A result on the Word Problem for free groups
MAT 690 Coxeter Groups Seminar

Let $\mathcal{A}$ be a set, and let $\mathcal{W}_A$ be the set of all finite-length words from the alphabet $\mathcal{A} \cup \mathcal{A}^{-1}$. We refer to each element of a word in $\mathcal{W}_A$ as a factor. Make $\mathcal{W}_A$ a monoid using concatenation as the operation and the empty word, denoted $\varepsilon$, as the identity. Let $l(w)$ be the length of any word $w \in \mathcal{W}_A$, with $l(\varepsilon) = 0$. A subword of $w$ is a sequence of consecutive factors of $w$.

Say a word $v \in \mathcal{W}_A$ is obtained from $u \in \mathcal{W}_A$ by an elementary reduction (respectively, elementary expansion) if $v$ is obtained from $u$ by deleting (resp. inserting) a subword of the form $aa^{-1}$ or $a^{-1}a$ for some $a \in \mathcal{A}$. Say $u \sim v$ if we can pass from $u$ to $v$ by a (possibly empty) finite sequence of elementary reductions or expansions. In this case say $u$ and $v$ are freely equivalent. Say a word $w \in \mathcal{W}_A$ is reduced if it contains no subwords of the form $aa^{-1}$ or $a^{-1}a$.

Exercise 1  Show $\sim$ is an equivalence relation on $\mathcal{W}_A$.

For $w \in \mathcal{W}_A$, denote by $[w]$ the equivalence class of $w$ with respect to the equivalence relation $\sim$ on $\mathcal{W}_A$. If $u, v \in [w]$, then write $u \rightarrow v$ if $v$ is obtained from $u$ by an elementary expansion (or, equivalently, if $u$ is obtained from $v$ by an elementary reduction). So we obtain a directed graph $\mathcal{R}(w)$ whose vertices are the set of words freely equivalent to $w$ and whose directed edges correspond to the application of one elementary expansion. Some observations:

Observation 1  The directed graph $\mathcal{R}(w)$ is connected. This follows from the fact that if $u, v \in \mathcal{R}(w)$, then $u \sim w$ and $v \sim w$ implies that $u$ and $v$ are freely equivalent. Thus there is a sequence of elementary expansions or reductions that move us from $u$ to $v$.

Observation 2  If $w_1 \rightarrow w_2$ in $\mathcal{R}(w)$, then $l(w_1) = l(w_2) - 2$. Thus, for a path in $\mathcal{R}(w)$ of the form $u = u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_p = v$, we have $p \leq \lfloor l(v)/2 \rfloor$.

Think of directed edges in $\mathcal{R}(w)$ as pointing “up.” We will view $\mathcal{R}(w)$ as a partially ordered set as follows: For $u, v \in \mathcal{R}(w)$, say $u \leq v$ if $u = v$ or there is some sequence $u \rightarrow \cdots \rightarrow v$ of directed edges from $u$ up to $v$.

Exercise 2  Show that $(\mathcal{R}(w), \leq)$ is a partially ordered set as follows:

(A) Show that $\leq$ is reflexive: $v \leq v$ for all $v \in \mathcal{R}(w)$.

(B) Show that $\leq$ is antisymmetric: If for any $u, v \in \mathcal{R}(w)$ we have $u \leq v$ and $v \leq u$, then $u = v$.

(C) Show that $\leq$ is transitive: If for any $w_1, w_2, w_3 \in \mathcal{R}(w)$ we have $w_1 \leq w_2$ and $w_2 \leq w_3$, then $w_1 \leq w_3$.

We continue with an observation about this partially ordered set $\mathcal{R}(w)$:

Observation 3  For any $v \in \mathcal{R}(w)$, there is a $u \in \mathcal{R}(w)$ for which $u \leq v$ and $u$ is minimal in $\mathcal{R}(w)$, i.e. if $u' \leq u$, then $u' = u$. The reason is as follows: Let $p$ be the largest integer such that there is a path in $\mathcal{R}(w)$ from some $u$ up to $v$ of the form $u = u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_p = v$. October 10, 2009
There is such a largest integer \( p \) since for any such path, we have \( p \leq \lfloor l(v)/2 \rfloor \) by Observation 2. For any such longest path, \( u \) is easily seen to be minimal.

**Exercise 3** Show that if \( u \to x \) and \( v \to x \) for \( u, v, x \in \mathcal{R}(w) \), then there exists \( y \in \mathcal{R}(w) \) for \( \boxtimes \) which \( y \to u \) and \( y \to v \). That is, whenever \( \circ \to \nabla \) (a “peak”) is part of a path in \( \mathcal{R}(w) \), then it can be replaced with some “valley” \( \circ \to \nabla \).

We now make the following observations:

**Observation 4** For any \( u, v \in \mathcal{R}(w) \), there is some \( y \in \mathcal{R}(w) \) such that \( y \leq u \) and \( y \leq v \). The reason is that for any path from \( u \) to \( v \), we may adjust any “peak” to become a “valley” as in Exercise 3. Apply this principle again to the resulting path, and again etc, to obtain a path which has only one valley, which therefore occurs at a lower bound \( y \).

**Observation 5** A word \( u \in \mathcal{R}(w) \) is reduced if and only if it is minimal. Both directions of this equivalence follow immediately from the definitions.

A sequence of elementary reductions applied to a word \( w \in \mathcal{W}_A \) is longest if, when the sequence is applied to \( w \), no further elementary reductions can be applied. Putting these pieces together, we have the following theorem.

**Theorem** For any \( w \in \mathcal{W}_A \), the poset \( \mathcal{R}(w) \) has a unique minimal element \( w_0 \). This word \( w_0 \) is the unique reduced word that is freely equivalent to \( w \). Moreover, any longest sequence of elementary reductions applied to \( w \) yields \( w_0 \).

**Proof.** Existence of some minimal element is guaranteed by Observation 3. If \( u \) and \( u' \) are both minimal, then use Observation 4 to get \( y \leq u' \) and \( y \leq u \). Since \( u' \) and \( u \) are minimal, then \( u' = y = u \). So, there is a unique minimal element \( w_0 \). By Observation 5, \( w_0 \) is the unique reduced word freely equivalent to \( w \). Now any longest sequence applied to \( w \) corresponds to a longest path of the form \( u = u_0 \to u_1 \to \cdots \to u_p = w \), cf. Observation 3. Then \( u \) is minimal, and hence \( u = w_0 \).

This theorem resolves the Word Problem for free groups, in that it can be used to conclude when a word \( w \) is freely equivalent to the empty word \( \varepsilon \): \( w \sim \varepsilon \) if and only if \( \varepsilon \) can be obtained by some sequence of elementary reductions of \( w \) if and only if every longest sequence of elementary reductions applied to \( w \) produces \( \varepsilon \).