Boolean lattices, revisited:

Let $\mathcal{B}_n := \{\text{subsets of } \{1, 2, \ldots, n\}\}$. We think of $\mathcal{B}_n$ as a partially ordered set with respect to subset containment $\subseteq$, so we have $S \subseteq T$ for $S, T \in \mathcal{B}_n$ iff $S$ is a subset of $T$ when we think of $S$ and $T$ as subsets of $\{1, 2, \ldots, n\}$. For $S$ and $T$ in $\mathcal{B}_n$, write $S \rightarrow T$ iff $S \subseteq T$ and $|T \setminus S| = 1$. These edges are depicted in the order diagram for $\mathcal{B}_3$ below:

All of the edges are taken to be pointing “up.”

Let $V = V[\mathcal{B}_n] = \text{span}_C \{v_S \mid S \in \mathcal{B}_n\}$, a $2^n$-dimensional complex vector space. The subalgebra of $\mathfrak{gl}(V)$ generated by the following linear transformations $X$, $Y$, and $H$ on $V$ is isomorphic to $\mathfrak{g}(A_1)$:

$$
X(v_S) := \sum_{T \in \mathcal{B}_n, S \rightarrow T} v_T \\
Y(v_S) := \sum_{R \in \mathcal{B}_n, R \rightarrow S} v_R \\
H(v_S) := (2|S| - n) v_S
$$

So in our example above, $X(v_{\{2\}}) = v_{\{1,2\}} + v_{\{2,3\}}$, $Y(v_{\{2\}}) = v_{\emptyset}$, and $H(v_{\{2\}}) = (2 \cdot 1 - 3)v_{\{2\}} = -v_{\{2\}}$.

**Question:**

Given any complex finite-dimensional representation of a complex finite-dimensional semisimple Lie algebra, is it possible to find a similar kind of combinatorial model?
Yes. Here’s how.

Let $g := g(\Gamma, A)$ for a GCM graph $(\Gamma, A)$ on $n$ nodes so that $g$ is finite-dimensional. In particular, $g$ is semisimple. Given any representation $\phi : g \to \mathfrak{gl}(V)$, there exists a basis $B = \{v_s\}_{s \in R}$ (for now $R$ is just an index set of the appropriate size) of common eigenvectors for the $\phi(h_i)$'s:

$$\phi(h_i)(v_s) = m_i(s)v_s.$$ 

The eigenvalues $m_i(s)$ are all integers.

We will use elements of $R$ as vertices. We declare that there is a colored directed edge $s \xrightarrow{i} t$ if when we write $\phi(x_i)(v_s)$ and $\phi(y_i)(v_t)$ respectively as linear combinations in the basis $B$

$$\phi(x_i)(v_s) = \cdots + c_{t,s} v_t + \cdots$$
$$\phi(y_i)(v_t) = \cdots + d_{s,t} v_s + \cdots$$

then $c_{t,s} \neq 0$ or $d_{s,t} \neq 0$.

The resulting edge-colored directed graph is the supporting graph for the basis $B$ for the $g$-module $V$. We sometimes attach the coefficients $(c_{t,s}, d_{s,t})$ to the edges $s \xrightarrow{i} t$ and call this the representation diagram for the basis $B$. Notice that:

$$\phi(x_i)(v_s) = \sum_{t \in R, s \xrightarrow{i} t} c_{t,s} v_t$$
$$\phi(y_i)(v_t) = \sum_{r \in R, t \xrightarrow{i} r} d_{r,s} v_r$$
$$\phi(h_i)(v_s) = m_i(s)v_s$$

Some facts about supporting graphs and representation diagrams:

1. Let $\mu = (m_1(s), \ldots, m_n(s))$ and $\nu = (m_1(t), \ldots, m_n(t))$. If $s \xrightarrow{i} t$ in a supporting graph $R$, then

   $$\mu + \alpha_i = \nu,$$

   where $\alpha_i$ is the $i$th row of the matrix $A$ for the GCM graph $(\Gamma, A)$.

2. The directed graph $R$ is the order diagram for a ranked poset.

3. For any $1 \leq i \leq n$ and any $s$ in $R$, we have $m_i(s) = 2\rho_i(s) - l_i(s)$, where $\rho_i(s)$ is the rank of $s$ in its $i$-component of $R$ and $l_i(s)$ is the length of this component. That is,

   $$\phi(h_i)(v_s) = (2\rho_i(s) - l_i(s))v_s.$$

4. If $V$ is irreducible, then $R$ is connected, has a unique maximal element (corresponding to the maximal vector for the $g$-module $V$), and has a unique minimal element.

5. If $R$ is connected, then $R$ is “rank symmetric, rank unimodal, and strongly Sperner.”
6. If $V$ is irreducible with highest weight $\lambda$, then $R$ has this many elements

$$\text{card}(R) \overset{\text{Theorem}}{=} \frac{\prod_{\alpha \in \Phi^+} \langle \lambda + \varrho, \alpha \rangle}{\prod_{\alpha \in \Phi^+} \langle \varrho, \alpha \rangle},$$

has rank generating function

$$\text{rgf}(R, q) := \sum_{s \in R} q^{\text{rank}(s)} \overset{\text{Theorem}}{=} \frac{\prod_{\alpha \in \Phi^+} (1 - q^{\langle \lambda + \varrho, \alpha \rangle})}{\prod_{\alpha \in \Phi^+} (1 - q^{\langle \varrho, \alpha \rangle})},$$

and has weight-multiplicity generating function or “character”

$$\text{char}(R) := \sum_{s \in R} e^{\text{wt}(s)} \overset{\text{Theorem}}{=} \frac{\sum_{\sigma \in W} \det(\sigma)e_{\sigma(\varrho + \lambda)}}{e_{\varrho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}. $$

The notation here might require some explanation...

7. Almost all weight bases for $V$ share the same supporting graph $M$ which contains all other supporting graphs for $V$ as edge-colored subgraphs. We call $M$ the maximal support for $V$.

Some combinatorial properties for a weight basis $B$ with supporting graph $R$:

**Edge-minimizing** Say $B$ is **edge-minimizing** if no other weight basis for $V$ has a supporting graph with fewer edges than $R$ has.

**Locally edge-minimizing** Say $B$ is **locally edge-minimizing** if no other weight basis for $V$ has a supporting graph that is contained in $R$ as a proper edge-colored subgraph.

**Modular lattice** Say $B$ is a **modular lattice** weight basis if the supporting graph $R$ is a modular lattice when viewed as a partially ordered set.

**Solitary** Say $B$ is **solitary** if any other weight basis for $V$ with supporting graph $R$ is “diagonally equivalent” to $B$ (i.e. comprised of scalar multiples of the basis vectors for $B$).

In some sense, solitary bases, if they exist, are uniquely determined by their supporting graphs.

There are a finite number of supporting graphs for $V$, and thus at most a finite number of solitary bases.

But, there’s a problem:

- Which comes first: the weight basis or the supporting graph?
- We will have nice posets with lots of nice combinatorial properties IF we have weight bases in hand.
- Before 1995 and aside from some special cases, the only “explicit” weight bases for irreducible representations that had been obtained were for the irreducible representations of $\mathfrak{g}(A_n)$ — by Gel’fand and Tsetlin (Moscow, 1950).
- If we start with the “right” posets, and somehow use information from the order diagrams to find edge coefficients, then we can realize representations.
Here’s some of what we know now:

<table>
<thead>
<tr>
<th>Family of representations</th>
<th>Bases considered</th>
<th>Solitary?</th>
<th>Locally edge-minimizing?</th>
<th>Modular lattice?</th>
<th>Edge-minimizing?</th>
</tr>
</thead>
<tbody>
<tr>
<td>The irreducible reps. of $\mathfrak{sl}(n, \mathbb{C})$</td>
<td>Both GT “left” and “right” bases</td>
<td>Yes [Don3]</td>
<td>Yes [Don3]</td>
<td>Yes [Don3]</td>
<td>Open</td>
</tr>
<tr>
<td>The fundamental reps. of $\mathfrak{sp}(2n, \mathbb{C})$</td>
<td>Both bases of [Don2]</td>
<td>Yes [Don3]</td>
<td>Yes [Don3]</td>
<td>Yes [Don1]</td>
<td>Open</td>
</tr>
<tr>
<td>Irreducible one-dimensional weight space reps.</td>
<td>The (essentially) unique weight basis</td>
<td>Yes [Don3]</td>
<td>Yes [Don3]</td>
<td>Yes [Don3]</td>
<td>Yes [Don3]</td>
</tr>
<tr>
<td>The fundamental reps. of $\mathfrak{so}(2n+1, \mathbb{C})$</td>
<td>Both bases of [Don5]</td>
<td>Yes [Don5]</td>
<td>Yes [Don5]</td>
<td>Yes [Don5]</td>
<td>Open</td>
</tr>
<tr>
<td>The “one-rowed” reps. of $\mathfrak{so}(2n+1, \mathbb{C})$</td>
<td>Both bases of [DLP1]</td>
<td>Yes [DLP1]</td>
<td>Yes [DLP1]</td>
<td>Yes [DLP1]</td>
<td>Open</td>
</tr>
<tr>
<td>The irreducible reps. of $\mathfrak{sp}(2n, \mathbb{C})$, $\mathfrak{so}(2n, \mathbb{C})$, and $\mathfrak{so}(2n+1, \mathbb{C})$</td>
<td>Molev’s bases in [Mol1], [Mol2], and [Mol3]</td>
<td>Open</td>
<td>Open</td>
<td>Open</td>
<td>Open</td>
</tr>
</tbody>
</table>


What is any of this “useful” for?

- **Solving combinatorics problems**
  For example, we have used certain of our symplectic and odd orthogonal representation constructions to resolve in [Don1] and [DLP1] respectively two “Sperner” conjectures of Vic Reiner and Dennis Stanton (Univ. of Minnesota) about some related families of distributive lattices. In [Don1] and elsewhere we obtain some generating function identities which are difficult to obtain combinatorially.

- **Constructing new bases for families of irreducible representations**
  For example, it appears that our constructions of the fundamental representations of the symplectic Lie algebra [Don2] were the first completely explicit constructions of a (non-routine) family of irreducible representations of semisimple Lie algebras found since the Gel’fand-Tsetlin construction for \( \mathfrak{sl}(n, \mathbb{C}) \) was given in 1950.

- **Studying existing bases from an extremal, combinatorial viewpoint**
  For example, in [Don3] we determined precisely when the two Gel’fand-Tsetlin bases for a given irreducible representation of \( \mathfrak{sl}(n, \mathbb{C}) \) coincide by looking at the combinatorics of their supporting graphs. In a similar way we have also made connections between some of Molev’s bases and some of our bases in [Don2], [Don3], [Don5], and [DLP1].

Some general questions:
The case by case results give some evidence for affirmative answers to the following questions:

- **Does every irreducible representation have a solitary basis?**
  Molev’s bases seem to be good candidates in the classical cases. Does locally edge-minimizing imply solitary, or vice-versa?

- **Does every irreducible representation have a modular lattice basis?**
  Does edge-minimizing imply modular lattice, or vice-versa?

- **For supporting graphs for \( \mathfrak{sl}(2, \mathbb{C}) \) modules, does locally edge-minimizing imply edge-minimizing?**
  An affirmative answer would yield surprising results: it would show that any supporting graph for any irreducible representation of a semisimple Lie algebra has a “symmetric chain decomposition.” This would resolve many longstanding combinatorics conjectures.

What’s my motivation?

- **To continue adding to a growing collection of beautiful and interesting combinatorial manifestations of semisimple Lie algebra representations.**

- **To enjoy this interaction between combinatorics and Lie representation theory.**

- **To obtain new combinatorial and algebraic results.**

- **Someday, combinatorialize Lie theory.**