_0 = f_x(x_0, y_0)(x - x_0) +$ $f_y(x_0, y_0)(y - y_0)$
Tangent vector for $y = f(x)$ at $x = x_0$: $\langle 1,\ f'(x_0) \rangle$ Normal vector for $y = f(x)$ at $x = x_0$: $\langle -f'(x_0),\ 1 \rangle$	Tangent vectors for $z = f(x, y)$ at $(x, y) = (x_0, y_0)$: $\langle 1,\ 0,\ \frac{\partial f}{\partial x}(x_0, y_0) \rangle$ $\langle 0,\ 1,\ \frac{\partial f}{\partial y}(x_0, y_0) \rangle$ Normal vectors for $z = f(x, y)$ at $(x, y) = (x_0, y_0)$: $\langle -\frac{\partial f}{\partial x},\ -\frac{\partial f}{\partial y},\ 1 \rangle$ "Upward normal" $\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ -1 \rangle$ "Downward normal"

Pre-calc/Calc I/Calc II	Calculus III
Two curves $f(x)$ and $g(x)$ are "the same" if $f'(x) = g'(x)$	Two curves $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are "the same" if *and only if* they have the same curvature and torsion. Two plane curves $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are "the same" if *and only if* they have the same curvature.
Local extreme values: If $f(x)$ has a local max or local min at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.	Local extreme values: If $f(x, y)$ has a local max or local min at $x = a$, $y = b$, and if $f_x(a, b)$ and $f_y(a, b)$ exist, then $f_x(a, b) = f_y(a, b) = 0$.
Continuous functions on closed, bounded intervals: If $y = f(x)$ is continuous on $[a, b]$, then f achieves an absolute max and an absolute min, and moreover these extreme values occur at critical points or endpoints.	Continuous functions on closed, bounded regions: If $z = f(x, y)$ is continuous on a closed, bounded region D, then f achieves an absolute max and an absolute min, and moreover these extreme values occur at critical points or boundary points.
Second derivative test: If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at $x = c$. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at $x = c$. If $f'(c) = 0$ and $f''(c) = 0$, the test is inconclusive.	Second derivatives test: If $f_x(a, b) = f_y(a, b) = 0$, then let $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$ If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local min at $x = a$, $y = b$. If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local max at $x = a$, $y = b$. If $D < 0$, then f has a saddle point at $x = a$, $y = b$. Otherwise the test is inconclusive.

Pre-calc/Calc I/Calc II	Calculus III
Chain Rule for $y = f(x)$ with $x = g(t)$: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = f'(g(t))\, g'(t)$	Chain Rule for $z = f(x, y)$ with $x = g(t)$ and $y = h(t)$: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = f_x\, g'(t) + f_y\, h'(t)$
<u>Rate of change</u> $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\frac{d}{dx}$: function $\longmapsto$ function	<u>Rates of change</u> $\frac{\partial}{\partial x} f(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$ $\frac{\partial}{\partial x}$: function $\longmapsto$ function $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$ ∇ : function $\longmapsto$ vector field $\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ div : vector field $\longmapsto$ function $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ curl : vector field $\longmapsto$ vector field

Pre-calc/Calc I/Calc II	Calculus III
Definite integral $\displaystyle\int_a^b f(x)\ dx$	Line integral $\displaystyle\int_C f\ ds$ Double integral $\displaystyle\iint_D f(x,y)\ dA$ Surface integral $\displaystyle\iint_S f(x,y,z)\ dS$ Triple integral $\displaystyle\iiint_E f(x,y,z)\ dV$
Length of interval: $\displaystyle\int_a^b 1\ dx$	Length of curve: $\displaystyle\int_C 1\ ds$ Area of region: $\displaystyle\iint_D 1\ dA$ Surface area $\displaystyle\iint_S 1\ dS$ Volume of region: $\displaystyle\iiint_E 1\ dV$
Mass of straight, thin rod with density function $\rho(x)$: $\displaystyle\int_a^b \rho(x)\ dx$	Mass of thin wire with shape C and density function ρ: $\displaystyle\int_C \rho\ ds$ Mass of thin lamina with density function $\rho(x,y)$: $\displaystyle\iint_D \rho(x,y)\ dA$ Mass of thin surface with shape S and density function ρ: $\displaystyle\iint_S \rho\ dS$ Mass of solid of shape E with density function $\rho(x,y,z)$: $\displaystyle\iiint_E \rho(x,y,z)\ dV$

Pre-calc/Calc I/Calc II	Calculus III
Probability density function $f(x)$ for a continuous random variable X:	Probability density function $f(x,y)$ for two continuous random variables X and Y:
$\int_{-\infty}^{\infty} f(x)\,dx = 1$	$\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$
$\text{Prob}(a \leq X \leq b) = \int_a^b f(x)\,dx$	$\text{Prob}(\,(X,Y) \in D) = \iint_D f(x,y)\,dA$
Mean (expected value) $= \int_{-\infty}^{\infty} x f(x)\,dx$	X-mean $= \iint_{\mathbb{R}^2} x f(x,y)\,dA$ Y-mean $= \iint_{\mathbb{R}^2} y f(x,y)\,dA$
Moment about origin $= \int_{-\infty}^{\infty} x^2 f(x)\,dx$	X-moment about origin $= \iint_{\mathbb{R}^2} x^2 f(x,y)\,dA$ Y-moment about origin $= \iint_{\mathbb{R}^2} y^2 f(x,y)\,dA$
Density function $\rho(x)$ for a thin, straight rod:	Density function $\rho(x,y)$ for a thin lamina:
Total mass $= \int_a^b \rho(x)\,dx$	Total mass $= \iint_D \rho(x,y)\,dA$
Moment $M_y = \int_a^b x\rho(x)\,dx$	Moment $M_y = \iint_D x\rho(x,y)\,dA$ Moment $M_x = \iint_D y\rho(x,y)\,dA$
Moment of inertia $I_y = \int_a^b x^2\rho(x)\,dx$	Moment of inertia $I_y = \iint_D x^2\rho(x,y)\,dA$ Moment of inertia $I_x = \iint_D y^2\rho(x,y)\,dA$
Center of mass $\overline{x} = M_y/(\text{Total mass})$	Center of mass $(\overline{x}, \overline{y})$ $\overline{x} = M_y/(\text{Total mass})$ $\overline{y} = M_x/(\text{Total mass})$

Pre-calc/Calc I/Calc II	Calculus III
Change of variables: $\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$	Change of variables: $\iint_D f(x,y)\,dA =$ $\iint_S f(x(u,v),y(u,v))\,\left\lvert\frac{\partial(x,y)}{\partial(u,v)}\right\rvert\,du\,dv$
Antiderivatives: An *antiderivative* for $g(x)$ is a function $f(x)$ whose derivative is g: $f'(x) = g(x)$	Potential functions: Let $\mathbf{F}(x,y,z)$ be a vector field. A *potential function* for $\mathbf{F}$ is a function $f(x,y,z)$ whose gradient is $\mathbf{F}$: $\nabla f(x,y,z) = \mathbf{F}(x,y,z)$
Work done in moving an object in a straight line with force $F(x)$: $\int_a^b F(x)\,dx$	Work done in moving an object through a force field $\mathbf{F}(x,y,z)$ along a curve C: $\int_C \mathbf{F}\cdot d\mathbf{r}$
<u>Fundamental Theorem of Calculus:</u> $\int_a^b f'(x)\,dx = f(b) - f(a)$	<u>Fundamental Theorem for Line Integrals:</u> Let C be a spacecurve with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$. Let $f(x,y,z)$ be a function whose domain is a region E in $\mathbb{R}^3$ which contains C. $\int_C \nabla f\cdot d\mathbf{r}$ $= \int_a^b \nabla f(x(t),y(t),z(t))\cdot\mathbf{r}'(t)dt$ $= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$