An enumerative problem from abstract algebra

Rob Donnelly, Supplement for an MSU course MAT 421 Intro to Algebraic Structures, Fall 2002

Question: How many different six-bead bracelets can one make if each bead can be either Red, Black, or Green?

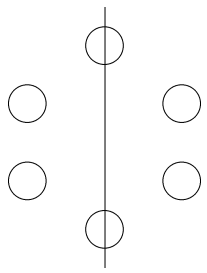
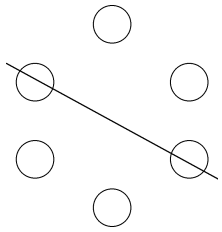
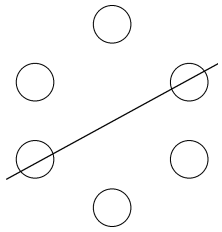
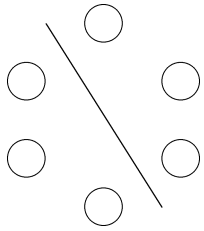
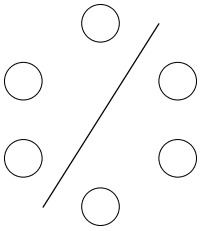
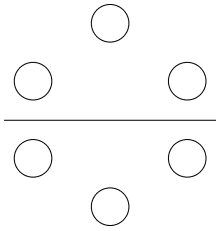
Example of two "D_6-equivalent" bracelets:

Burnside's Theorem: *The number of equivalence classes in a finite set S under the action of a finite group G is*

$$\frac{1}{|G|}\sum_{g\in G}|\mathsf{fix}(g)|,$$

where $\mathsf{fix}(g) := \{x \in S \mid g.x = x\}$.

Enumerative problem (continued)

D_6 element g	$\lvert\mathrm{fix}(g)\rvert$	D_6 element g	$\lvert\mathrm{fix}(g)\rvert$	D_6 element g	$\lvert\mathrm{fix}(g)\rvert$
e	$3^6 = 729$	r_{60}	$3^1 = 3$	r_{120}	$3^2 = 9$
r_{180}	$3^3 = 27$	r_{240}	$3^2 = 9$	r_{300}	$3^1 = 3$
	$3^4 = 81$		$3^4 = 81$		$3^4 = 81$
	$3^3 = 27$		$3^3 = 27$		$3^3 = 27$

Total: $$\frac{1}{|D_6|}\sum_{g\in D_6}|\mathrm{fix}(g)| = \frac{1}{12}(729+3+9+27+9+3+81+81+81+27+27+27)$$

$$= \frac{1}{12}(1104) = 92 \text{ different bracelets.}$$