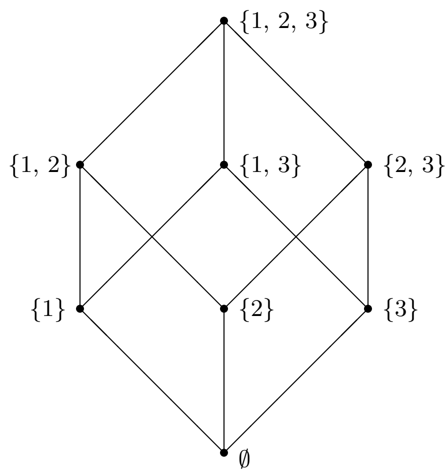
Combinatorial models for Lie algebra representations
Rob Donnelly   April 14, 2006

Boolean lattices, revisited:

Let $\mathfrak{B}_n := \Big\{\text{subsets of } \{1, 2, \ldots, n\}\Big\}$. We think of $\mathfrak{B}_n$ as a partially ordered set with respect to subset containment "$\subseteq$," so we have $S \subseteq T$ for $S, T \in \mathfrak{B}_n$ iff S is a subset of T when we think of S and T as subsets of $\{1, 2, \ldots, n\}$. For S and T in $\mathfrak{B}_n$, write $S \to T$ iff $S \subseteq T$ and $|T \setminus S| = 1$. These edges are depicted in the *order diagram* for $\mathfrak{B}_3$ below:



The "Boolean Lattice" $\mathfrak{B}_3$

All of the edges are taken to be pointing "up."

Let $V = V[\mathfrak{B}_n] = \text{span}_{\mathbb{C}}\{v_S \,\big|\, S \in \mathfrak{B}_n\}$, a 2^n-dimensional complex vector space. The subalgebra of $\mathfrak{gl}(V)$ generated by the following linear transformations X, Y, and H on V is isomorphic to $\mathfrak{g}(A_1)$:

$$\begin{aligned} X(v_S) &:= \sum_{T \in \mathfrak{B}_n, S \to T} v_T \\ Y(v_S) &:= \sum_{R \in \mathfrak{B}_n, R \to S} v_R \\ H(v_S) &:= (2\underset{\substack{\uparrow \\ \text{rank}}}{|S|} - \underset{\substack{\uparrow \\ \text{length}}}{n})\, v_S \end{aligned}$$

So in our example above, $X(v_{\{2\}}) = v_{\{1,2\}} + v_{\{2,3\}}$, $Y(v_{\{2\}}) = v_{\emptyset}$, and $H(v_{\{2\}}) = (2 \cdot 1 - 3)v_{\{2\}} = -v_{\{2\}}$.

Question:

Given any complex finite-dimensional representation of a complex finite-dimensional semisimple Lie algebra, is it possible to find a similar kind of combinatorial model?

Answer:

Yes. Here's how.

Let $\mathfrak{g} := \mathfrak{g}(\Gamma, A)$ for a GCM graph (Γ, A) on n nodes so that $\mathfrak{g}$ is finite-dimensional. In particular, $\mathfrak{g}$ is semisimple. Given any representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, there exists a basis $\mathcal{B} = \{v_{\mathbf{s}}\}_{\mathbf{s}\in R}$ (for now R is just an index set of the appropriate size) of common eigenvectors for the $\phi(h_i)$'s:

$$\phi(h_i)(v_{\mathbf{s}}) = m_i(\mathbf{s})v_{\mathbf{s}}.$$

The eigenvalues $m_i(\mathbf{s})$ are all integers.

We will use elements of R as vertices. We declare that there is a colored directed edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ if when we write $\phi(x_i)(v_{\mathbf{s}})$ and $\phi(y_i)(v_{\mathbf{t}})$ respectively as linear combinations in the basis $\mathcal{B}$

$$\begin{aligned} \phi(x_i)(v_{\mathbf{s}}) &= \cdots + c_{\mathbf{t},\mathbf{s}}v_{\mathbf{t}} + \cdots \\ \phi(y_i)(v_{\mathbf{t}}) &= \cdots + d_{\mathbf{s},\mathbf{t}}v_{\mathbf{s}} + \cdots \end{aligned}$$

then $c_{\mathbf{t},\mathbf{s}} \neq 0$ or $d_{\mathbf{s},\mathbf{t}} \neq 0$.

The resulting edge-colored directed graph is the _supporting graph_ for the basis $\mathcal{B}$ for the $\mathfrak{g}$-module V. We sometimes attach the coefficients $(c_{\mathbf{t},\mathbf{s}}, d_{\mathbf{s},\mathbf{t}})$ to the edges $\mathbf{s} \xrightarrow{i} \mathbf{t}$ and call this the _representation diagram_ for the basis $\mathcal{B}$. Notice that:

$$\begin{aligned} \phi(x_i)(v_{\mathbf{s}}) &= \sum_{\mathbf{t}\in R, \mathbf{s}\xrightarrow{i}\mathbf{t}} c_{\mathbf{t},\mathbf{s}}\, v_{\mathbf{t}} \\ \phi(y_i)(v_{\mathbf{s}}) &= \sum_{\mathbf{r}\in R, \mathbf{r}\xrightarrow{i}\mathbf{s}} d_{\mathbf{r},\mathbf{s}}\, v_{\mathbf{r}} \\ \phi(h_i)(v_{\mathbf{s}}) &= m_i(\mathbf{s})\, v_{\mathbf{s}} \end{aligned}$$

Some facts about supporting graphs and representation diagrams:

1. Let $\mu = (m_1(\mathbf{s}), \ldots, m_n(\mathbf{s}))$ and $\nu = (m_1(\mathbf{t}), \ldots, m_n(\mathbf{t}))$. If $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in a supporting graph R, then

$$\mu + \alpha_i = \nu,$$

   where α_i is the ith row of the matrix A for the GCM graph (Γ, A).

2. The directed graph R is the order diagram for a ranked poset.

3. For any $1 \leq i \leq n$ and any $\mathbf{s}$ in R, we have $m_i(\mathbf{s}) = 2\rho_i(\mathbf{s}) - l_i(\mathbf{s})$, where $\rho_i(\mathbf{s})$ is the rank of $\mathbf{s}$ in its i-component of R and $l_i(\mathbf{s})$ is the length of this component. That is,

$$\phi(h_i)(v_{\mathbf{s}}) = (2\rho_i(\mathbf{s}) - l_i(\mathbf{s})\,)\, v_{\mathbf{s}}.$$

4. If V is irreducible, then R is connected, has a unique maximal element (corresponding to the maximal vector for the $\mathfrak{g}$-module V), and has a unique minimal element.

5. If R is connected, then R is "rank symmetric, rank unimodal, and strongly Sperner."

6. If V is irreducible with highest weight λ, then R has this many elements

$$\mathrm{card}(R) \stackrel{\text{Theorem}}{=\!=} \frac{\Pi_{\alpha\in\Phi^+}\langle\lambda+\varrho,\alpha\rangle}{\Pi_{\alpha\in\Phi^+}\langle\varrho,\alpha\rangle},$$

has rank generating function

$$\mathrm{rgf}(R,q) := \sum_{\mathbf{s}\in R} q^{\mathrm{rank}(\mathbf{s})} \stackrel{\text{Theorem}}{=\!=} \frac{\Pi_{\alpha\in\Phi^+}(1-q^{\langle\lambda+\varrho,\alpha\rangle})}{\Pi_{\alpha\in\Phi^+}(1-q^{\langle\varrho,\alpha\rangle})},$$

and has weight-multiplicity generating function or "character"

$$\mathrm{char}(R) := \sum_{\mathbf{s}\in R} e_{wt(\mathbf{s})} \stackrel{\text{Theorem}}{=\!=} \frac{\sum_{\sigma\in W}\det(\sigma)e_{\sigma(\varrho+\lambda)}}{e_\varrho\Pi_{\alpha\in\Phi^+}(1-e_{-\alpha})}.$$

The notation here might require some explanation...

7. Almost all weight bases for V share the same supporting graph M which contains all other supporting graphs for V as edge-colored subgraphs. We call M the *maximal support* for V.

Some combinatorial properties for a weight basis $\mathcal{B}$ with supporting graph R:

Edge-minimizing Say $\mathcal{B}$ is *edge-minimizing* if no other weight basis for V has a supporting graph with fewer edges than R has.

Locally edge-minimizing Say $\mathcal{B}$ is *locally edge-minimizing* if no other weight basis for V has a supporting graph that is contained in R as a proper edge-colored subgraph.

Modular lattice Say $\mathcal{B}$ is a *modular lattice* weight basis if the supporting graph R is a modular lattice when viewed as a partially ordered set.

Solitary Say $\mathcal{B}$ is *solitary* if any other weight basis for V with supporting graph R is "diagonally equivalent" to $\mathcal{B}$ (i.e. comprised of scalar multiples of the basis vectors for $\mathcal{B}$).

In some sense, solitary bases, if they exist, are uniquely determined by their supporting graphs.

There are a finite number of supporting graphs for V, and thus at most a finite number of solitary bases.

But, there's a problem:

- Which comes first: the weight basis or the supporting graph?
- We will have nice posets with lots of nice combinatorial properties IF we have weight bases in hand.
- Before 1995 and aside from some special cases, the only "explicit" weight bases for irreducible representations that had been obtained were for the irreducible representations of $\mathfrak{g}(A_n)$ — by Gel'fand and Tsetlin (Moscow, 1950).
- If we start with the "right" posets, and somehow use information from the order diagrams to find edge coefficients, then we can realize representations.

Here's some of what we know now:

Family of representations	Bases considered	Solitary?	Locally edge-minimizing?	Modular lattice?	Edge-minimizing?
The irreducible reps. of $\mathfrak{sl}(n, \mathbb{C})$	Both GT "left" and "right" bases	Yes [Don3]	Yes [Don3]	Yes [Don3]	Open
The fundamental reps. of $\mathfrak{sp}(2n, \mathbb{C})$	Both bases of [Don2]	Yes [Don3]	Yes [Don3]	Yes [Don1]	Open
Irreducible one-dimensional weight space reps.	The (essentially) unique weight basis	Yes [Don3]	Yes [Don3]	Yes [Don3]	Yes [Don3]
Adjoint reps. of the simple Lie algebras	The n extremal bases of [Don4]	Yes [Don4]	Yes [Don4]	Yes [Don4]	Yes [Don4]
The fundamental reps. of $\mathfrak{so}(2n+1, \mathbb{C})$	Both bases of [Don5]	Yes [Don5]	Yes [Don5]	Yes [Don5]	Open
The "one-rowed" reps. of $\mathfrak{so}(2n+1, \mathbb{C})$	Both bases of [DLP1]	Yes [DLP1]	Yes [DLP1]	Yes [DLP1]	Open
The "one-rowed" reps. of G_2	Both bases of [DLP1]	Yes [DLP2]	Yes [DLP2]	Yes [DLP1]	Open
The irreducible reps. of $\mathfrak{sp}(2n, \mathbb{C})$, $\mathfrak{so}(2n, \mathbb{C})$, and $\mathfrak{so}(2n+1, \mathbb{C})$	Molev's bases in [Mol1], [Mol2], and [Mol3]	Open	Open	Open	Open

[Don1] R. G. Donnelly, "Symplectic analogs of $L(m, n)$," *J. Comb. Th. Series A* **88** (1999), 217-234.

[Don2] R. G. Donnelly, "Explicit constructions of the fundamental representations of the symplectic Lie algebras," *J. Algebra*, **233** (2000), p. 37-64.

[Don3] R. G. Donnelly, "Extremal properties of bases for representations of semisimple Lie algebras," *J. Algebraic Comb.* **17** (2003), 255–282.

[Don4] R. G. Donnelly, "Extremal bases for the adjoint representations of the simple Lie algebras," *Comm. Alg.*, to appear.

[Don5] R. G. Donnelly, "Explicit constructions of the fundamental representations of the odd orthogonal Lie algebras," in preparation.

[DLP1] R. G. Donnelly, S. J. Lewis, and R. Pervine, "Constructions of representations of $\mathfrak{o}(2n+1, \mathbb{C})$ that imply Molev and Reiner-Stanton lattices are strongly Sperner," *Discrete Math.* **263** (2003), 61–79.

[DLP2] R. G. Donnelly, S. J. Lewis, and R. Pervine, "Solitary and edge-minimal bases for representations of the simple Lie algebra G_2," *Discrete Math.*, to appear.

[Mol1] A. Molev, "A basis for representations of symplectic Lie algebras," *Comm. Math. Phys.* **201** (1999), 591-618.

[Mol2] A. Molev, "A weight basis for representations of even orthogonal Lie algebras," in "Combinatorial Methods in Representation Theory," *Adv. Studies in Pure Math.* **28** (2000), 221-240.

[Mol3] A. Molev, "Weight bases of Gel'fand-Tsetlin type for representations of classical Lie algebras," *J. Phys. A: Math. Gen.* **33** (2000), 4143–4168.

What is any of this "useful" for?

- Solving combinatorics problems

  For example, we have used certain of our symplectic and odd orthogonal representation constructions to resolve in [Don1] and [DLP1] respectively two "Sperner" conjectures of Vic Reiner and Dennis Stanton (Univ. of Minnesota) about some related families of distributive lattices. In [Don1] and elsewhere we obtain some generating function identities which are difficult to obtain combinatorially.

- Constructing new bases for families of irreducible representations

  For example, it appears that our constructions of the fundamental representations of the symplectic Lie algebra [Don2] were the first completely explicit constructions of a (non-routine) family of irreducible representations of semisimple Lie algebras found since the Gel'fand-Tsetlin construction for $\mathfrak{sl}(n, \mathbb{C})$ was given in 1950.

- Studying existing bases from an extremal, combinatorial viewpoint

  For example, in [Don3] we determined precisely when the two Gel'fand-Tsetlin bases for a given irreducible representation of $\mathfrak{sl}(n, \mathbb{C})$ coincide by looking at the combinatorics of their supporting graphs. In a similar way we have also made connections between some of Molev's bases and some of our bases in [Don2], [Don3], [Don5], and [DLP1].

Some general questions:

The case by case results give some evidence for affirmative answers to the following questions:

- Does every irreducible representation have a solitary basis?

  Molev's bases seem to be good candidates in the classical cases. Does locally edge-minimizing imply solitary, or vice-versa?

- Does every irreducible representation have a modular lattice basis?

  Does edge-minimizing imply modular lattice, or vice-versa?

- For supporting graphs for $\mathfrak{sl}(2, \mathbb{C})$ modules, does locally edge-minimizing imply edge-minimizing?

  An affirmative answer would yield surprising results: it would show that any supporting graph for any irreducible representation of a semisimple Lie algebra has a "symmetric chain decomposition." This would resolve many longstanding combinatorics conjectures.

What's my motivation?

- To continue adding to a growing collection of beautiful and interesting combinatorial manifestations of semisimple Lie algebra representations.
- To enjoy this interaction between combinatorics and Lie representation theory.
- To obtain new combinatorial and algebraic results.
- Someday, combinatorialize Lie theory.