“Mars Attacks” and some classification problems

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Goals

- Show you a simple combinatorial game.
- Answer a "finiteness" question about the game.
- Say how the game and this answer are connected to certain algebraic structures called "Kac-Moody Lie algebras" and "Weyl groups."

Questions

1. For which graphs, for which initial assignments of numbers to the nodes, and for which sequences of firings will the process terminate in a finite number of steps?
2. Is there a *good* science fiction narrative we could use to interpret this process?

Mars Attacks

The Year: 20,006. Earth learns that Martians are planning to attack.

Martians have two weapons:

Atmospheric Disruptor (This will irradiate the atmosphere, making it poisonous to humans)

The "Mutinator" (A cyborg who infiltrates the population and turns humans into mutants)

Earth strategizes a defense:

1. Enclose cities, and connect as many of them as possible by enclosed passageways, or "conduits." (But, time is limited!!)

2. Distribute a "scrambling" device to all earthlings. (This will use the cyborg's energy to clone the person just before he or she is "mutinated.")

GCM graphs

Let Γ be a finite simple graph (no loops, no multiple edges) and with nodes indexed by a set N.

Assign *amplitudes* to pairs of nodes (i, j) from Γ according to the rules:

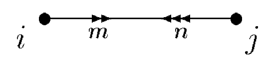
$$a_{i,j} = \begin{cases} \text{negative integer} & \text{if } i \text{ and } j \text{ are adjacent nodes in } \Gamma \\ 2 & \text{if } i = j \\ 0 & \text{if } i \text{ and } j \text{ are not adjacent} \end{cases}$$

The amplitude matrix $A = (a_{i,j})$ is a *Generalized Cartan Matrix (GCM)*.

We call the pair (Γ, A) a *GCM graph*.

GCM graphs, continued

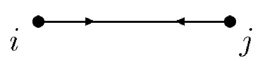
A typical connected two-node GCM graph:



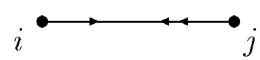
In this graph, $m = |a_{i,j}|$ and $n = |a_{j,i}|$.

Some special GCM graphs (for each, $m = 1$ and $n = 1$, 2, or 3):

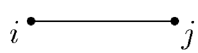
A_2



C_2



G_2

We also use the graph 

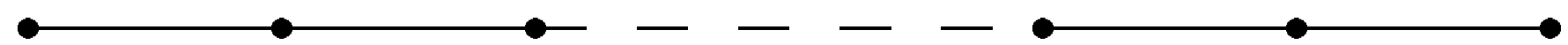
to represent the GCM graph A_2.

We use $A_1 \times A_1$ to denote the disconnected two-node GCM graph.

A GCM graph (Γ, A) is a *Dynkin diagram* if each connected component of (Γ, A) is one of the graphs that follows. . .

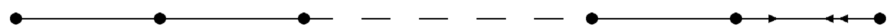
Connected Dynkin diagrams

A_n $(n \geq 1)$

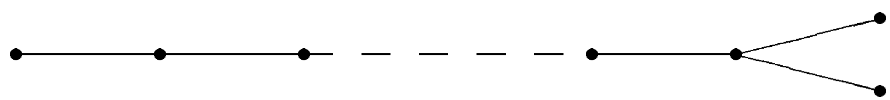


B_n $(n \geq 3)$

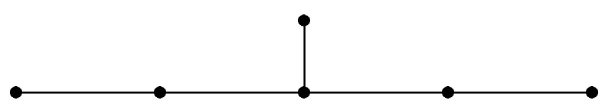
C_n $(n \geq 2)$



D_n $(n \geq 4)$

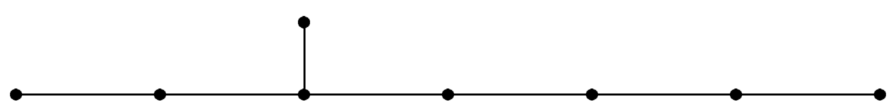


E_6

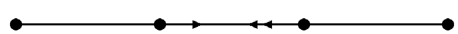


E_7

E_8



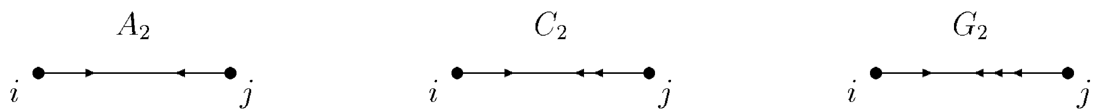
F_4



G_2

Some amplitude matrices (GCM′s)

The amplitude matrices for the two-node Dynkin diagrams



plus $A_1 \times A_1$ are:

$A_1 \times A_1$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	A_2	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
C_2	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	G_2	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Populations and firings

A *population* $\lambda = (\lambda_i)_{i \in N}$ is an assignment of integers to the nodes of the GCM graph (Γ, A); the number λ_i is the *population at node* i.

The population λ is *nonzero* if at least one $\lambda_i \neq 0$. It is *dominant* if all $\lambda_i \geq 0$.

For $i \in N$, the *fundamental population* ω_i is the assignment of population 1 at node i and population 0 at all other nodes.

Given a population λ for a GCM graph (Γ, A), to *fire at node* i is to change the population at each node j of Γ by the transformation

$$\lambda_j \longmapsto \lambda_j - a_{i,j}\lambda_i,$$

provided the population at node i is positive; otherwise node i cannot be fired.

Mars attacks

Mars attacks is a one-player game in which a player fires at the nodes of the GCM graph (Γ, A) in some sequence, given some initial population.

Specifically, to play Mars attacks on (Γ, A) a player must:

(1) Assign a nonzero dominant population (the *initial population*) to the nodes of Γ;

(2) Choose a node with a positive population and fire the node to obtain a new population;

(3) Repeat step (2) for the new population if there is at least one node with a positive population.

A GCM graph is *Mars-friendly* if there is an initial population (nonzero, dominant) for which the game terminates after a finite number of firings, i.e. there are no nodes with positive populations to fire.

The Mars Attacks Problem: Find all Mars-friendly GCM graphs

Progress on the problem

In an undergraduate research project, John Eveland (B.S. MSU May 2001) used the following algorithm to obtain concrete data useful for the study of Lie algebra representations: Given a GCM graph (Γ, A) and dominant population λ, "iterate" the following "transformation":

$$\lambda \longmapsto \{\lambda - A_i, \lambda - 2A_i, \ldots, \lambda - \lambda_i A_i\}$$

where A_j is the jth row vector of the matrix A. Work on this procedure led to an early version of the Mars Attacks Problem.

In 2001, I learned that the transformation $\lambda \longmapsto \lambda - \lambda_i A_i$ defines the "Numbers Game" studied by Shahar Mozes (1990) and independently by Bob Proctor (1984) and Norman Wildberger (2000).

In 2001-04, it became clear that data from the Mars Attacks version of the Numbers Game was useful for producing certain combinatorial models of Lie algebra representations.

In 2005, I found results in papers by Kimmo Eriksson (1995, 1996) which led to the following solution of the Mars Attacks Problem.

The Mars Attacks Theorem

Theorem (D., 2005) *A connected GCM graph* (Γ, A) *is Mars-friendly if and only if* (Γ, A) *is a connected Dynkin diagram.*

NOTES:

- I've known the two-node version of this theorem since 2002 — it can actually be found in one of Eriksson's papers. I needed the two-node version to solve a poset classification problem I talked about at last year's meeting.

- The requirement that populations be *integers* can be relaxed to *real numbers*.

- For a connected Dynkin diagram, any Mars attacks game will terminate in a finite number of steps. In particular, finite termination does not depend on the choice of initial population or on the choice of the firing sequence.

- What follows is a discussion of the "only if" part of the proof. The proof is by induction on the number of nodes. We state some of the key results that make the proof work.

Eriksson's results

STRONG CONVERGENCE

Theorem (Eriksson, 1996) *The Mars Attacks game on a GCM graph is* strongly convergent, *which means that if a firing sequence for a given initial population λ converges to some terminal population, then all firing sequences for λ will converge to the same terminal population in the same number of steps.*

Corollary *In any GCM graph, if a firing sequence for an initial population λ diverges, then all firing sequences for λ diverge.*

Eriksson's results, continued

COMPARISON

Theorem (Eriksson, 1995) *Suppose that a firing sequence for an initial population* $\lambda = (\lambda_i)_{i \in N}$ *converges. Suppose that an initial population* $\lambda' := (\lambda'_i)_{i \in N}$ *has the property that* $\lambda'_i \leq \lambda_i$ *for all* $i \in N$*. Then a firing sequence for* λ' *also converges.*

Corollary *Suppose that a firing sequence for a dominant initial population* $\lambda = (\lambda_i)_{i \in N}$ *converges. For some* $i \in N$*, suppose that* $\lambda_i > 0$*. Then a firing sequence for the fundamental population* ω_i *also converges.*

A CONSEQUENCE OF STRONG CONVERGENCE AND COMPARISON

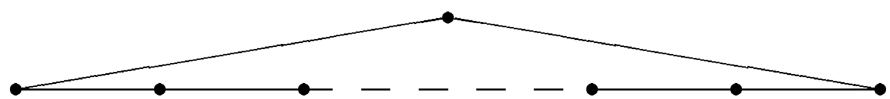
Corollary *A GCM graph is not Mars-friendly if and only if for each fundamental population there is a divergent firing sequence.*

Not Mars-friendly

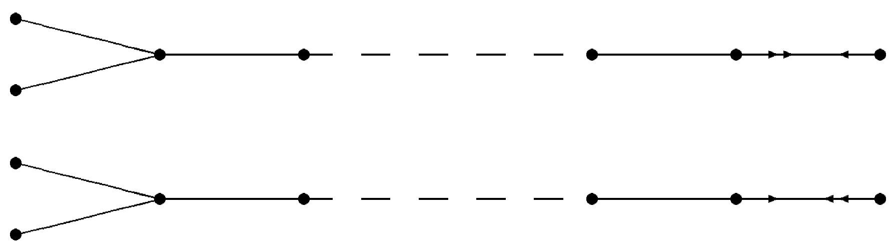
SOME GCM GRAPHS THAT ARE NOT MARS-FRIENDLY

Lemma *The following connected GCM graphs are not Mars-friendly:*

The "$\widetilde{A}$" family of GCM graphs

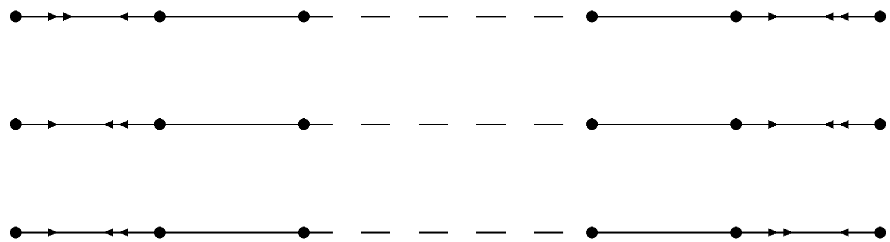


The "$\widetilde{B}$" family of GCM graphs

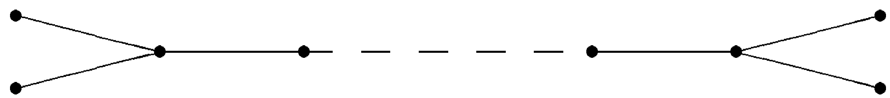


Not Mars-friendly, continued

The "$\widetilde{C}$" family of GCM graphs

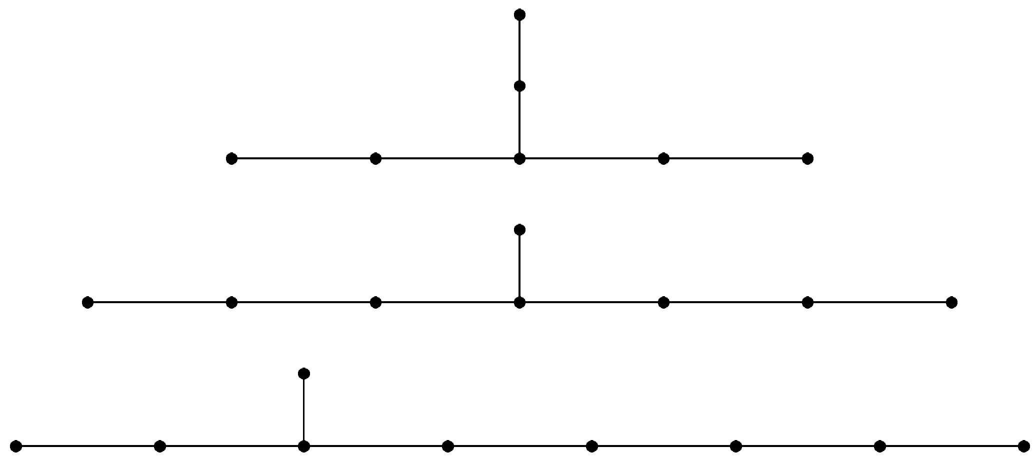


The "$\widetilde{D}$" family of GCM graphs



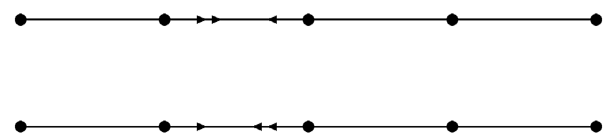
Not Mars-friendly, continued

The "$\widetilde{E}$" family of GCM graphs

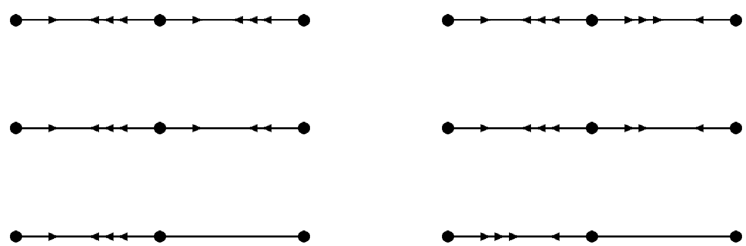


Not Mars-friendly, continued

The "$\widetilde{F}$" family of GCM graphs

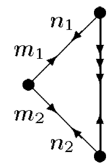


The "$\widetilde{G}$" family of GCM graphs

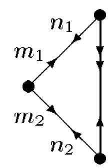


Not Mars-friendly, continued

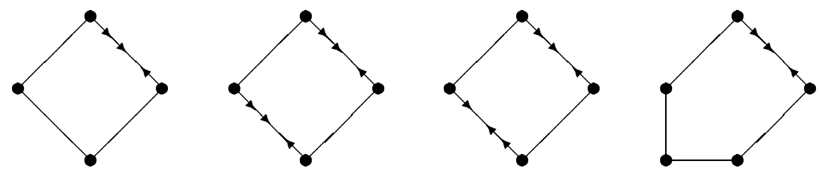
Families of small cycles



$m_1 n_1 \leq 3, m_2 n_2 \leq 3$



$m_1 n_1 \leq 2, m_2 n_2 \leq 2$



Other reductions

EVERY NODE IS FIRED

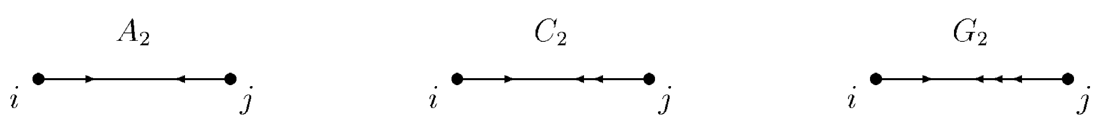
Lemma *For a connected GCM graph, in any firing sequence for a given initial population λ, every node is fired at least once.*

SUBGRAPHS OF MARS-FRIENDLY GRAPHS ARE MARS-FRIENDLY

Lemma *If a connected GCM graph is Mars-friendly, then any connected GCM subgraph is also Mars-friendly.*

CONNECTED TWO-NODE SUBGRAPHS ARE A_2, C_2, OR G_2

Lemma (Eriksson, 1996) *In a connected Mars-friendly graph, suppose i and j are adjacent nodes with $|a_{i,j}| \leq |a_{j,i}|$. Then the amplitude product $a_{i,j}a_{j,i}$ is 1, 2, or 3, and the GCM subgraph with nodes i and j is one of*



Weyl groups

The <u>*Weyl group*</u> W associated to (Γ, A) is the quotient of the free group generated by $\{S_i\}_{i\in N}$ modulo the relations (W1) and (W2):

(W1) $S_i^2 = 1$ for all i, and

(W2) $(S_iS_j)^{m_{i,j}} = 1$ whenever $i \neq j$, where

$$m_{i,j} = \begin{cases} 2 & \text{if } a_{i,j}a_{j,i} = 0 \\ 3 & \text{if } a_{i,j}a_{j,i} = 1 \\ 4 & \text{if } a_{i,j}a_{j,i} = 2 \\ 6 & \text{if } a_{i,j}a_{j,i} = 3 \\ \infty & \text{if } a_{i,j}a_{j,i} \geq 4 \end{cases}$$

Finite Weyl groups and finite-dimensional Kac-Moody Lie algebras

Theorem (Eriksson, 1995) *Form a word* $w = S_{i_k} \cdots S_{i_1}$ *in* W *of Weyl group generators from the sequence of the first* k *nodes* $(i_1, \ldots, i_k)$ *that are fired in a Mars attacks game on a connected GCM graph* (Γ, A). *Then* w *is a reduced word.*

Corollary *The Weyl group* W *for a connected GCM graph* (Γ, A) *is finite only if* (Γ, A) *is Mars-friendly.*

Proof. Previous theorem plus the Mars Attacks Theorem.

Corollary *The Kac-Moody Lie algebra* $\mathfrak{g}$ *associated to a connected GCM graph* (Γ, A) *is finite-dimensional only if* (Γ, A) *is Mars-friendly.*

Proof. A theorem due to Kac says that W is finite if and only if $\mathfrak{g}$ is finite-dimensional.