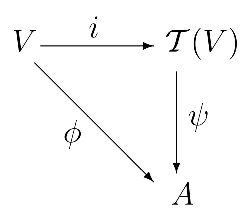
Generators and Relations
Rob Donnelly March 17, 2006

In our last meeting there was some discussion about what was meant by "generators and relations" in the context of Lie algebras. Here I'll give you my understanding of the apparatus that makes this idea sound mathematically, and then I'll introduce to you a large class of Lie algebras defined by generators and relations: the Kac-Moody Lie algebras.

To begin, here are some algebras that are characterized by universal mapping properties. It should be kept in mind that any associative $\mathbb{F}$-algebra A can be made into a Lie algebra by defining the bracket $[x, y] := xy - yx$ for all $x, y \in A$.

The Tensor Algebra

Let V be an $\mathbb{F}$-vector space. Then there is a unique pair $(\mathcal{T}(V), i)$ such that $\mathcal{T}(V)$ is an associative $\mathbb{F}$-algebra with unity, $i : V \to \mathcal{T}(V)$ is an $\mathbb{F}$-linear mapping, and which satisfies the following universal mapping property: If $\phi : V \to A$ is any $\mathbb{F}$-linear mapping from V to an associative $\mathbb{F}$-algebra A with unity, then there exists a unique $\mathbb{F}$-algebra homomorphism $\psi : \mathcal{T}(V) \to A$ such that $\psi(1) = 1$ and the following diagram commutes:



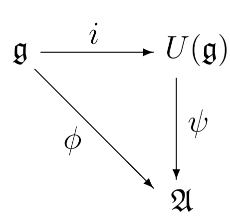
That is, $\phi = \psi \circ i$. The pair $(\mathcal{T}(V), i)$ is the *tensor algebra on* V. Uniqueness of the pair $(\mathcal{T}(V), i)$ is established in the usual way any time you have a universal mapping property around. A construction of the tensor algebra using an associative product defined on the space $\mathcal{T}(V) = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$ actually shows that the mapping i is injective. For simple tensors $v_1 \otimes \cdots \otimes v_j$ and $w_1 \otimes \cdots \otimes w_k$, then

$$(v_1 \otimes \cdots \otimes v_j)(w_1 \otimes \cdots \otimes w_k) \stackrel{\text{(def)}}{=} v_1 \otimes \cdots \otimes v_j \otimes w_1 \otimes \cdots \otimes w_k.$$

Extend bilinearly to get the associative product on $\mathcal{T}(V)$.

The Universal Enveloping Algebra

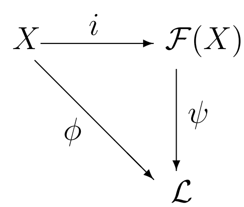
Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then there exists a unique pair $(U(\mathfrak{g}), i)$ such that $U(\mathfrak{g})$ is an associative $\mathbb{F}$-algebra with unity, $i : \mathfrak{g} \to U(\mathfrak{g})$ is a Lie algebra homomorphism, and which satisfies the following universal mapping property: If $\phi : \mathfrak{g} \to \mathfrak{A}$ is any Lie algebra mapping from $\mathfrak{g}$ to an associative $\mathbb{F}$-algebra $\mathfrak{A}$ with unity, then there exists a unique $\mathbb{F}$-algebra homomorphism $\psi : U(\mathfrak{g}) \to \mathfrak{A}$ such that $\psi(1) = 1$ and the following diagram commutes:



That is, $\phi = \psi \circ i$. The pair $(U(\mathfrak{g}), i)$ is the *universal enveloping algebra of* $\mathfrak{g}$. Uniqueness of the pair $(U(\mathfrak{g}), i)$ is established in the usual way. If J is the two-sided (ring theoretic) ideal in $\mathcal{T}(\mathfrak{g})$ generated by all $x \otimes y - y \otimes x - [x, y]$ $(x, y \in \mathfrak{g})$, then $U(\mathfrak{g})$ can be identified with $\mathcal{T}(\mathfrak{g})/J$ and i with the projection map. It is a consequence of the famous **Poincaré-Birkhoff-Witt Theorem** that the mapping i is injective.

Free Lie Algebras

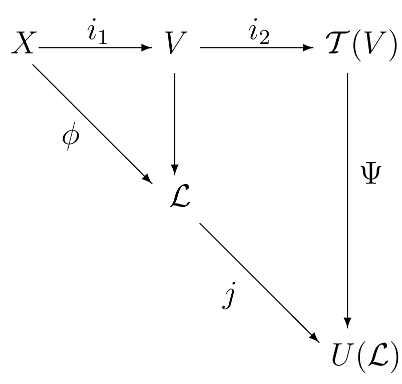
Let $X = \{x_\alpha\}_{\alpha \in A}$ be any set. Then there exists a unique pair $(\mathcal{F}(X), i)$ such that $\mathcal{F}(X)$ is a Lie algebra over $\mathbb{F}$, $i : X \to \mathcal{F}(X)$ is a mapping of sets, and which satisfies the following universal mapping property: If $\phi : X \to \mathcal{L}$ is any set mapping from X to a Lie algebra $\mathcal{L}$ over $\mathbb{F}$, then there exists a unique Lie algebra homomorphism $\psi : \mathcal{F}(X) \to \mathcal{L}$ such that the following diagram commutes:



That is, $\phi = \psi \circ i$. The pair $(\mathcal{F}(X), i)$ is the *free Lie algebra for* X. Uniqueness of the pair $(\mathcal{F}(X), i)$ is established in the usual way. It is a consequence of the construction of the pair $(\mathcal{F}(X), i)$ that the mapping i is injective.

Construction of $(\mathcal{F}(X), i)$: Let V be the $\mathbb{F}$-vector space freely generated by X. Then consider the tensor algebra $\mathcal{T}(V)$. Let $\mathcal{F}(X)$ be the Lie subalgebra of $\mathcal{T}(V)$ (thought of as a Lie algebra) generated by X. We have injections $X \xrightarrow{i_1} V \xrightarrow{i_2} \mathcal{T}(V)$, and the image of X in $\mathcal{T}(V)$ is in $\mathcal{F}(X)$. Let $i = i_2 \circ i_1$.

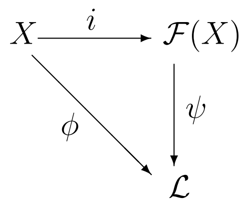
Existence of ψ: Given $\phi : X \to \mathcal{L}$. Let $(U(\mathcal{L}), j)$ be the universal enveloping algebra of $\mathcal{L}$. Then in the following diagram we have the mapping $\Psi : \mathcal{T}(V) \to U(\mathcal{L})$ of $\mathbb{F}$-algebras induced by universal mapping properties:



Now let $\mathcal{K}$ be the Lie subalgebra of $\mathcal{L}$ generated by $\phi(X)$. By thinking about how elements of $\mathcal{F}(X)$ and $\mathcal{K}$ are obtained and by considering the above diagram, one sees that $\Psi(\mathcal{F}(X)) = j(\mathcal{K})$. By the Poincaré-Birkhoff-Witt Theorem we can identify $\mathcal{L}$ with $j(\mathcal{L})$. Now if we set

$$\psi := \Psi|_{\mathcal{F}(X)} : \mathcal{F}(X) \longrightarrow j(\mathcal{L}) \approx \mathcal{L},$$

then the following diagram commutes:



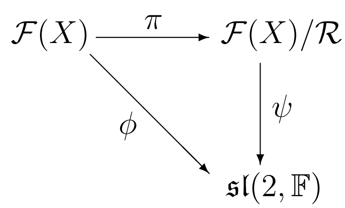
Uniqueness of ψ: Since X generates $\mathcal{F}(X)$, then knowledge of $j(\phi(x_\alpha))$ fixes $\psi(x_\alpha)$, and hence ψ.

Remarks:

1. The tensor algebra $\mathcal{T}(V)$ is substantially larger than $\mathcal{F}(X)$ here, since $\mathcal{F}(X)$ is a Lie subalgebra generated just by the x_α's in $\mathcal{T}(V)$. In fact $\mathcal{T}(V)$ is the universal enveloping algebra for $\mathcal{F}(X)$.

2. To construct Lie algebras by generators and relations, one may now proceed in the usual way: Relations $\{r_\beta\}$ will correspond to generating elements of an ideal $\mathcal{R}$ in $\mathcal{F}(X)$. Then the Lie algebra with generators $\{x_\alpha\}$ subject to relations $\{r_\beta\}$ is just $\mathcal{F}(X)/\mathcal{R}$.

3. An example? Consider the set $X = \{x, y, h\}$ and the free Lie algebra $\mathcal{F}(X)$. Now impose relations $[x, y] - h = 0$, $[h, x] - 2x = 0$, and $[h, y] + 2y = 0$. Identify $x + \mathcal{R}$, $y + \mathcal{R}$, and $h + \mathcal{R}$ in the quotient $\mathcal{F}(X)/\mathcal{R}$ with the elements x, y, and h. Evidently everything in $\mathcal{F}(X)/\mathcal{R}$ is in the span of x, y, and h, so $\mathcal{F}(X)/\mathcal{R}$ has at most dimension three. However, last time we found a three-dimensional matrix representation of $\mathcal{F}(X)/\mathcal{R}$, namely $\mathfrak{sl}(2, \mathbb{F})$. We have the following commutative diagram of Lie algebra homomorphisms:

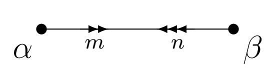


The mapping ψ is induced from the universal mapping property enjoyed by the quotient $\mathcal{F}(X)/\mathcal{R}$ together with the fact that $\mathcal{R} \subset \ker(\phi)$. Since ϕ is surjective, then ψ is surjective. Then we have a surjective linear mapping from a space $\mathcal{F}(X)/\mathcal{R}$ of at most dimension three to a space $\mathfrak{sl}(2, \mathbb{F})$ of dimension three. Then ψ is an isomorphism of vector spaces, and since ψ is also a Lie algebra homomorphism, then we see that

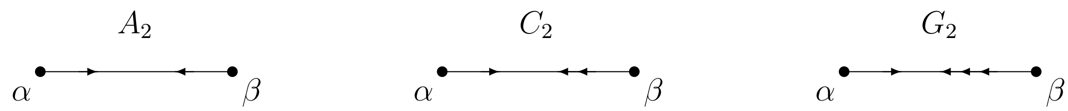
$$\mathcal{F}(X)/\mathcal{R} \approx \mathfrak{sl}(2, \mathbb{F}).$$

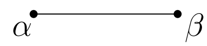
Kac-Moody Lie Algebras

Start with a finite graph Γ with no loops and no multiple edges. Nodes are to be indexed by a set $N = N(\Gamma)$. For each pair $\alpha, \beta \in N$ of adjacent nodes in Γ assign two negative integer *amplitudes* $a_{\alpha,\beta}$ and $a_{\beta,\alpha}$; set $a_{\alpha,\alpha} := 2$, and set $a_{\alpha,\beta} = a_{\beta,\alpha} = 0$ when α and β are distinct and not adjacent. A *GCM graph* is such a graph Γ together with the matrix $A = (a_{\alpha,\beta})_{(\alpha,\beta)\in N\times N}$ of amplitudes. We consider two GCM graphs $(\Gamma, A = (a_{\alpha,\beta})_{(\alpha,\beta)\in N\times N})$ and $(\Gamma', A' = (a'_{\alpha',\beta'})_{(\alpha',\beta')\in N'\times N'})$ to be the same if there is a one-to-one correspondence between their index sets N and N' such that $a_{\alpha,\beta} = a'_{\alpha',\beta'}$ whenever α and β in N correspond to α' and β' respectively in N'. We will depict a connected two-node GCM graph as follows:



In this graph, $m = |a_{\alpha,\beta}|$ and $n = |a_{\beta,\alpha}|$. We give special names to GCM graphs which have $m = 1$ and $n = 1$, 2, or 3:



When $m = 1$ and $n = 1$ it is convenient to use the graph 

to represent the GCM graph A_2. We use $A_1 \times A_1$ to denote the disconnected two-node GCM graph. A GCM graph (Γ, A) is a *Dynkin diagram* if each connected component of (Γ, A) is one of the graphs of Figure 1. The amplitude matrices for the two-node Dynkin diagrams are

$A_1 \times A_1$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	A_2	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
C_2	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	G_2	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

The abbreviation "GCM" is short for *Generalized Cartan Matrix*. Any GCM has the following form: 2's on the main diagonal, nonpositive integer entries everywhere else subject to the requirement that entries in transpose positions must both be zero if one is. So the amplitude matrix A is a generalized Cartan matrix. Given A, the graph Γ really provides no additional information. However, the graph will be crucial for our perspective in a forthcoming talk. One can associate to each GCM graph (or generalized Cartan matrix) a *Kac-Moody Lie algebra* and a *Weyl group*.

Given a GCM graph (Γ, A) with n nodes, choose a complex vector space $\mathfrak{h}$ of dimension $n+\text{corank}(A)$. Choose n linearly independent vectors $\{\alpha_i^\vee\}_{1\le i\le n}$ in $\mathfrak{h}$, and find n linearly independent functionals $\{\alpha_i\}_{1\le i\le n}$ in $\mathfrak{h}^*$ satisfying $\alpha_j(\alpha_i^\vee) = a_{i,j}$. It turns out that the construction that follows does not depend on the specific choices made here, only on the fact that such choices can be made.

The *Kac-Moody Lie algebra* $\mathfrak{g}(\Gamma, A)$ is the Lie algebra over $\mathbb{C}$ generated by the set $\mathfrak{h} \cup \{x_i, y_i\}_{1\le i\le n}$ with relations:

(R1) $[\mathfrak{h}, \mathfrak{h}] = 0$,
(R2) $[h, x_i] = \alpha_i(h)x_i$; $[h, y_i] = -\alpha_i(h)y_i$ for all $h \in \mathfrak{h}$,
(R3) $[x_i, y_j] = \delta_{i,j}\alpha_i^\vee$,
(R4) $(\text{ad}x_i)^{1-a_{j,i}}(x_j) = 0$ for $i \ne j$, and
(R5) $(\text{ad}y_i)^{1-a_{j,i}}(y_j) = 0$ for $i \ne j$,

where $(\text{ad}z)^k(w) = [z, [z, \cdots, [z, w] \cdots]]$.

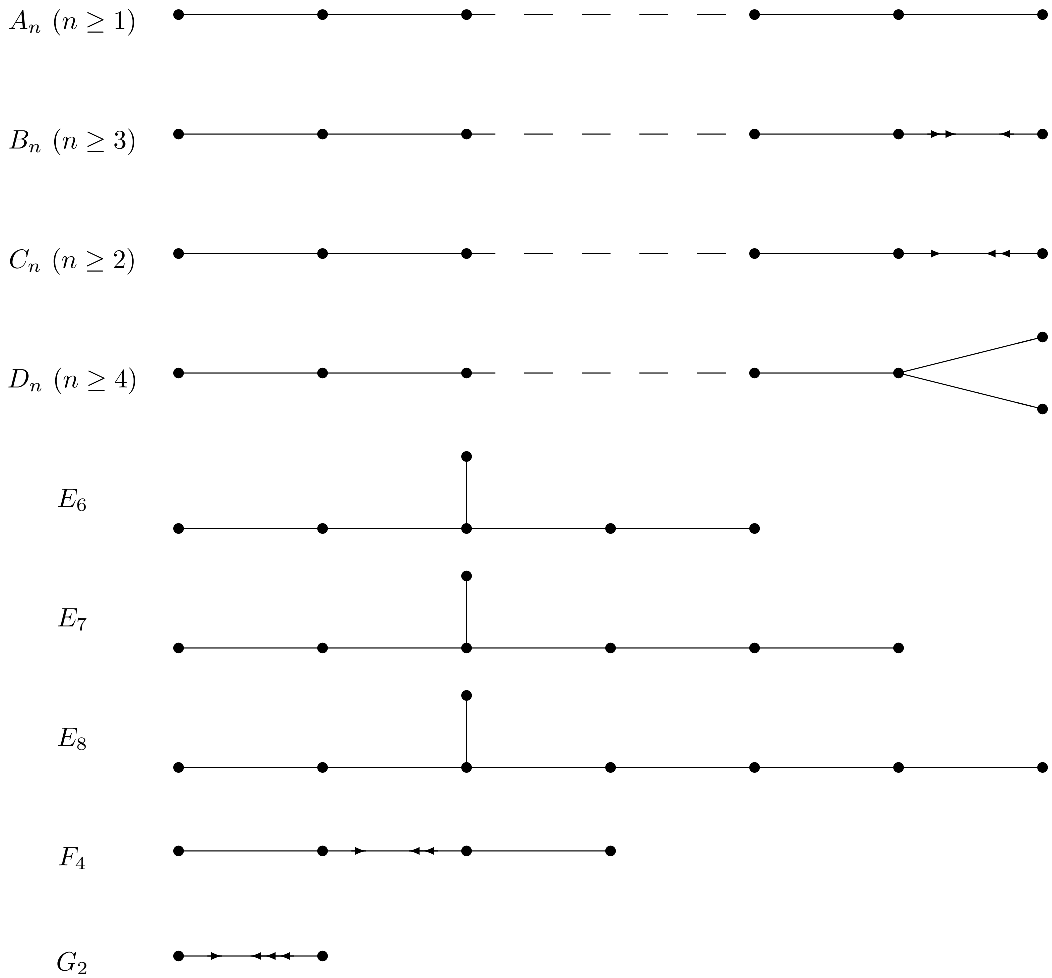
It turns out that if the graph Γ has components $\Gamma_1, \ldots, \Gamma_k$ with corresponding amplitude matrices $A_1, \ldots, A_k$, then

$$\mathfrak{g}(\Gamma, A) \approx \mathfrak{g}(\Gamma_k, A_k) \oplus \cdots \oplus \mathfrak{g}(\Gamma_k, A_k)$$

Now for each $1 \le i \le n$, define a transformation $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$

$$s_i(\mu) := \mu - \mu(\alpha_i^\vee)\alpha_i$$

Figure 1: Connected Dynkin diagrams



for all $\mu \in \mathfrak{h}^*$. For any μ in the hyperplane $\{\nu \,|\, \nu(\alpha_i^\vee) = 0\}$, then $s_i(\mu) = \mu$. Also, $s_i(\alpha_i) = \alpha_i - 2\alpha_i = -\alpha_i$. So in particular, s_i is a reflection, and $s_i^2 = id$. The subgroup $W \in Aut(\mathfrak{h}^*)$ generated by the s_i's is the *Weyl group* associated to the Kac-Moody Lie algebra $\mathfrak{g}(\Gamma, A)$, and the s_i's are *simple reflections*.

Theorem *The Weyl group W associated to $\mathfrak{g}(\Gamma, A)$ is isomorphic to the quotient of the free group generated by $\{S_i\}_{1 \leq i \leq n}$ modulo the relations* (W1) *and* (W2):

(W1) $S_i^2 = 1$ *for all i, and*
(W2) *whenever $i \neq j$, $(S_i S_j)^{m_{i,j}} = 1$ where $m_{i,j} = 2$ if $a_{i,j}a_{j,i} = 0$, $m_{i,j} = 3$ if $a_{i,j}a_{j,i} = 1$, $m_{i,j} = 4$ if $a_{i,j}a_{j,i} = 2$, $m_{i,j} = 6$ if $a_{i,j}a_{j,i} = 3$, $m_{i,j} = \infty$ if $a_{i,j}a_{j,i} \geq 4$.*

In particular, Weyl groups of Kac-Moody Lie algebras are precisely the crystallographic Coxeter groups with finite generating sets.

A question of possible interest

For which GCM graphs are the Kac-Moody Lie algebras finite-dimensional? are the Weyl groups finite?