Word Processing in Coxeter Groups∗

Benjiman C. Tharp

Murray State University
Department of Mathematics and Statistics
MAT691 - Seminar in Coxeter Groups
Fall 2009

Abstract

Several definitions, facts, and theorems stated during the recent talks given by Dr. Donnelly relating to the “word problem” for Coxeter groups are restated in §1 to provide some background information and notation. Three other facts, due to Casselman, are also stated and proved as lemmas in this section. In §2, we state and prove Tits’ Theorem for the word problem on Coxeter groups. Finally, we consider some applications and corollaries of Tits’ Theorem in §3.

1 Preliminaries

Let $W = \langle S | R \rangle$ be a Coxeter group with $S = \{s_i\}_{i \in I_n}$, where $I_n$ is a finite index set, and $R = \{(s_is_j)^{m_{ij}} : m_{ii} = 1, m_{ij} \in \{\infty, 2, 3, 4, \ldots \} \text{ for } i \neq j, m_{ij} = m_{ji}\}$. Let $I_n^*$ denote the set of all finite sequences from $I_n$, including the empty sequence $()$. For $x \in I_n^*$ with $x = (j_1, \ldots, j_k)$, say the length of $x$, denoted by $l(x)$, is $k$. Define the following two elementary simplifications on elements of $I_n^*$:

- Length-reducing: Replace a subsequence of the form $(i, i)$ with the empty sequence;
- Braid: Replace a subsequence of the form $(i, j, i, \ldots)$ with $(j, i, j, \ldots)$.

Now, for $x \in I_n^*$, let $S(x)$ be the set of sequences $y \in I_n^*$ that are obtainable from $x$ by a finite sequence of elementary simplifications. Note that $y \in S(x)$ may be obtained from $x$ by an empty sequence of elementary simplifications, hence $x \in S(x)$. Also, we must have the following

**Fact 1.** If $y \in S(x)$ for some $x \in I_n^*$, then $l(y) \leq l(x)$.

Notice that $S(x)$ is a finite set since it is a subset of the set of all sequences with length less than or equal to $l(x)$, which is a finite set. We are now prepared to prove the following three lemmas, which will be required in the proof of Tits’ Theorem.

---

∗The title “Word Processing” is due to Casselman.
Lemma 1. If \( y \in S(x) \), then \( S(y) \subseteq S(x) \).

*Proof.* Let \( z \in S(y) \). Then, \( z \) can be obtained from \( y \) by a finite sequence of elementary simplifications. Now, \( y \) can be obtained from \( x \) by a finite sequence of elementary simplifications, hence \( z \) can be obtained from \( x \) by a finite sequence of elementary simplifications by first passing from \( x \) to \( y \). Thus, \( z \in S(x) \).

Lemma 2. If \( y \in S(x) \) and \( l(x) = l(y) \), then \( S(x) = S(y) \).

*Proof.* By Lemma 1, \( S(y) \subseteq S(x) \). Since \( l(x) = l(y) \), it follows that \( x \) can be obtained from \( y \) by a finite sequence of braid relations, so \( x \in S(y) \). Applying Lemma 1 again, we have \( S(x) \subseteq S(y) \). Hence, \( S(x) = S(y) \).

Lemma 3. Let \( x = (i, x') \), for some \( i \in I_n \), \( x' \in I_n^* \). Then, \( (i, y) \in S(x) \) for all \( y \in S(x') \).

*Proof.* Focusing on the subsequence \( x' \) of \( x \), we may pass to \( y \in S(x') \) by a finite sequence of elementary simplifications, obtaining \( (i, y) \) from \( (i, x') = x \). Hence, \( (i, y) \in S(x) \).

Now, let \( T : I_n^* \to W \) be given by \( T(j_1, \ldots, j_k) = s_{j_k} \cdots s_{j_1} \). Note that

Fact 2. If \( y \in S(x) \), then \( T(x) = T(y) \).

Denote by \( |w| \) the length of the word \( w \in W \), which is defined to be the smallest number \( p \) such that \( w \) can be written as a product of \( p \) generators. If \( |w| = p \) and \( w = s_{i_p} \cdots s_{i_1} \), then we say \( s_{i_p} \cdots s_{i_1} \) is reduced. Also, for any \( w, w' \in W \), we observe

Fact 3. \( |w| = 0 \) if and only if \( w = \varepsilon \);

Fact 4. \( |w| = 1 \) if and only if \( w = s_i \), for some \( s_i \in S \); and

Fact 5. For all \( s_i \in S \), \( |ws_i| = |s_iw| = |w| \pm 1 \).

Let \( J \subseteq I_n \). Denote by \( W_J \) the subgroup of \( W \) generated by \( \{s_i\}_{i \in J} \) (we call this a **parabolic** subgroup), which is the smallest subgroup of \( W \) containing \( \{s_i\}_{i \in J} \). Let \( W^J = \{w \in W : |ws_j| > |w|, \forall j \in J \} \). The following theorem can be found in Chapter 2 of the Bjorner and Brenti textbook *Combinatorics of Coxeter Groups* (see also [3], page 13).

**Theorem 1.** For all \( w \in W \), there exist unique elements \( w_J \in W_J \) and \( w^J \in W^J \) such that \( w = w^J w_J \). In this case, \( |w| = |w^J| + |w_J| \) and \( w^J \) is the unique smallest length element of the coset \( w W_J \).

Finally, we have the following proposition, the proof of which is a direct consequence of the proof of Tits’ Theorem.

**Proposition.** Proving Tits’ theorem is not so easy.

*Proof.* See §2...
2 Theorem and Its Proof

We state Tits’ Theorem as in [2] and adapt a proof given in [1] through changes in notation and addition of details.

**Theorem.** [Tits’ Theorem for the Word Problem on Coxeter Groups] For \( x, y \in I_n^* \), \( T(x) = T(y) \) if and only if \( S(x) \cap S(y) \neq \emptyset \).

**Proof.** Suppose \( S(x) \cap S(y) \neq \emptyset \). Then, there exists \( z \in S(x) \cap S(y) \), and by Fact 2 we obtain \( T(z) = T(z) = T(y) \).

Now, suppose \( T(x) = T(y) \). Without loss of generality, assume \( l(y) \leq l(x) \). Define the index of the integer pair \((l(x), l(y)) \) to be \( \rho(l(x), l(y)) := \frac{1}{2}l(x)[l(x) + 1] + l(y) \). We induct on this index. If \( \rho(l(x), l(y)) = 0 \), then if follows that \( l(x) = l(y) = 0 \), so \( x = y = () \) and \( S(x) \cap S(y) = \{()\} \neq \emptyset \). Assume now that \( \rho(l(x), l(y)) > 0 \), i.e. \( l(x) > 0 \). We take the following as our induction hypothesis: if \((x_s, y_s)\) is any pair of sequences in \(I_n^*\) with \( T(x_s) = T(y_s), l(y_s) \leq l(x_s), \) and \( \rho(l(x_s), l(y_s)) < \rho(l(x), l(y)) \), then \( S(x_s) \cap S(y_s) \neq \emptyset \). Note that the condition \( \rho(l(x_s), l(y_s)) < \rho(l(x), l(y)) \) can be expressed as: either \( l(x_s) < l(x) \) or \( l(x_s) = l(x) \) and \( l(y_s) < l(y) \).

Suppose there exists \( z \in S(x) \) with \( l(z) < l(x) \). Then, one of the pairs \((y, z)\) or \((z, y)\) satisfies the induction hypothesis, so \( S(z) \cap S(y) \neq \emptyset \). But, \( S(z) \subseteq S(x) \) by Lemma 1, so \( S(z) \cap S(y) \neq \emptyset \). Hence, we may now assume that \( l(z) = l(x) \) for all \( z \in S(x) \).

Since \( l(x) > 0 \), \( x = (i, x_s) \), for some \( i \in I_n, x_s \in I_n^* \). Suppose there exists \( z \in S(x_s) \) with \( l(z) < l(x_s) \). Then, \((i, z) \in S(x) \) by Lemma 3. But, \( l(i, z) \leq l(i) + l(z) < 1 + l(x_s) = l(x) \), so \( S(x) \cap S(y) \neq \emptyset \). Hence, \( l(z) = l(x_s) = l(x) - 1 \) for all \( z \in S(x_s) \).

We now show by deriving contradiction that \( l(y) > 0 \). Suppose \( l(y) = 0 \). Then, \( T(x) = T(y) = \varepsilon \) and \( l(x) > 0 \), so we obtain \( l(x) \geq 2 \), for, otherwise, if \( l(x) = 1 \), then \( T(x) = s_i \) for some \( s_i \in S, \) and \( s_i \neq \varepsilon \) (see [2], page 6). With \( x = (i, x_s) \) as before, we have that \( S(x_s) \) contains no sequence shorter than \( x_s \), and \( T(x) = T(i, x_s) = T(x_s)s_i \). But, \( \varepsilon = T(x) = T(x_s)s_i \), hence \( T(x_s) = s_i \). Note that \( l(x_s) = l(x) - 1 \geq 1 \). Now, suppose \( l(x_s) > 1 \) and consider the sequence \( x' := (i) \), so \( l(x') = 1 \). Then, we have \( T(x') = s_i = T(x_s), l(x') < l(x_s), \) and \( l(x_s) < l(x) \), so the induction hypothesis applies to the pair \((x_s, x')\) to yield \( S(x_s) \cap S(x') \neq \emptyset \). But, \( S(x') = \{(i)\} \), so \((i) \in S(x_s) \), which is a contradiction since \( S(x_s) \) contains no sequence shorter than \( x_s \) and \( l(x_s) > 1 \). Thus, it follows that \( l(x_s) = 1 \), so \( x_s = (i) \) and \( x = (i, x_s) = (i, i) \), whence \(() \in S(x) \), a contradiction since \( l(z) = l(x) \) for all \( z \in S(x) \). Therefore, it follows that \( l(y) = 0 \) is impossible, so \( 0 < l(y) \leq l(x) \).

Now, we may write \( y = (i, y_s) \), where \( y_s \in I_n^* \) and \( i \in I_n \) is the same as in \( x = (i, x_s) \), or \( y = (j, y^*) \), where \( y^* \in I_n^* \) and \( j \in I_n \) is different from the \( i \) in \( x = (i, x_s) \). First, suppose \( y = (i, y_s) \), where \( y_s \in I_n^* \) and \( i \in I_n \) is the same as in \( x = (i, x_s) \). Then, \( T(x_s)s_i = T(i, x_s) = T(x) = T(y) = T(i, y_s) = T(y_s)s_i \), so \( T(x_s) = T(y_s) \), and \( l(y_s) - l(x_s) \geq 1 \). Hence, \( l(x_s) = l(x) - 1 \leq l(x) - 1 = l(x_s) < l(x) \), so the induction hypothesis applies to the pair \((x_s, y_s)\) to yield \( S(x_s) \cap S(y_s) \neq \emptyset \). Let \( z \in S(x_s) \cap S(y_s) \). By Lemma 3, it follows that \((i, z) \in S(x) \) and \((i, z) \in S(y) \), so \( S(x) \cap S(y) \neq \emptyset \).

Finally, suppose \( y = (j, y^*) \) where \( y_s \in I_n^* \) and \( i \neq j \) as in \( x = (i, x_s) \). Suppose \( l(y) < l(x) \). Then, \( T(i, y) = T(y)s_i = T(x)s_i = T(i, x_s)s_i = T(x_s)s_i = T(x) \) and either \( l(i, y) < l(x_s) < l(x) \) or \( l(x_s) < l(i, y) = l(x) \), so \((x_s, (i, y))\) or \((i, y, x_s)\) is a pair satisfying the induction hypothesis, so \( S(i, y) \cap S(x_s) \neq \emptyset \). Let \( z \in S(i, y) \cap S(x_s) \). Then, we must have \( l(x_s) = l(z) \leq l(i, y) \leq l(x) = l(x_s) + 1 \). But, \( l(i, y) = l(x_s) + 1 \) is impossible since elementary simplifications reduce length by zero or two only. Thus, \( l(i, y) = l(x_s) \), so it follows that we may pass from \( x_s \) to \((i, y)\) through \( z \) by a finite number of braid relations, hence \((i, y) \in S(x_s) \) so \( S(x_s) = S(i, y) \) by Lemma
2. Now, since \( z \in S(x_\ast) \) and \( x = (i,x_\ast) \), we have \((i,z) \in S(x)\) by Lemma 3, so \((i,i,y) \in S(x)\) since \((i,y) \in S(x_\ast)\). But, a length-reducing simplification applied to \((i,i,y)\) then places \( y \in S(x)\), which is a contradiction since \( l(y) < l(x)\). Hence, we must have \( l(x) = l(y) \). Now, let \( J := \{i,j\} \subseteq I_n \) and consider \( W_J \), the subgroup of \( W \) generated by \( s_i \) and \( s_j \). Note that \( W_J \) is a dihedral group. Then, \( T(j,x) = T(x)s_j = T(y)s_j = T(j,y), \) so \( |T(j,x)| = |T(j,y)| < |T(y)| = |T(x)| \). Similarly, we obtain \( |T(i,y)| < |T(y)| \). Let \( w := T(x) = T(y), \) so that \( |w| > |ws_i| \) and \( |w| > |ws_j| \). By Theorem 1, we may write \( w = w^J w_J \), where \( |w| = |w^J| + |w_J| \) and \( w^J \) is the unique smallest length element of the coset \( wW_J \). Suppose \( w_J \) is not the longest element in \( W_J \). Then, a shortest expression for \( w_J \) is written uniquely as a product of \( s_i \)'s and \( s_j \)'s. Without loss of generality, assume \( w_J \) ends in \( s_i \). Then, \( ws_j = w^J w_J s_j \), so \( |ws_j| = |w^J| + |w_J s_j| > |w^J| + |w_J| = |w| \), which is a contradiction. Thus, \( w_J \) must be longest in \( W_J \) and so \( m_{ij} < \infty \). From this, we see that \( w_J \) can be written to end in either \( s_i \) or \( s_j \). Now, since \( w = T(y) = T(j,y_s) = T(y_s) s_j \), we may write \( w = w^J (\cdots s_js_j s_i s_j) s_j \) so that the expression \((\cdots s_js_j s_i s_j)\) has \( m_{ij} \) factors. Let \( y' := (i,j,i,\ldots,z) \), where \( z \) is a fixed sequence for \( w^J \) and without \( z \) the initial \((i,j,i,\ldots)\) portion of \( y' \) has length \( m_{ij} - 1 \). Then, \( T(y') = T(y_s) \). Similarly, we obtain a sequence \( x' := (j,i,j,\ldots,z) \), for the same \( z \) as before, with \( T(x') = T(x_\ast) \). Then, by the induction hypothesis, \( S(y_s) \cap S(y') \neq \emptyset \) and \( S(x_\ast) \cap S(x') \neq \emptyset \). Let \( z_1 \in S(x_\ast) \cap S(x') \) and \( z_2 \in S(y_s) \cap S(y') \). Now, everything in \( S(x') \) must have the same length, for otherwise if \( v \in S(x') \) is shorter than \( x' \), then \( l(i,v) \leq l(v) + 1 < l(x') + 1 = |w| \) and \( T(i,v) = T(v) s_j = T(x') s_j = w \), which is a contradiction. So, it follows that we may obtain \( x' \) from \( x_\ast \) by a finite sequence of braid simplifications through \( z_1 \), hence \( x' \in S(x_\ast) \). Similarly, we find \( y' \in S(y_s) \). But, every sequence in \( S(x_\ast) \) has the same length, so we have \( l(x') = l(x_\ast) \), so \( S(x_\ast) = S(x') \) by Lemma 2. Now, \( l(y_s) \geq l(y') = l(x') = l(x_\ast) \), so \( l(y_s) = l(y') \) since \( l(y) = l(x) \), we have \( l(y_s) = l(x_\ast) \) and so \( l(y_s) = l(y') \). Thus, \( S(y_s) = S(y') \) by Lemma 2. Now, \((j,y') \in S(y) \) and \((i,x') \in S(x) \) by Lemma 3, and \((i,x') = (i,j,i,j,\ldots,z)\) can be obtained from \((j,y') = (i,j,i,j,\ldots,z)\) by a braid relation, so we have \((i,x') \in S(y) \). Therefore, \( S(x) \cap S(y) \neq \emptyset \).

This completes the proof.

3 Some Applications of Tits' Theorem

Corollary 1. Let \( y \) and \( z \) be shortest-length sequences in \( S(x) \), for some \( x \in I_n^\ast \). Then, we may pass from \( y \) to \( z \) (and vice versa) by a finite sequence of braid relations.

Proof. Since \( y,z \in S(x) \), we have \( S(y) \subseteq S(x) \) and \( S(z) \subseteq S(x) \) by Lemma 1, so \( T(y) = T(x) = T(z) \) and \( S(y) \cap S(z) \neq \emptyset \) by Tits' Theorem. Note that \( l(y) = l(z) \), since \( y \) and \( z \) are both shortest sequences in \( S(x) \). Let \( w \in S(y) \cap S(z) \). Then, \( l(w) = l(y) = l(z) \) since \( l(w) \leq l(y) = l(z) \) by Fact 1 and \( w \in S(x) \). Hence, \( y \) and \( w \) and \( z \) and \( w \) are related by finite sequences of braid relations, so it follows that \( y \) and \( z \) are related (by passing through \( w \)) by a finite sequence of braid relations. 

Thus, we see that if a word has more than one reduced expression, then we may move between these reduced expressions by braid relations. Furthermore, for any \( z \in S(x) \) and any shortest \( y \in S(x) \), for some \( x \in I_n^\ast \), we may obtain \( y \) from \( z \) by applying braid relations to a shortest-length representation of \( z \) in \( S(x) \). Notice that we may need to perform these braid relations in an appropriate dihedral subgroup of the original Coxeter group if \( m_{ij} \) is larger than the length of the shortest sequence(s).
Next, we show that for \( w \in W \), \(|w|\), which we defined to be the smallest number \( p \) such that \( w \) can be written as a product of \( p \) generators, is the length of any shortest sequence in any \( S(x) \) for which \( w = T(x) \).

**Corollary 2.** Let \( x \in I_n^* \) and let \( y = (i_1, \ldots, i_p) \) be a shortest length sequence in \( S(x) \). Let \( w = T(x) \). Then, \( w = s_{i_p} \cdots s_{i_1} \), and any expression of \( w \) as a product of generators must use at least \( p \) generators.

**Proof.** Since \( y \in S(x) \), \( S(x) \cap S(y) \neq \emptyset \), so \( w = T(x) = T(y) = s_{i_p} \cdots s_{i_1} \) by Tits’ Theorem. Now, suppose \( s_{j_k} \cdots s_{j_1} \) where \( k < p \) is an expression for \( w \) as a product of generators. Then, \( T(j_1, \ldots, j_k) = s_{j_k} \cdots s_{j_1} = w = T(x) \), so \( S(j_1, \ldots, j_k) \cap S(x) \neq \emptyset \) by Tits’ Theorem. Thus, since \( (j_1, \ldots, j_k) \) is shorter than the shortest sequence in \( S(x) \), it follows that we may obtain \( (j_1, \ldots, j_k) \) from \( x \) by a finite sequence of elementary simplifications, so that \( (j_1, \ldots, j_k) \in S(x) \), which contradicts the minimality of \( y \) in \( S(x) \). Therefore, any expression of \( w \) as a product of generators must use at least \( p \) generators.

Thus, for \( x \in I_n^* \), if \( w = T(x) \) and \( y \) is a shortest length sequence in \( S(x) \), we have \(|w| = l(y)\). Moreover, since elementary simplifications change sequence length by zero or two, we also have \(|w| \) and \( l(z) \) have the same parity for all sequences \( z \in S(x) \).

**Corollary 3.** Let \( x = (j_1, \ldots, j_k) \in I_n^* \), and let \( w := T(x) \). Then \( w = \varepsilon \) if and only if \( () \in S(x) \).

**Proof.** If \( () \in S(x) \), then \( S(x) \cap S(() \neq \emptyset \), so \( w = T(x) = T(() \varepsilon \) by Tits’ Theorem. Conversely, if \( T(x) = w = \varepsilon = T(() \), then \( S(x) \cap S(() \neq \emptyset \) by Tits’ Theorem, so \( () \in S(x) \).

Therefore, we see that the word problem on Coxeter groups is solvable. Indeed, given two words \( w, w' \in W \), we may determine if \( w = w' \) using Corollary 3 to check if \( w'w^{-1} = \varepsilon \) by computing elementary simplifications of \( x \in I_n^* \), where \( T(x) = w'w^{-1} \), until we either encounter \( () \) or exhaust all elements of \( S(x) \) (recall that this set is finite) without encountering \( () \).

**Theorem.** [Eriksson’s Reduced Word Theorem] Suppose \( (\gamma_{i_1}, \ldots, \gamma_{i_p}) \) is a legal firing sequence from some start position \( \lambda \) on the SC-graph \( G = (\Gamma, A) \). Then, \( s_{i_p} \cdots s_{i_1} \) is a reduced expression in \( W = W(G) \).

Here, the proof given in [3] proceeds by contrapositive and Tit’s Theorem is used to obtain equivalent words in \( W \) by performing elementary simplifications on \( (i_1, \ldots, i_p) \). Thus, there is a connection between Tit’s Theorem and the “numbers game” discussed in Unit 1 of Dr. Donnelly’s lecture notes, which hints at the deeper connection between Coxeter groups and the numbers game.

**References**

