The numbers game and Dynkin diagram classification results

Rob Donnelly
Murray State University

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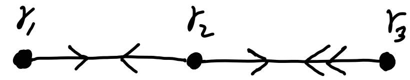
The numbers game



γ_1	γ_2	γ_3
2	0	3
-2	2	3
0	-2	5
0	8	-5
8	-8	3

etc

$$M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

Definitions / Notation

Γ a finite simple graph with n nodes

I_n an index set of size n for the nodes of Γ

$M = (M_{ij})_{i,j \in I_n}$ an $n \times n$ matrix of integers satisfying

1. $M_{ii} = 2$
2. $M_{ij} \leq 0$ for $i \neq j$
3. For $i \neq j$,

   $M_{ij} < 0 \iff M_{ji} < 0 \iff$ nodes γ_i and γ_j are adjacent in Γ,

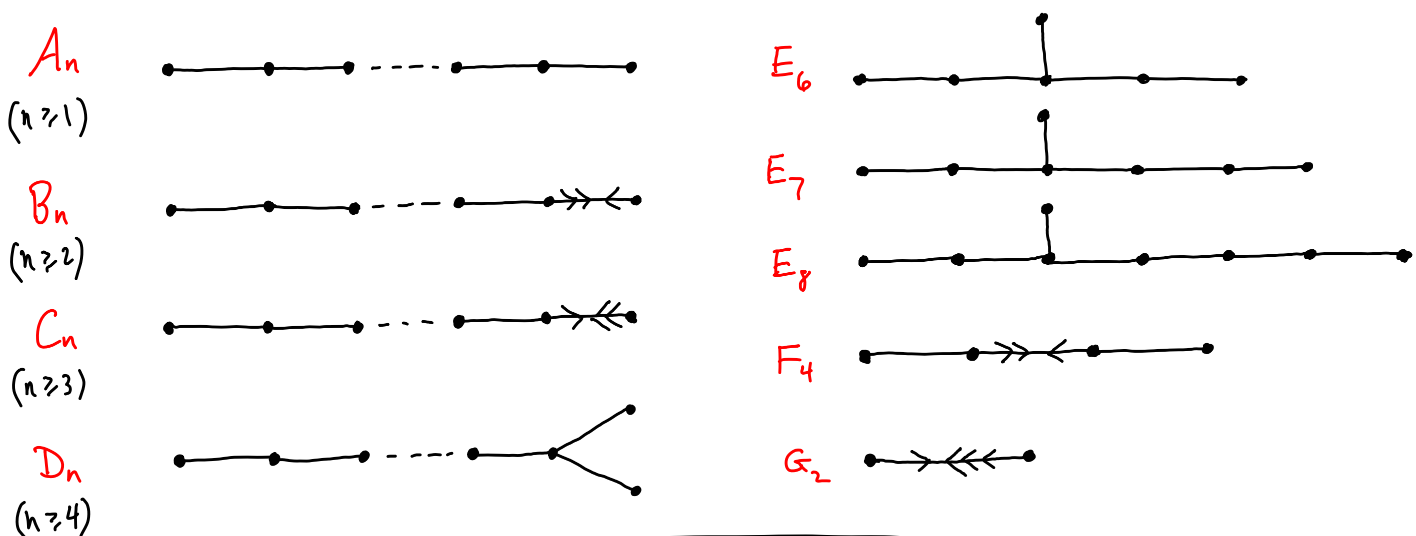
   in which case $|M_{ij}| = \#$ of arrows from γ_i to γ_j

M is a "generalized Cartan matrix" or GCM

$\mathcal{G} = (\Gamma, M)$ is a GCM graph

Question For which connected GCM graphs does there exist an initial position of nonnegative real numbers (not all zero) for which the numbers game terminates?

Theorem [D.] A GCM graph $\mathcal{G}$ that answers our question must be one of the following:



$\boxed{\text{Connected Dynkin diagrams of finite type}}$

Proofs

① Combinatorial proof [D. 2006]

[The classifications of finite Weyl groups and finite-dimensional Kac-Moody Lie algebras follow from this proof.]

② Proof using geometric representations of Weyl groups [D. 2006]

[This proof requires the classification of finite Weyl groups. But, it generalizes to a Coxeter groups setting.]

③ Proof using Perron-Frobenius theory [D., Kimmo Eriksson 2007]

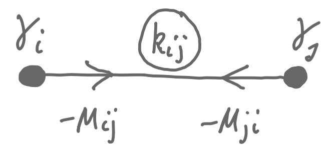
[This proof generalizes to a Coxeter groups setting.

The classifications of finite Weyl groups, finite-dimensional Kac-Moody Lie algebras, and finite Coxeter groups follow from this proof.]

Comments on Proof ②

$$M = (M_{ij})_{i,j \in I_n} \quad , \quad \text{an } \underline{E\text{-}GCM}$$

M is an $n \times n$ matrix of real numbers satisfying



① $M_{ii} = 2$

② $M_{ij} \leq 0$ for $i \neq j$

③ For $i \neq j$, $M_{ij} < 0 \iff M_{ji} < 0$

④ For all i, j,

$$M_{ij} M_{ji} < 4 \implies M_{ij} M_{ji} = 4 \cos^2(\pi / k_{ij})$$

for some positive integer k_{ij}.

NOTE: If M is a GCM, then

$M_{ij} M_{ji} = 1, 2, 3 \implies k_{ij} = 3, 4, 6$

$\mathcal{G} = (\Gamma, M)$ is an E-GCM graph.

$\mathcal{G}$ is admissible $\iff$ $\mathcal{G}$ is connected and there is an initial position of nonnegative real numbers (not all zero) for which a numbers game on $\mathcal{G}$ terminates.

W Coxeter group (Weyl group if M is a GCM)

$\langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = \varepsilon \rangle$, where

$$m_{ij} = \begin{cases} 1 & \text{if } i=j \\ k_{ij} & \text{if } i \neq j \text{ and } M_{ij}M_{ji} = 4\cos^2(\pi/k_{ij}) < 4 \\ & \quad \text{for a positive integer } k_{ij} \\ \infty & \text{if } i \neq j \text{ and } M_{ij}M_{ji} \geq 4 \end{cases}$$

NOTE: $m_{ij} = 3, 4, 6$ if $M_{ij}M_{ji} = 1, 2, 3$.

$V := \operatorname{span}_{\mathbb{R}} \{\alpha_i\}_{i \in I_n}$ α_i's are simple roots

$\sigma : W \to GL(V)$ given by $\sigma(s_i)(\alpha_j) := \alpha_j - M_{ij}\alpha_i$

is a faithful representation of W.

Std. geom. rep if $-M_{ij} = 2\cos(\pi/m_{ij})$

$\sigma^* : W \longrightarrow GL(V^*)$ given by $\langle \sigma^*(w)(\lambda), v \rangle = \langle \lambda, \sigma(w^{-1})(v) \rangle$

for all $\lambda \in V^*$, $v \in V$, the dual representation

$D := \{ \lambda \in V^* \mid \langle \lambda, \alpha_i \rangle \geq 0 \quad \forall\, i \in I_n \}$

$\mathcal{U} := \bigcup_{w \in W} wD$, Tits cone (a convex cone)

Facts

1. (Eriksson) The set of initial positions from which a numbers game on $\mathcal{G}$ converges is $-\mathcal{U}$.

2. (D.) If $\mathcal{G}$ is connected and "unital ON-cyclic," and if W is infinite, then $\mathcal{U} \cap (-\mathcal{U}) = \{0\}$.

- "Unital ON-cyclic" is guaranteed if ... M is a GCM
  M is symmetrizable
  $\mathcal{G}$ is a tree

- "
$$\left(\frac{-M_{i_0 i_{p-1}}}{2\cos(\pi/m_{i_0 i_{p-1}})}\right)\left(\frac{-M_{i_{p-1} i_{p-2}}}{2\cos(\pi/m_{i_{p-1} i_{p-2}})}\right)\cdots\left(\frac{-M_{i_1 i_0}}{2\cos(\pi/m_{i_1 i_0})}\right) = 1$$
whenever $(\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_{p-1}}, \gamma_{i_p = i_0})$ is a cycle in Γ for which each $m_{i_{q-1} i_q}$ is odd."

3. It can be shown by hand that certain families of cyclic E-GCM's are not admissible.

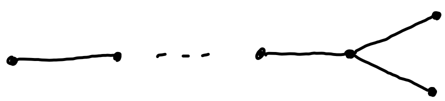
Combining these facts in an induction argument on the number of nodes...

<u>Conclusion</u> $\mathcal{G}$ is admissible $\iff$ $\mathcal{G}$ is one of

$\mathcal{A}_n$ $(n \geq 1)$

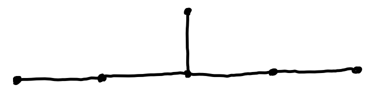
$\mathcal{B}_n$ $(n \geq 3)$

$\mathcal{D}_n$ $(n \geq 4)$

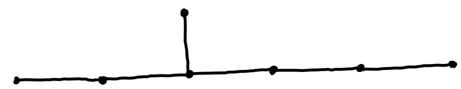


$\mathcal{I}_2^{(m)}$ $(m \geq 4)$

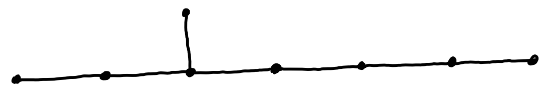
$\mathcal{E}_6$



$\mathcal{E}_7$

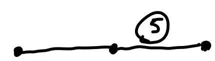


$\mathcal{E}_8$



$\mathcal{F}_4$

$\mathcal{H}_3$



$\mathcal{H}_4$

$\boxed{\text{Families of connected "E-Coxeter" graphs}}$

Application — Ranked posets and the M-structure property

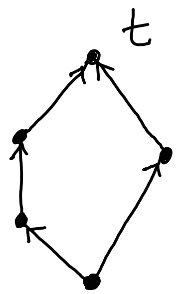
- $(R, \le_R)$ a finite partially ordered set

- The Hasse diagram for $(R, \le_R)$ is the directed graph with vertex set R and edges $s \longrightarrow t$ whenever t covers s, i.e. $\nexists\, x \in R$ such that $s < x < t$.

- $\rho : R \xrightarrow{\text{surjective}} \{0, 1, \dots, \ell\}$ is a rank function for R if
$$\rho(s) + 1 = \rho(t) \quad \text{whenever} \quad s \longrightarrow t \text{ in } R$$

(The depth function is the function $\delta : R \longrightarrow \{0, \dots, \ell\}$ such that $\delta(t) = \ell - \rho(t)$. The number ℓ is the length of R.)

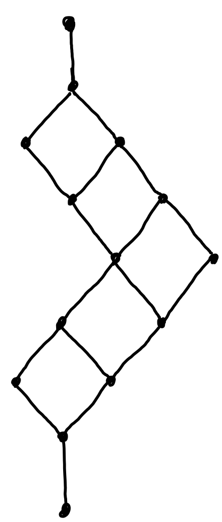
If such a function ρ exists, then R is ranked.



This five-element poset is NOT ranked.

(Would $\rho(t)$ be 2 or 3?)

This 14-element poset is ranked



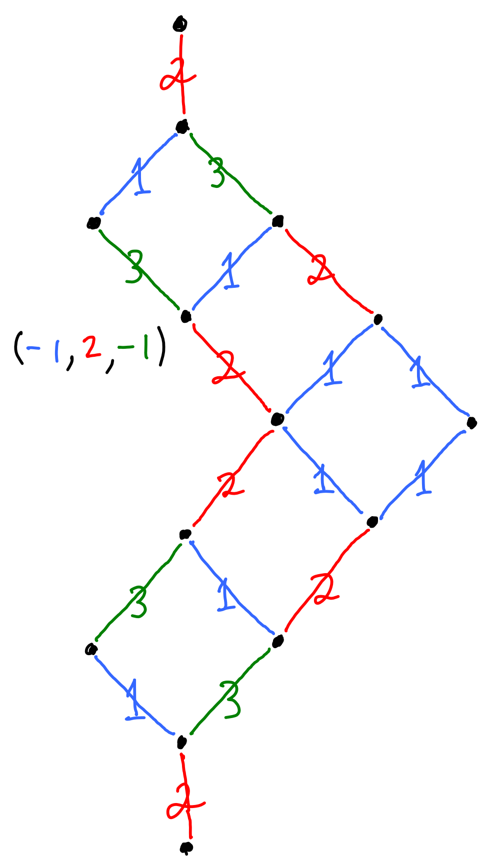
Rank ρ	Depth δ	Difference $\rho - \delta$
8	0	8
7	1	6
6	2	4
5	3	2
4	4	0
3	5	-2
2	6	-4
1	7	-6
0	8	-8

Edge colors, weights

- Color the edges of a ranked poset R using elements of I_n as edge colors.

- The color i components of R are ranked

- Define: $wt_R(t) = \left(\rho_i(t) - \delta_i(t) \right)_{i \in I_n}$
  "Combinatorial weight rule"



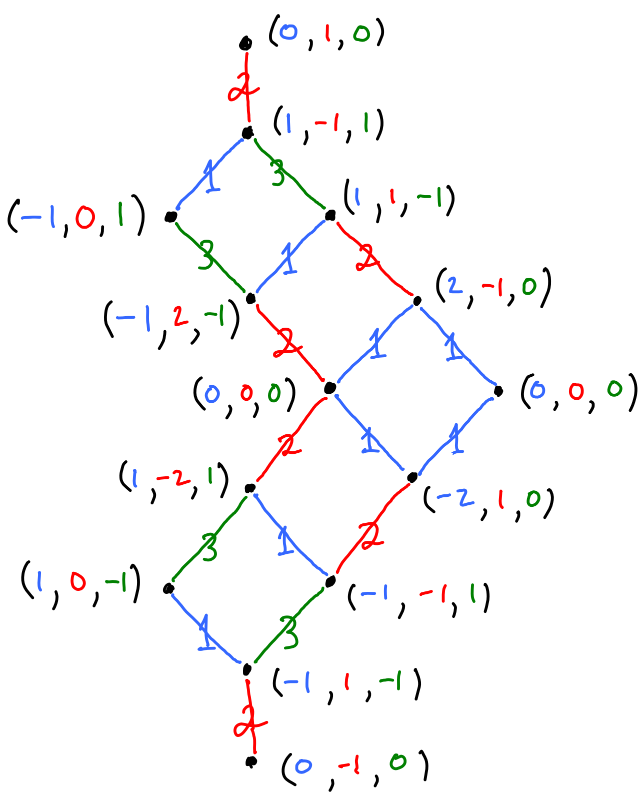
The M-stucture property

Let $M = (M_{ij})_{i,j \in I_n}$ a matrix of real numbers.

An edge-colored ranked poset R is M-structured if

$$s \xrightarrow{i} t \implies wt_R(t) - wt_R(s) = i^{th} \text{ row of } M$$

for all edges in the Hasse diagram for R.



$$M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

__Further examples__ Admissible systems (crystal graphs) and supporting graphs for finite-dimensional representations of finite-dim. complex semisimple Lie algebras have the M-structure property, where M is the associated Cartan matrix.

__Theorem [D, 2007]__

- Let $\mathfrak{g} = (\Gamma, M)$ be a connected GCM graph.

  Suppose there exists an M-structured ranked poset with at least one edge.

  Then $\mathfrak{g}$ is a connected Dynkin diagram of finite type.

- Let $\mathfrak{g} = (\Gamma, M)$ be a GCM graph.

  Suppose there exists an M-structured ranked poset which uses at least one edge color for each connected component of $\mathfrak{g}$.

  Then $\mathfrak{g}$ is a Dynkin diagram of finite type.

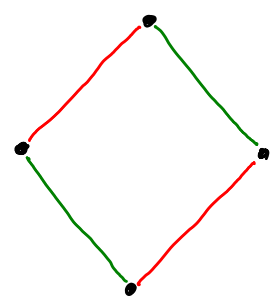
A distributive lattice is a partially ordered set $(L, \leq_L)$ such that for any two elements s and t in L

- s and t have a unique least upper bound $s \vee t$
- t and s have a unique greatest lower bound $s \wedge t$
- $\wedge$ distributes over $\vee$ and vice-versa:

$$r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$$

$$r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$$

An edge-colored distributive lattice is diamond-colored if opposite edges in "diamonds" have the same color:



Theorem [D, 2007]

Let $M = (M_{ij})_{i,j \in I_n}$ be a real matrix.

Let L be an M-structured diamond-colored distributive lattice with at least one edge of color i for each $i \in I_n$.

Then M is a generalized Cartan matrix.

(So by our previous result, (Γ, M) is a Dynkin diagram of finite type.)

NOTES:

- I know of no E_8-structured diamond-colored distributive lattice
- There are M-structured diamond-colored distributive lattices for all other connected Dynkin diagrams of finite type (Γ, M).
- For a Dynkin diagram $\mathfrak{g} = (\Gamma, M)$ and an M-structured diamond-colored distributive lattice L, let $\lambda := wt_L(\max)$.

  We say L is $(\mathfrak{g}, \lambda)$-structured.

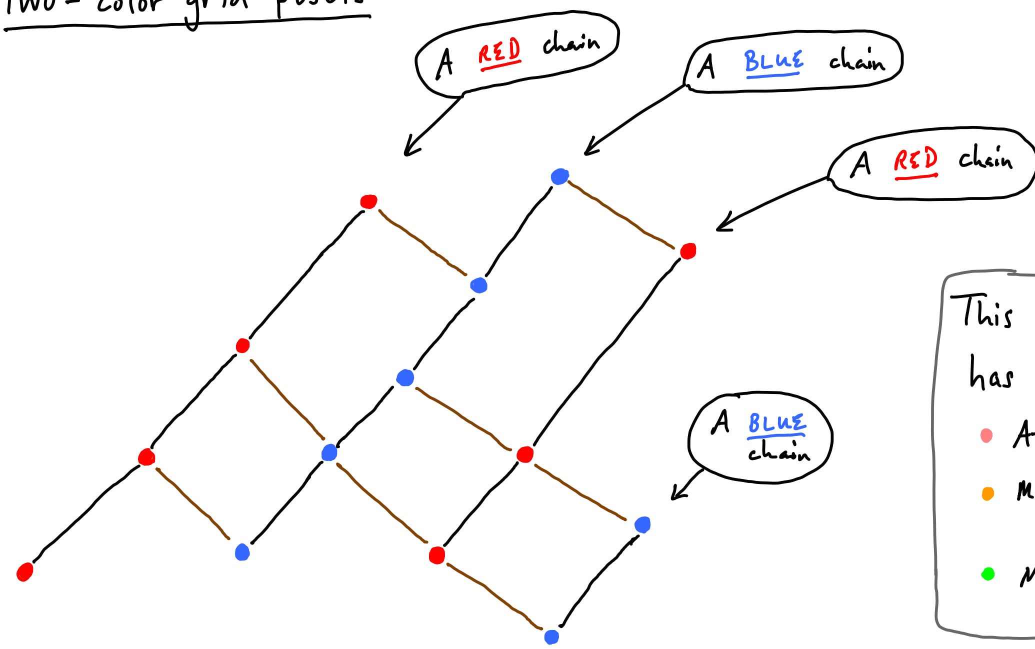
Motivation

Might help lead to combinatorial characterizations of posets and distributive lattices which have Weyl character / Lie representation theoretic significance.

Example Classifying those $(\mathfrak{g}, \lambda)$ for which there is a $(\mathfrak{g}, \lambda)$-structured distributive lattice would be a big step in classifying which "irreducible Weyl characters" have distributive lattice models.

Example A combinatorial characterization of semistandard lattices...

Two-color grid posets



This two-color grid poset has the "max property":

- At most two maximal elements
- Max elements are in the top two chains
- Max elements have different colors

"$\mathfrak{g}$-semistandard" lattices

For $\mathfrak{g} \in \{A_1 \oplus A_1, A_2, B_2, G_2\}$ and for any irreducible $\mathfrak{g}$-Weyl character χ_λ we⊛ have two $\mathfrak{g}$-structured diamond-colored distributive lattices $L_{\mathfrak{g}}^{(1)}(\lambda)$ and $L_{\mathfrak{g}}^{(2)}(\lambda)$ such that

$$\chi_\lambda = \sum_{t \in L} x_1^{\rho_1(t) - \delta_1(t)} x_2^{\rho_2(t) - \delta_2(t)}$$

⊛ L.W. Alverson, MSU, R.G. Donnelly, MSU, S.J. Lewis, MSU, M. McClard, UT, R. Pervine, MSU, R.A. Proctor, UNC, N.J. Wildberger, UNSW

Theorem [D.]

Let M be a 2×2 real matrix.

Let L be an M-structured diamond-colored distributive lattice.

The vertex-colored poset of irreducibles for L is isomorphic to a two-color grid poset with the max property

$\Longleftrightarrow$ L is a $\mathfrak{g}$-semistandard lattice for some $\mathfrak{g} \in \{A_1 \oplus A_1, A_2, B_2, G_2\}$.

edge-color isomorphic to