Lie groups and Lie algebras
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One of my goals for this series of talks has been to introduce you to the algebraic notion of a Lie algebra and (perhaps) to convince you that these are familiar and naturally occurring objects, whatever their other uses might be.

Another goal has been to share with you what Lie algebras might be "useful" for. Some uses, such as Lie algebra actions in the study of knot invariants, I know almost nothing about and will only mention in passing (this counts as the "mention in passing"). Other uses — such as the role played in the classification of finite simple groups by those finite simple groups arising as subgroups of the automorphism group ("specialized" to a finite field) of a simple Lie algebra — are not really in the vein of our discussion.

I hoped to demonstrate that there are natural combinatorial manifestations of Lie algebras realized as spaces of operators "acting on" some very beautiful (in the eyes of some beholders) finite partially ordered sets. In these notes I hope you will see that Lie algebras are useful for studying representations of Lie groups (which might be the most fundamental mathematical objects in the universe).

- $\boxed{\text{Formal definitions}}$ A *Lie group* G is a smooth (i.e. C^∞) n-manifold which is also endowed with a group structure for which the multiplication mapping $G \times G \to G$ (given by $(x, y) \mapsto xy$) and the inverse mapping $G \to G$ (given by $x \mapsto x^{-1}$) are smooth.

  A *topological n-manifold* is a 2nd countable Hausdorff topological space X that is locally Euclidean of dimension n, i.e. for any x in X there is an open neighborhood U and a mapping ϕ such that ϕ is a homeomorphism from U onto an open subset of $\mathbb{R}^n$.

  A *differentiable (or smooth) structure* on a topological n-manifold X is a collection $\mathcal{A} = \{(U_\alpha, \phi_\alpha) \,|\, \alpha \in A\}$ for some index set A and for which (1) $\bigcup_\alpha U_\alpha = X$, (2) each U_α is a connected open subset of X, (3) each ϕ_α is a homeomorphism from U_α onto some open subset of $\mathbb{R}^n$, (4) whenever $U_\alpha \cap U_\beta \neq \emptyset$ then $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is smooth, and (5) $\mathcal{A}$ is maximal in the sense that if U is any connected open subset of X with homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$ and such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha^{-1} \circ \phi$ are smooth mappings whenever $U \cap U_\alpha \neq \emptyset$, then $(U, \phi) \in \mathcal{A}$. The $(U_\alpha, \phi_\alpha)'$s are called *coordinate systems* or *coordinate charts* for X. The pair $(X, \mathcal{A})$ (and by abuse of notation just X) is called a *smooth n-manifold*.

  A *complex Lie group* is defined analogously, replacing "smooth manifold" with "complex manifold," smooth mappings with analytic mappings, etc.

- $\boxed{\text{Preliminary examples}}$

  1. Euclidean spaces:

     Consider $\mathbb{R}^n$ as an inner product space in the usual way, with topology generated by the open balls relative to the usual metric. Then $\mathbb{R}^n$ under vector addition is a Lie group.

     (Similarly, the complex inner product space $\mathbb{C}^n$ is a complex Lie group with the usual vector addition as the group operation.)

  2. *General linear groups*:

     Now identify $Mat_{n\times n}(\mathbb{R})$ with $\mathbb{R}^{n^2}$. The mapping $\det : Mat_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is smooth since it is n-linear. Then the set of invertible $n \times n$ matrices $GL(n, \mathbb{R})$ is $\det^{-1}(\mathbb{R} \setminus \{0\})$, hence is open.

In this way $GL(n, \mathbb{R})$ is a smooth n-manifold. Since each of the n^2 entries of a product matrix AB are polynomials in the entries of A and B, then multiplication is a smooth mapping. Since det is smooth, we could use Cramer's Rule to see that the inverse mapping is also smooth. Thus $GL(n, \mathbb{R})$ is a real n^2-dimensional Lie group. Similarly, we can see that $GL(n, \mathbb{C})$ is an n^2-dimensional complex Lie group.

The *general linear Lie groups* are $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$.

3. General principle: An abstract subgroup of a real Lie group G that is closed as a subset of G is a real Lie group relative to the subspace topology. This is not true in general if we replace the adjective "real" with "complex," as the next example shows.

   Thus in particular, since the normal subgroup $SL(n, \mathbb{R}) = \det^{-1}(\{1\}) = \ker(\det)$ is closed as a subset of $GL(n, \mathbb{R})$, it is a Lie group relative to the subspace topology.

4. The circle group $\mathbb{S}^1$:

   Let $\mathbb{S}^1 := \{e^{2\pi i x} \,|\, x \in \mathbb{R}\}$, the set of complex numbers of modulus 1. Then $\mathbb{S}^1$ is a subgroup of the multiplicative group $\mathbb{C}^* = GL(1, \mathbb{C})$. It is also closed as a subspace of $GL(1, \mathbb{C})$. Regarding $GL(1, \mathbb{C})$ to be a real Lie group, we see that $\mathbb{S}^1$ is a real Lie group relative to the subspace topology of $\mathbb{C}^*$. Notice that $\mathbb{S}^1$ is not a complex Lie group, however.

5. General principle: For any Lie group G, the connected component G_0 containing the identity is itself a Lie group. In fact, G_0 is normal as a subgroup of G.

   The general linear Lie group $GL(n, \mathbb{R})$ is not connected since its image in $\mathbb{R}^*$ under the continuous mapping $\det : GL(n, \mathbb{R}) \to \mathbb{R}^*$ is not connected. The connected component containing the identity matrix I is denoted $GL(n, \mathbb{R})^+$ and consists of all real $n \times n$ matrices with positive determinant.

- $\boxed{\text{Automorphism group of a bilinear form}}$

  Given an invertible matrix M in $GL(n, \mathbb{F})$, define

$$\begin{aligned} G_M &:= \{A \in GL(n, \mathbb{F}) \,\big|\, A^T M A = M\} \\ SG_M &:= \{A \in SL(n, \mathbb{F}) \,\big|\, A^T M A = M\} = G_M \cap SL(n, \mathbb{F}) \end{aligned}$$

  One can see that G_M and SG_M are abstract subgroups of $GL(n, \mathbb{F})$ of $SL(n, \mathbb{F})$ respectively. Use a sequence argument to see that each is closed as a subset of $GL(n, \mathbb{F})$. Since $\det(M) \neq 0$ and $\det(A^T) = \det(A)$, one can see that $\det(A) = \pm 1$ for any $A \in G_M$.

  Special cases:

1. $M = I$. Then G_M is the *orthogonal group* $O(n, \mathbb{F})$, and SG_M is the *special orthogonal group* $SO(n, \mathbb{F})$. The special orthogonal group is connected, and thus the orthogonal group has exactly two connected components. When $\mathbb{F} = \mathbb{R}$, one can see that $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ are bounded, hence compact. When $\mathbb{F} = \mathbb{C}$, one can see that neither $O(n, \mathbb{R})$ nor $SO(n, \mathbb{R})$ is bounded.

2. $M = M_{p,q} = \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix}$ with $\mathbb{F} = \mathbb{R}$. Then G_M is the *pseudo-orthogonal group* $O(p, q, \mathbb{R})$, and SG_M is the *special pseudo-orthogonal group* $SO(p, q, \mathbb{R})$. These are not compact if $p \geq 1$ and $q \geq 1$ since in this case it can be seen that $SO(p, q, \mathbb{R})$ is not bounded; in this case it also turns out that $SO(p, q, \mathbb{R})$ has exactly two connected components.

3. $M = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$. Then G_M and SG_M coincide (this fact is not obvious) and are the *symplectic group* $Sp(2n, \mathbb{F})$. This Lie group is not compact since it can be seen that $Sp(2n, \mathbb{F})$ is not bounded. The symplectic group is connected.

When $\mathbb{F} = \mathbb{C}$, we can similarly define the group of automorphisms of a nondegenerate Hermitian form:

$$\begin{aligned} G_M^* &:= \{A \in GL(n,\mathbb{F}) \,\big|\, A^*MA = M\} \\ SG_M^* &:= \{A \in SL(n,\mathbb{F}) \,\big|\, A^*MA = M\} = G_M \cap SL(n,\mathbb{F}) \end{aligned}$$

It is easy to see that G_M^* and SG_M^* are abstract subgroups of $GL(n,\mathbb{C})$ of $SL(n,\mathbb{C})$ respectively. Use a sequence argument to see that each is closed as a subset of $GL(n,\mathbb{C})$. Since $\det(M) \neq 0$, one can see that $|\det(A)| = 1$ for any $A \in G_M^*$. However, these turn out to be real, NOT complex, Lie groups.

1. $M = I$. Then G_M^* is the *unitary group* $U_n = U(n,\mathbb{C})$, and SG_M^* is the *special unitary group* $SU_n = SU(n,\mathbb{C})$. These are compact, connected Lie groups.
2. $M = M_{p,q}$. Then G_M^* is the *pseudo-unitary group* $U_{p,q} = U(p,q,\mathbb{C})$, and SG_M^* is the *special pseudo-unitary group* $SU_{p,q} = SU(p,q,\mathbb{C})$. For $p \geq 1$ and $q \geq 1$, I don't think these are compact or connected.

Real vs. Complex: Notice that $U(1,\mathbb{C}) \approx \mathbb{S}^1$, which is evidently a real but not a complex manifold. This suggests that $U(n,\mathbb{C})$ and $SU(n,\mathbb{C})$ might not be complex manifolds in general. There are several other clues. First, the associated Lie algebras $\mathfrak{u}(n,\mathbb{C})$ and $\mathfrak{su}(n,\mathbb{C})$ discussed previously and again below are real, not complex, Lie subalgebras of $\mathfrak{gl}(n,\mathbb{C})$. Second, conjugation $z \mapsto \overline{z}$ is well-known to be a real C^∞ but not a complex analytic transformation of the complex plane, and so the defining equations for $U(n,\mathbb{C})$ and $SU(n,\mathbb{C})$ are not complex analytic. Third, and decisively, a result about Lie groups says that a compact, connected, complex Lie group must be abelian.

- $\boxed{\text{The matrix exponential}}$

  Take $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. With $A \in \mathfrak{gl}(n,\mathbb{F})$, define the *matrix exponential function* by the rule

  $$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$$

  The following properties of the matrix exponential are fairly routine to justify (but #5 might be a little trickier). Here A and B are any $n \times n$ matrices in $\mathfrak{gl}(n,\mathbb{F})$:

  1. For any $P \in GL(n,\mathbb{F})$, $e^{P^{-1}AP} = P^{-1}e^AP$.
  2. $\det(e^A) = e^{\mathrm{trace}(A)}$
  3. $e^A \in GL(n,\mathbb{F})$. If $\mathbb{F} = \mathbb{R}$, then $e^A \in GL(n,\mathbb{R})^+$.
  4. $e^{(A^T)} = (e^A)^T$
  5. If $AB = BA$, then $e^{A+B} = e^Ae^B$.

  We will use the matrix exponential below to derive correspondences between Lie groups and Lie algebras.

- $\boxed{\text{Tangent space at the identity is a Lie algebra!}}$

  Now consider the tangent space $T_I(G)$ at the identity for one of the matrix groups we have seen so far. We'll think of these as real Lie groups right now. We identify a tangent vector as the derivative $f'(0)$ for a smooth curve $f : (-\epsilon,\epsilon) \to G$ for which $f(0) = I$.

  $\boxed{G = GL(n,\mathbb{F}) \longleftrightarrow \mathfrak{g} = \mathfrak{gl}(n,\mathbb{F})}$ Since $G = GL(n,\mathbb{F})$ is open as a subset of $\mathfrak{gl}(n,\mathbb{F})$, then we can make the identification of vector spaces $T_I(G) = \mathfrak{gl}(n,\mathbb{F})$.

$\boxed{G = SL(n,\mathbb{F}) \longleftrightarrow \mathfrak{g} = \mathfrak{sl}(n,\mathbb{F})}$ Now consider $G = SL(n,\mathbb{F})$. We will think of the determinant as an alternating n-linear form on $\bigwedge^n(\mathbb{F}^n)$. The kth exterior power $\bigwedge^k(\mathbb{F}^n)$ is an $\binom{n}{k}$-dimensional vector space over $\mathbb{F}$ with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \,\big|\, 1 \le i_1 < \cdots < i_k \le n\}$ where the wedge product is subject to the condition that

$$v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_k$$

for any $v_1, \ldots, v_k$ in $\mathbb{F}^n$. Then $\bigwedge^n(\mathbb{F}^n)$ is an one-dimensional vector space over $\mathbb{F}$ with basis $\{e_1 \wedge \cdots \wedge e_n\}$. If $A \in \mathfrak{gl}(n,\mathbb{F})$, then $\det(A)$ is the unique scalar such that $A^{(1)} \wedge \cdots \wedge A^{(n)} = \det(A) e_1 \wedge \cdots \wedge e_n$, where $A^{(i)}$ is the ith column of the matrix A.

So let $f : (-\epsilon, \epsilon) \to SL(n,\mathbb{F})$ be a smooth curve with $f(0) = I$ and $f'(0) = X$. Then we have $f(t)^{(1)} \wedge \cdots \wedge f(t)^{(n)} = e_1 \wedge \cdots \wedge e_n$ for all t. Differentiating, we obtain by the "product rule"

$$\sum_{i=1}^{n} f(t)^{(1)} \wedge \cdots \wedge f'(t)^{(i)} \wedge \cdots \wedge f(t)^{(n)} = 0,$$

which at $t = 0$ becomes

$$\sum_{i=1}^{n} X^{(1)} \wedge \cdots \wedge e_i \wedge \cdots \wedge X^{(n)} = 0,$$

which simplifies to

$$\sum_{i=1}^{n} X_{ii} = 0,$$

and hence $\operatorname{trace}(X) = 0$. Thus if $X \in T_I(SL(n,\mathbb{F}))$, then $\operatorname{trace}(X) = 0$.

Conversely suppose $X \in T_I(SL(n,\mathbb{F}))$. Then consider the smooth function $f : \mathbb{R} \to SL(n,\mathbb{F})$ given by $f(t) = e^{tX}$. Clearly $f(0) = I$ and $f'(0) = X$. Then $X \in \mathfrak{sl}(n,\mathbb{F})$.

$\boxed{G = SG_M \longleftrightarrow \mathfrak{g} = \mathfrak{g}_M}$ Here M is invertible. Consider a smooth curve $f : (-\epsilon, \epsilon) \to SG_M$ for which $f(0) = I$ and $f'(0) = X$. Taking the derivative of both sides of $f(t)^T M f(t) = M$ we get $f'(t)^T M f(t) + f(t)^T M f'(t) = O$, and at $t = 0$ this becomes $X^T M + MX = O$, and hence $X \in \mathfrak{g}_M$.

Conversely, if $X \in \mathfrak{g}_M$, then define $f : \mathbb{R} \to SG_M$ by $f(t) = e^{tX}$. Now M invertible implies $\mathfrak{g}_M \subset \mathfrak{sl}(n,\mathbb{F})$, and hence $\det(e^A) = 1$ for any $A \in \mathfrak{g}_M$. So indeed $f(t) \in SG_M$ for all time t. Then $f'(0) = X$, and hence $X \in T_I(SG_M)$. Thus,

$$\begin{aligned} G = SO(n,\mathbb{F}) \quad &\longleftrightarrow \quad \mathfrak{g} = \mathfrak{so}(n,\mathbb{F}) \\ G = Sp(n,\mathbb{F}) \quad &\longleftrightarrow \quad \mathfrak{g} = \mathfrak{sp}(n,\mathbb{F}) \\ G = SO(p,q,\mathbb{R}) \quad &\longleftrightarrow \quad \mathfrak{g} = \mathfrak{so}(p,q,\mathbb{R}) \end{aligned}$$

$\boxed{\text{Unitary and pseudo-unitary groups}}$ As in the previous cases one can check that the unitary and pseudo-unitary groups $U_n = U(n,\mathbb{C})$, $SU_n = SU(n,\mathbb{C})$, $U_{p,q} = U(p,q,\mathbb{C})$, and $SU_{p,q} = SU(p,q,\mathbb{C})$ have real Lie algebras $\mathfrak{u}_n = \mathfrak{u}(n,\mathbb{C})$, $\mathfrak{su}_n = \mathfrak{su}(n,\mathbb{C})$, $\mathfrak{u}_{p,q} = \mathfrak{u}(p,q,\mathbb{C})$, and $\mathfrak{su}_{p,q} = \mathfrak{su}(p,q,\mathbb{C})$ respectively.

- $\boxed{\text{More examples}}$

  1. *The pseudo-orthogonal groups* $O(1,1,\mathbb{R})$ *and* $SO(1,1,\mathbb{R})$.

     First, $SO(1,1,\mathbb{R})$ has precisely two connected components. Set

$$SO(1,1,\mathbb{R})^+ := \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}, SO(1,1,\mathbb{R})^- := \left\{ \begin{pmatrix} -\cosh t & \sinh t \\ \sinh t & -\cosh t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

It is not too hard to check that

$$SO(1,1,\mathbb{R}) = SO(1,1,\mathbb{R})^+ \cup SO(1,1,\mathbb{R})^-$$

(Check that the minimum distance between elements of the two sets is positive.) The only other coset of $SO(1,1,\mathbb{R})$ in $O(1,1,\mathbb{R})$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} SO(1,1,\mathbb{R}) = \left\{ \begin{pmatrix} \cosh t & -\sinh t \\ \sinh t & -\cosh t \end{pmatrix} \right\} \bigcup \left\{ \begin{pmatrix} -\cosh t & -\sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\}$$

Since $SO(1,1,\mathbb{R})$ is an index two subgroup of $O(1,1,\mathbb{R})$, it is therefore a normal subgroup. Also, clearly $SO(1,1,\mathbb{R})$ is not bounded, and hence not compact. (Thanks to Tan Zhang for suggesting this example.)

2. *Irrational line on the torus.*

   We think of the torus as $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, a Lie group relative to the product topology. Consider the subgroup $H = \{(e^{2\pi i t}, e^{2\pi i \alpha t}) \,|\, t \in \mathbb{R}\}$, where α is some given irrational number. Then H is dense in $\mathbb{T}$, and is therefore not a topologically closed subset of $\mathbb{T}$. Moreover, H is not a Lie group relative to the subspace topology. However, H is an *immersed* Lie subgroup of $\mathbb{T}$, as an immersion of the Lie group $(\mathbb{R},+)$ in $\mathbb{T}$ via the mapping $f : (\mathbb{R},+) \to \mathbb{T}$ with $f(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$.

3. *The Heisenberg group.*

   Consider the group G of *unipotent* 3×3 matrices. We have

$$G := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\Big|\, a, b, c \in \mathbb{R} \right\}$$

   This is topologically closed as a subgroup of $GL(3,\mathbb{R})$. Hence G is a Lie group relative to the subspace topology; it is the three-dimesnional Heisenberg group (there are higher dimension analogs). The center of G is

$$Z(G) := \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\Big|\, b \in \mathbb{R} \right\}$$

   Identify $\mathbb{Z}$ as the subgroup of $Z(G)$ given by taking $b \in \mathbb{Z}$. Then $\mathbb{Z}$ is a discrete subgroup of the center. As it turns out, the quotient $G/\mathbb{Z}$ does not have a faithful finite-dimensional representation as a subgroup of some general linear group $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$.

- $\boxed{\text{A general picture for the study of Lie group/Lie algebra representations}}$

  A *(complex) representation* of a (real or complex) Lie group G is a Lie group homomorphism $\phi : G \to GL(n,\mathbb{C})$. A *(complex) representation* of a (real or complex) Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(n,\mathbb{C})$.

  The general theory (some key results are outlined on the following pages) gives us the following picture that helps connect the study of representations of Lie groups and representations of complex Lie algebras:

$$\begin{tikzcd}
G \arrow[dr, "\phi"'] & \tilde{G} \arrow[l, "\pi"'] \arrow[r, leftrightarrow, "\exp, \log"] \arrow[d, "\psi"] & \mathfrak{g} \arrow[r, "j"] \arrow[d, "d\psi"] & \mathfrak{g}_{\mathbb{C}} \arrow[dl, "(d\psi)_{\mathbb{C}} = \rho"] \\
 & GL(n,\mathbb{C}) \arrow[r, leftrightarrow, "\exp, \log"] & \mathfrak{gl}(n,\mathbb{C}) &
\end{tikzcd}$$

Here G is a connected Lie group, $\phi : G \to GL(n, \mathbb{C})$ is a representation of G, and $\tilde{G}$ is the simply connected form for G. So $\tilde{G}$ is simply connected, and G and $\tilde{G}$ share the same Lie algebra $\mathfrak{g}$.

- $\boxed{\text{Determinants and the one-dimensional real representations of the general linear group } GL(n, \mathbb{R})^+}$

  Our goal is to describe the one-dimensional real representations $\phi : GL(n, \mathbb{R})^+ \to GL(1, \mathbb{R})$. It is sufficient to assume that the mapping ϕ is continuous since this implies, by a general principle about Lie groups, that the mapping is in fact smooth. In this case, observe that the image of ϕ must be a connected subset of $GL(1, \mathbb{R})$, and hence must reside in $GL(1, \mathbb{R})^+$. For convenience we'll let $\mathbb{R}^+ \approx GL(1, \mathbb{R})^+$ denote the multiplicative group of positive real numbers. So, our aim is to find all continuous homomorphisms $\phi : GL(n, \mathbb{R})^+ \to \mathbb{R}^+$.

  We start with what is probably a familiar fact: Suppose $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous homomorphism. Then there exists a real number a such that $\phi(x) = x^a$. To see this, use a standard "bootstrapping" argument. Notice that this nicely illustrates the general principle that a continuous homomorphism of Lie groups is necessarily smooth. Thus we've resolved our question in the case that $n = 1$.

  Next we consider the continuous homomorphisms $\phi : SL(n, \mathbb{R}) \to \mathbb{R}^+$. Since $SL(n, \mathbb{R})$ is connected and the continuous mapping ϕ is necessarily smooth, then by general principles this mapping is uniquely determined by its differential $d\phi : \mathfrak{sl}(n, \mathbb{R}) \to \mathbb{R}$, which is a mapping of Lie algebras where the target set here is the one-dimensional real Lie algebra. In a previous talk we asserted that any one-dimensional Lie algebra is abelian. We also learned that the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is simple, so that $[\mathfrak{sl}(n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{R})] = \mathfrak{sl}(n, \mathbb{R})$. In particular, it must be the case that $d\phi$ is trivial. But now this implies that ϕ is trivial. We conclude that $\phi(X) = 1$ for all X in $SL(n, \mathbb{R})$.

  Now we consider continuous homomorphisms $\phi : GL(n, \mathbb{R})^+ \to \mathbb{R}^+$. Since $SL(n, \mathbb{R}) \subseteq \ker(\phi)$, then there is an induced continuous homomorphism $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ that makes the following diagram commute:

$$\begin{tikzcd} GL(n, \mathbb{R})^+ \arrow[r, "\det"] \arrow[dr, "\phi"'] & \mathbb{R}^+ \arrow[d, "\psi"] \\ & \mathbb{R}^+ \end{tikzcd}$$

  But we already know that ψ must take the form of a power function $\psi(x) = x^a$ for some real exponent a. Thus, $\phi(X) = (\det(X))^a$. We summarize this as follows:

  If $\phi : GL(n, \mathbb{R})^+ \to \mathbb{R}^+$ is a continuous homomorphism, then there is a real number a such that

$$\phi(X) = (\det(X))^a$$

Some data for real Lie groups

Real Lie group	G_0, connected component of identity	$\pi_1(G_0)$, the fundamental group	Closed subgroup of appropriate $GL(n,\mathbb{F})$?	Compact?
$GL(1,\mathbb{R}) \overset{\text{Lie grp}}{\approx} \mathbb{R}^*$	$(\mathbb{R}^+,\cdot)$	$\pi_1(\mathbb{R}^+) \approx \{0\}$	Open subset of $\mathfrak{gl}(1,\mathbb{R}) \overset{\text{diffeo}}{\approx} \mathbb{R}$	No
$GL(2,\mathbb{R})$	$GL(2,\mathbb{R})^+$	$\pi_1(GL(2,\mathbb{R})^+) \approx \mathbb{Z}$	Open subset of $\mathfrak{gl}(2,\mathbb{R}) \overset{\text{diffeo}}{\approx} \mathbb{R}^4$	No
$GL(n,\mathbb{R}), n \geq 3$	$GL(n,\mathbb{R})^+$	$\pi_1(GL(n,\mathbb{R})^+) \approx \mathbb{Z}_2$	Open subset of $\mathfrak{gl}(n,\mathbb{R}) \overset{\text{diffeo}}{\approx} \mathbb{R}^{n^2}$	No
$SL(1,\mathbb{R}) \overset{\text{Lie grp}}{\approx} \{1\}$	$SL(1,\mathbb{R})$	$\{0\}$	Yes	No
$SL(2,\mathbb{R})$	$SL(2,\mathbb{R})$	$\mathbb{Z}$	Yes	No
$SL(n,\mathbb{R}), n \geq 3$	$SL(n,\mathbb{R})$	$\mathbb{Z}_2$	Yes	No
$SO(1,\mathbb{R}) \overset{\text{Lie grp}}{\approx} \{1\}$	$\{1\}$	$\{0\}$	Yes	Yes
$SO(2,\mathbb{R}) \overset{\text{Lie grp}}{\approx} \mathbb{S}^1$	$SO(2,\mathbb{R})$	$\mathbb{Z}$	Yes	Yes
$SO(n,\mathbb{R}), n \geq 3$	$SO(n,\mathbb{R})$	$\mathbb{Z}_2$	Yes	Yes
$Spin(n,\mathbb{R}), n \geq 3$	$Spin(n,\mathbb{R})$	$\{0\}$	Yes	Yes
$O(1,\mathbb{R}) \overset{\text{Lie grp}}{\approx} \{\pm 1\}$	$\{1\}$	$\pi_1(\{1\}) \approx \{0\}$	Yes	Yes
$O(2,\mathbb{R}) \overset{\text{Lie grp}}{\approx} SO(2,\mathbb{R}) \times_{\phi_R} \{I,R\}$	$SO(2,\mathbb{R})$	$\pi_1(SO(2,\mathbb{R})) \approx \mathbb{Z}$	Yes	Yes
$O(n,\mathbb{R}) \overset{\text{Lie grp}}{\approx} SO(n,\mathbb{R}) \times_{\phi_R} \{I,R\}$ $n \geq 3$	$SO(n,\mathbb{R})$	$\pi_1(SO(n,\mathbb{R})) \approx \mathbb{Z}_2$	Yes	Yes
$Sp(2n,\mathbb{R})$	$Sp(2n,\mathbb{R})$	$\mathbb{Z}$	Yes	No
$SO(p,q,\mathbb{R}) \overset{\text{Lie grp}}{\approx} SO(q,p,\mathbb{R})$ $p,q \geq 1$	$SO(p,q,\mathbb{R})^+$	???	Yes	No
$SU(n,\mathbb{C})$	$SU(n,\mathbb{C})$	$\{0\}$	Yes	Yes
$U(n,\mathbb{C}) \overset{\text{diffeo}}{\approx} SU(n,\mathbb{C}) \times \mathbb{S}^1$	$U(n,\mathbb{C})$	$\mathbb{Z}$	Yes	Yes

NOTES:

- In the above table, the matrix R is any matrix from the appropriate orthogonal group such that $R^2 = I$ and $\det(R) = -1$.
- I don't believe that the simply connected forms for $SL(n,\mathbb{R})$ and $GL(n,\mathbb{R})^+$ have faithful finite-dimensional representations. However, $Spin(n,\mathbb{R})$ does.

Some data for complex Lie groups

Complex Lie group	G_0, connected component of identity	$\pi_1(G_0)$, the fundamental group	Closed subgroup of appropriate $GL(n, \mathbb{F})$?	Compact?
$GL(n, \mathbb{C})$	$GL(n, \mathbb{C})$	$\mathbb{Z}$	Open subset of $\mathfrak{gl}(n, \mathbb{C}) \overset{\text{diffeo}}{\approx} \mathbb{C}^{n^2}$	No
$SL(n, \mathbb{C})$	$SL(n, \mathbb{C})$	$\{0\}$	Yes	No
$SO(1, \mathbb{C}) \overset{\text{Lie grp}}{\approx} \{1\}$	$\{1\}$	$\{0\}$	Yes	Yes
$SO(2, \mathbb{C}) \overset{\text{Lie grp}}{\approx} \mathbb{C}^*$	$SO(2, \mathbb{C})$	$\mathbb{Z}$	Yes	No
$SO(n, \mathbb{C}), n \geq 3$	$SO(n, \mathbb{C})$	$\mathbb{Z}_2$	Yes	No
$Spin(n, \mathbb{C}), n \geq 3$	$Spin(n, \mathbb{C})$	$\{0\}$	Yes	No
$O(1, \mathbb{C}) \overset{\text{Lie grp}}{\approx} \{\pm 1\}$	$\{1\}$	$\pi_1(\{1\}) \approx \{0\}$	Yes	Yes
$O(2, \mathbb{C}) \overset{\text{Lie grp}}{\approx} SO(2, \mathbb{C}) \times_{\phi_R} \{I, R\}$	$SO(2, \mathbb{C})$	$\pi_1(SO(2, \mathbb{C})) \approx \mathbb{Z}$	Yes	No
$O(n, \mathbb{C}) \overset{\text{Lie grp}}{\approx} SO(n, \mathbb{C}) \times_{\phi_R} \{I, R\}$ $n \geq 3$	$SO(n, \mathbb{C})$	$\pi_1(SO(n, \mathbb{C})) \approx \mathbb{Z}_2$	Yes	No
$Sp(2n, \mathbb{C})$	$Sp(2n, \mathbb{C})$	$\{0\}$	Yes	No

NOTES:

- In the above table, the matrix R is any matrix from the appropriate orthogonal group such that $R^2 = I$ and $\det(R) = -1$.