Word Processing in Coxeter Groups*

Benjiman C. Tharp

Murray State University
Department of Mathematics and Statistics
MAT691 - Seminar in Coxeter Groups
Fall 2009

Abstract

Several definitions, facts, and theorems stated during the recent talks given by Dr. Donnelly relating to the "word problem" for Coxeter groups are restated in §1 to provide some background information and notation. Three other facts, due to Casselman, are also stated and proved as lemmas in this section. In §2, we state and prove Tits' Theorem for the word problem on Coxeter groups. Finally, we consider some applications and corollaries of Tits' Theorem in §3.

1 Preliminaries

Let $W = \langle \mathcal{S} | R \rangle$ be a Coxeter group with $\mathcal{S} = \{s_i\}_{i \in I_n}$, where I_n is a finite index set, and $R = \{(s_i s_j)^{m_{ij}} : m_{ii} = 1, m_{ij} \in \{\infty, 2, 3, 4, \dots\} \text{ for } i \neq j, m_{ij} = m_{ji}\}$. Let I_n^* denote the set of all finite sequences from I_n, including the empty sequence (). For $x \in I_n^*$ with $x = (j_1, \dots, j_k)$, say the length of x, denoted by $l(x)$, is k. Define the following two *elementary simplifications* on elements of I_n^*:

- Length-reducing: Replace a subsequence of the form (i, i) with the empty sequence;
- Braid: Replace a subsequence of the form $\underbrace{(i, j, i, \dots)}_{\text{length } m_{ij}}$ with $\underbrace{(j, i, j, \dots)}_{\text{length } m_{ij}}$.

Now, for $x \in I_n^*$, let $S(x)$ be the set of sequences $y \in I_n^*$ that are obtainable from x by a finite sequence of elementary simplifications. Note that $y \in S(x)$ may be obtained from x by an empty sequence of elementary simplifications, hence $x \in S(x)$. Also, we must have the following

Fact 1. *If $y \in S(x)$ for some $x \in I_n^*$, then $l(y) \leq l(x)$.*

Notice that $S(x)$ is a finite set since it is a subset of the set of all sequences with length less than or equal to $l(x)$, which is a finite set. We are now prepared to prove the following three lemmas, which will be required in the proof of Tits' Theorem.

*The title "Word Processing" is due to Casselman.

Lemma 1. *If $y \in S(x)$, then $S(y) \subseteq S(x)$.*

Proof. Let $z \in S(y)$. Then, z can be obtained from y by a finite sequence of elementary simplifications. Now, y can be obtained from x by a finite sequence of elementary simplifications, hence z can be obtained from x by a finite sequence of elementary simplifications by first passing from x to y. Thus, $z \in S(x)$. $\square$

Lemma 2. *If $y \in S(x)$ and $l(x) = l(y)$, then $S(x) = S(y)$.*

Proof. By Lemma 1, $S(y) \subseteq S(x)$. Since $l(x) = l(y)$, it follows that x can be obtained from y by a finite sequence of braid relations, so $x \in S(y)$. Applying Lemma 1 again, we have $S(x) \subseteq S(y)$. Hence, $S(x) = S(y)$. $\square$

Lemma 3. *Let $x = (i, x')$, for some $i \in I_n$, $x' \in I_n^*$. Then, $(i, y) \in S(x)$ for all $y \in S(x')$.*

Proof. Focusing on the subsequence x' of x , we may pass to $y \in S(x')$ by a finite sequence of elementary simplifications, obtaining (i, y) from $(i, x') = x$. Hence, $(i, y) \in S(x)$. $\square$

Now, let $T : I_n^* \to W$ be given by $T(j_1, \ldots, j_k) = s_{j_k} \cdots s_{j_1}$. Note that

Fact 2. *If $y \in S(x)$, then $T(x) = T(y)$.*

Denote by $|w|$ the length of the word $w \in W$, which is defined to be the smallest number p such that w can be written as a product of p generators. If $|w| = p$ and $w = s_{i_p} \cdots s_{i_1}$, then we say $s_{i_p} \cdots s_{i_1}$ is *reduced*. Also, for any $w, w' \in W$, we observe

Fact 3. *$|w| = 0$ if and only if $w = \varepsilon$;*

Fact 4. *$|w| = 1$ if and only if $w = s_i$, for some $s_i \in \mathcal{S}$; and*

Fact 5. *For all $s_i \in \mathcal{S}$, $|ws_i| = |s_i w| = |w| \pm 1$.*

Let $J \subseteq I_n$. Denote by W_J the subgroup of W generated by $\{s_i\}_{i \in J}$ (we call this a *parabolic* subgroup), which is the smallest subgroup of W containing $\{s_i\}_{i \in J}$. Let $W^J = \{w \in W : |ws_j| > |w|, \forall j \in J\}$. The following theorem can be found in Chapter 2 of the Bjorner and Brenti textbook *Combinatorics of Coxeter Groups* (see also [3], page 13).

Theorem 1. *For all $w \in W$, there exist unique elements $w_J \in W_J$ and $w^J \in W^J$ such that $w = w^J w_J$. In this case, $|w| = \left|w^J\right| + |w_J|$ and w^J is the unique smallest length element of the coset wW_J.*

Finally, we have the following proposition, the proof of which is a direct consequence of the proof of Tits' Theorem.

Proposition. *Proving Tits' theorem is not so easy.*

Proof. See §2... $\square$

2 The Theorem and Its Proof

We state Tits' Theorem as in [2] and adapt a proof given in [1] through changes in notation and addition of details.

Theorem. [Tits' Theorem for the Word Problem on Coxeter Groups] *For* $x, y \in I_n^*$, $T(x) = T(y)$ *if and only if* $S(x) \bigcap S(y) \neq \varnothing$.

Proof. Suppose $S(x) \bigcap S(y) \neq \varnothing$. Then, there exists $z \in S(x) \bigcap S(y)$, and by Fact 2 we obtain $T(x) = T(z) = T(y)$.

Now, suppose $T(x) = T(y)$. Without loss of generality, assume $l(y) \leq l(x)$. Define the index of of the integer pair $(l(x), l(y))$ to be $\rho\,(l(x), l(y)) := \frac{1}{2} l(x)\,[l(x) + 1] + l(y)$. We induct on this index. If $\rho\,(l(x), l(y)) = 0$, then if follows that $l(x) = l(y) = 0$, so $x = y = ()$ and $S(x) \bigcap S(y) = \{()\} \neq \varnothing$. Assume now that $\rho\,(l(x), l(y)) > 0$, i.e. $l(x) > 0$. We take the following as our induction hypothesis: if (x_*, y_*) is any pair of sequences in I_n^* with $T(x_*) = T(y_*)$, $l(y_*) \leq l(x_*)$, and $\rho\,(l(x_*), l(y_*)) < \rho\,(l(x), l(y))$, then $S(x_*) \bigcap S(y_*) \neq \varnothing$. Note that the condition $\rho\,(l(x_*), l(y_*)) < \rho\,(l(x), l(y))$ can be expressed as: either $l(x_*) < l(x)$ or $l(x_*) = l(x)$ and $l(y_*) < l(y)$.

Suppose there exists $z \in S(x)$ with $l(z) < l(x)$. Then, one of the pairs (y, z) or (z, y) satisfies the induction hypothesis, so $S(z) \bigcap S(y) \neq \varnothing$. But, $S(z) \subseteq S(x)$ by Lemma 1, so $S(x) \bigcap S(y) \neq \varnothing$. Hence, we may now assume that $l(z) = l(x)$ for all $z \in S(x)$.

Since $l(x) > 0$, $x = (i, x_*)$, for some $i \in I_n$, $x_* \in I_n^*$. Suppose there exists $z \in S(x_*)$ with $l(z) < l(x_*)$. Then, $(i, z) \in S(x)$ by Lemma 3. But, $l\,(i, z) \leq l(i) + l(z) < 1 + l(x_*) = l(x)$, so $S(x)$ contains an element whose length is less that $l(x)$, which is a contradiction. Thus, $l(z) = l(x_*) = l(x) - 1$ for all $z \in S(x_*)$.

We now show by deriving contradiction that $l(y) > 0$. Suppose $l(y) = 0$. Then, $T(x) = T(y) = \varepsilon$ and $l(x) > 0$, so we obtain $l(x) \geq 2$, for, otherwise, if $l(x) = 1$, then $T(x) = s_i$ for some $s_i \in \mathcal{S}$, and $s_i \neq \varepsilon$ (see [2], page 6). With $x = (i, x_*)$ as before, we have that $S(x_*)$ contains no sequence shorter than x_* and $T(x) = T(i, x_*) = T(x_*)s_i$. But, $\varepsilon = T(x) = T(x_*)s_i$, hence $T(x_*) = s_i$. Note that $l(x_*) = l(x) - 1 \geq 1$. Now, suppose $l(x_*) > 1$ and consider the sequence $x' := (i)$, so $l(x') = 1$. Then, we have $T(x') = s_i = T(x_*)$, $l(x') < l(x_*)$, and $l(x_*) < l(x)$, so the induction hypothesis applies to the pair (x_*, x') to yield $S(x_*) \bigcap S(x') \neq \varnothing$. But, $S(x') = \{(i)\}$, so $(i) \in S(x_*)$, which is a contradiction since $S(x_*)$ contains no sequence shorter than x_* and $l(x_*) > 1$. Thus, it follows that $l(x_*) = 1$, so $x_* = (i)$ and $x = (i, x_*) = (i, i)$, whence $() \in S(x)$, a contradiction since $l(z) = l(x)$ for all $z \in S(x)$. Therefore, it follows that $l(y) = 0$ is impossible, so $0 < l(y) \leq (l(x)$.

Now, we may write $y = (i, y_*)$, where $y_* \in I_n^*$ and $i \in I_n$ is the same as in $x = (i, x_*)$, or $y = (j, y^*)$, where $y^* \in I_n^*$ and $j \in I_n$ is different from the i in $x = (i, x_*)$. First, suppose $y = (i, y_*)$, where $y_* \in I_n^*$ and $i \in I_n$ is the same as in $x = (i, x_*)$. Then, $T(x_*)s_i = T(i, x_*) = T(x) = T(y) = T(i, y_*) = T(y_*)s_i$, so $T(x_*) = T(y_*)$, and $l(y_*) = l(y) - 1 \leq l(x) - 1 = l(x_*) < l(x)$, so the induction hypothesis applies to the pair (x_*, y_*) to yield $S(x_*) \bigcap S(y_*) \neq \varnothing$. Let $z \in S(x_*) \bigcap S(y_*)$. By Lemma 3, it follows that $(i, z) \in S(x)$ and $(i, z) \in S(y)$, so $S(x) \bigcap S(y) \neq \varnothing$.

Finally, suppose $y = (j, y_*)$ where $y_* \in I_n^*$ and $i \neq j$ as in $x = (i, x_*)$. Suppose $l(y) < l(x)$. Then, $T(i, y) = T(y)s_i = T(x)s_i = T(i, x_*)s_i = T(x_*)s_is_i = T(x_*)$ and either $l(i, y) \leq l(x_*) < l(x)$ or $l(x_*) < l(i, y) = l(x)$, so $(x_*, (i, y))$ or $((i, y), x_*)$ is a pair satisfying the induction hypothesis, so $S(i, y) \bigcap S(x_*) \neq \varnothing$. Let $z \in S(i, y) \bigcap S(x_*)$. Then, we must have $l(x_*) = l(z) \leq l(i, y) \leq l(x) = l(x_*) + 1$. But, $l(i, y) = l(x_*) + 1$ is impossible since elementary simplifications reduce length by zero or two only. Thus, $l(i, y) = l(x_*)$, so it follows that we may pass from x_* to (i, y) through z by a finite number of braid relations, hence $(i, y) \in S(x_*)$ so $S(x_*) = S(i, y)$ by Lemma

2. Now, since $z \in S(x_*)$ and $x = (i, x_*)$, we have $(i, z) \in S(x)$ by Lemma 3, so $(i, i, y) \in S(x)$ since $(i, y) \in S(x_*)$. But, a length-reducing simplification applied to (i, i, y) then places $y \in S(x)$, which is a contradiction since $l(y) < l(x)$. Hence, we must have $l(x) = l(y)$. Now, let $J := \{i, j\} \subseteq I_n$ and consider W_J, the subgroup of W generated by s_i and s_j. Note that W_J is a dihedral group. Then, $T(j, x) = T(x)s_j = T(y)s_j = T(j, y)$, so $|T(j, x)| = |T(j, y)| < |T(y)| = |T(x)|$. Similarly, we obtain $|T(i, y)| < |T(y)|$. Let $w := T(x) = T(y)$, so that $|w| > |ws_i|$ and $|w| > |ws_j|$. By Theorem 1, we may write $w = w^J w_J$, where $|w| = \left|w^J\right| + |w_J|$ and w^J is the unique smallest length element of the coset wW_J. Suppose w_J is not the longest element in W_J. Then, a shortest expression for w_J is written uniquely as a product of s_i's and s_j's. Without loss of generality, assume w_J ends in s_i. Then, $ws_j = w^J w_J s_j$, so $|ws_j| = \left|w^J\right| + |w_J s_j| > \left|w^J\right| + |w_J| = |w|$, which is a contradiction. Thus, w_J must be longest in W_J and so $m_{ij} < \infty$. From this, we see that w_J can be written to end in either s_i or s_j. Now, since $w = T(y) = T(j, y_*) = T(y_*)s_j$, we may write $w = w^J (\cdots s_i s_j s_i s_j) = w^J (\cdots s_i s_j s_i) s_j$ so that the expression $(\cdots s_i s_j s_i s_j)$ has m_{ij} factors. Let $y' := (i, j, i, \ldots, z)$, where z is a fixed sequence for w^J and without z the initial $(i, j, i, \ldots)$ portion of y' has length $m_{ij} - 1$. Then, $T(y') = T(y_*)$. Similarly, we obtain a sequence $x' := (j, i, j, \ldots, z)$, for the same z as before, with $T(x') = T(x_*)$. Then, by the induction hypothesis, $S(y_*) \bigcap S(y') \neq \varnothing$ and $S(x_*) \bigcap S(x') \neq \varnothing$. Let $z_1 \in S(x_*) \bigcap S(x')$ and $z_2 \in S(y_*) \bigcap S(y')$. Now, everything in $S(x')$ must have the same length, for otherwise if $v \in S(x')$ is shorter than x', then $l(i, v) \leq l(v) + 1 < l(x') + 1 = |w|$ and $T(i, v) = T(v)s_i = T(x')s_i = w$, which is a contradiction. So, it follows that we may obtain x' from x_* by a finite sequence of braid simplifications through z_1, hence $x' \in S(x_*)$. Similarly, we find $y' \in S(y_*)$. But, every sequence in $S(x_*)$ has the same length, so we have $l(x') = l(x_*)$, so $S(x_*) = S(x')$ by Lemma 2. Now, $l(y_*) \geq l(y') = l(x') = l(x_*)$, so $l(y_*) = l(y')$ since $l(y) = l(x)$, we have $l(y_*) = l(x_*)$ and so $l(y_*) = l(y')$. Thus, $S(y_*) = S(y')$ by Lemma 2. Now, $(j, y') \in S(y)$ and $(i, x') \in S(x)$ by Lemma 3, and $(i, x') = (\underbrace{i, j, i, j, \ldots}_{\text{length } m_{ij}}, z)$ can be obtained from $(j, y') = (\underbrace{j, i, j, i, \ldots}_{\text{length } m_{ij}}, z)$ by a braid relation, so we have $(i, x') \in S(y)$. Therefore, $S(x) \bigcap S(y) \neq \varnothing$.

This completes the proof. □

3 Some Applications of Tits' Theorem

Corollary 1. *Let y and z be shortest-length sequences in $S(x)$, for some $x \in I_n^*$. Then, we may pass from y to z (and vice versa) by a finite sequence of braid relations.*

Proof. Since $y, z \in S(x)$, we have $S(y) \subseteq S(x)$ and $S(z) \subseteq S(x)$ by Lemma 1, so $T(y) = T(x) = T(z)$ and $S(y) \bigcap S(z) \neq \varnothing$ by Tits' Theorem. Note that $l(y) = l(z)$, since y and z are both shortest sequences in $S(x)$. Let $w \in S(y) \bigcap S(z)$. Then, $l(w) = l(y) = l(z)$ since $l(w) \leq l(y) = l(z)$ by Fact 1 and $w \in S(x)$. Hence, y and w and z and w are related by finite sequences of braid relations, so it follows that y and z are related (by passing through w) by a finite sequence of braid relations. □

Thus, we see that if a word has more than one reduced expression, then we may move between these reduced expressions by braid relations. Furthermore, for any $z \in S(x)$ and any shortest $y \in S(x)$, for some $x \in I_n^*$, we may obtain y from z by applying braid relations to a shortest-length representation of z in $S(x)$. Notice that we may need to perform these braid relations in an appropriate dihedral subgroup of the original Coxeter group if m_{ij} is larger than the length of the shortest sequence(s).

Next, we show that for $w \in W$, $|w|$, which we defined to be the smallest number p such that w can be written as a product of p generators, is the length of any shortest sequence in any $S(x)$ for which $w = T(x)$.

Corollary 2. *Let $x \in I_n^*$ and let $y = (i_1, \ldots, i_p)$ be a shortest length sequence in $S(x)$. Let $w = T(x)$. Then, $w = s_{i_p} \cdots s_{i_1}$, and any expression of w as a product of generators must use at least p generators.*

Proof. Since $y \in S(x)$, $S(x) \bigcap S(y) \neq \varnothing$, so $w = T(x) = T(y) = s_{i_p} \cdots s_{i_1}$ by Tits' Theorem. Now, suppose $s_{j_k} \cdots s_{j_1}$ where $k < p$ is an expression for w as a product of generators. Then, $T(j_1, \ldots, j_k) = s_{j_k} \cdots s_{j_1} = w = T(x)$, so $S(j_1, \ldots, j_k) \bigcap S(x) \neq \varnothing$ by Tits' Theorem. Thus, since $(j_1, \ldots, j_k)$ is shorter than the shortest sequence in $S(x)$, it follows that we may obtain $(j_1, \ldots, j_k)$ from x by a finite sequence of elementary simplifications, so that $(j_1, \ldots, j_k) \in S(x)$, which contradicts the minimality of y in $S(x)$. Therefore, any expression of w as a product of generators must use at least p generators. □

Thus, for $x \in I_n^*$, if $w = T(x)$ and y is a shortest length sequence in $S(x)$, we have $|w| = l(y)$. Moreover, since elementary simplifications change sequence length by zero or two, we also have $|w|$ and $l(z)$ have the same parity for all sequences $z \in S(x)$.

Corollary 3. *Let $x = (j_1, \ldots, j_k) \in I_n^*$, and let $w := T(x)$. Then $w = \varepsilon$ if and only if $() \in S(x)$.*

Proof. If $() \in S(x)$, then $S(x) \bigcap S(()) \neq \varnothing$, so $w = T(x) = T(()) = \varepsilon$ by Tits' Theorem. Conversely, if $T(x) = w = \varepsilon = T(())$, then $S(x) \bigcap \{()\} = S(x) \bigcap S(()) \neq \varnothing$ by Tits' Theorem, so $() \in S(x)$. □

Therefore, we see that the word problem on Coxeter groups is solvable. Indeed, given two words $w, w' \in W$, we may determine if $w = w'$ using Corollary 3 to check if $w'w^{-1} = \varepsilon$ by computing elementary simplifications of $x \in I_n^*$, where $T(x) = w'w^{-1}$, until we either encounter () or exhaust all elements of $S(x)$ (recall that this set is finite) without encountering ().

Theorem. [Eriksson's Reduced Word Theorem] *Suppose $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ is a legal firing sequence from some start position λ on the SC-graph $\mathcal{G} = (\Gamma, A)$. Then, $s_{i_p} \cdots s_{i_1}$ is a reduced expression in $W = W(\mathcal{G})$.*

Here, the proof given in [3] proceeds by contrapositive and Tit's Theorem is used to obtain equivalent words in W by performing elementary simplifications on $(i_1, \ldots, i_p)$. Thus, there is a connection between Tits' Theorem and the "numbers game" discussed in Unit 1 of Dr. Donnelly's lecture notes, which hints at the deeper connection between Coxeter groups and the numbers game.

References

[1] Casselman, B. Lecture notes on Coxeter groups. CRM Winter School on Coxeter Groups, 2002. Archived online: http://www.math.ubc.ca/~cass/coxeter/crm.html

[2] Donnelly, R. G. "Unit 2: More about Generators and Relations." Lecture notes on Coxeter Groups and Combinatorics, Fall 2009. Archived online: http://campus.murraystate.edu/academic/faculty/rob.donnelly/CoxeterGroupsSeminar/index.htm

[3] Donnelly, R. G. "Unit 3: Asymmetric Geometric Representations of Coxeter Groups." Lecture notes on Coxeter Groups and Combinatorics, Fall 2009. Archived online: same as above.