Coxeter groups and Combinatorics

Rob Donnelly
MSU
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Unit 3: Asymmetric Geometric Representations of Coxeter Groups — PART 1

Throughout these notes, "References" come from the annoted references webpage linked to from the site where these notes are posted.

We will show how there are many ways to view an arbitrary Coxeter group as a collection of invertible linear transformations on a real vector space whose geometry is given by a possibly asymmetric bilinear form. These representations were discovered independently by Vinberg (1970's) and Eriksson (1990's). One object of interest for us will be a convex cone (the so-called Tits cone, named after Jacques Tits, a Belgian/French mathematician, Abel prize winner in 2008, and progenitor of much of the basic theory of Coxeter groups) created by an associated action of the Coxeter group on a certain "polyhedral" fundamental domain. We will connect these Coxeter group representations/actions to the numbers game of Unit 1.

- **Set-up**

$\mathcal{G} = (\Gamma, A)$ an SC graph with nodes $\{\gamma_i\}_{i \in I_n}$ indexed by an n-set I_n.

Let $W := W(\mathcal{G})$ be the Coxeter group $\langle \mathcal{S} \mid \mathcal{R} \rangle$,

where $\mathcal{S} = \{s_i\}_{i \in I_n}$ and $\mathcal{R} = \{(s_i s_j)^{m_{ij}} = \varepsilon\}_{i,j \in I_n}$

$$\text{with} \quad m_{ij} = \begin{cases} 1 & \text{if } i = j \\ k_{ij} & \text{if } i \neq j \text{ and } a_{ij}a_{ji} = 4\cos^2(\pi/k_{ij}) \text{ with } k_{ij} \in \{2,3,4,\dots\} \\ \infty & \text{if } i \neq j \text{ and } a_{ij}a_{ji} \geq 4 \end{cases}$$

- A representing space for W

Let V be a real vector space freely generated by a basis $\mathcal{B} = \{\alpha_i\}_{i \in I_n}$.

The α_i's will be called "simple roots".

Equip V with the bilinear form $B : V \times V \to \mathbb{R}$ for which

$$[B]_{\mathcal{B}} = \frac{1}{2} A ,$$

i.e. $B(\alpha_i, \alpha_j) = \frac{1}{2} a_{ij}$ for all $i, j \in I_n$, and then extend B bilinearly.

Example $A = \begin{bmatrix} 2 & -2 \\ -1/2 & 2 \end{bmatrix} \Rightarrow \frac{1}{2} A = \begin{bmatrix} 1 & -1 \\ -1/4 & 1 \end{bmatrix}$

Let $u = 3\alpha_1 + \alpha_2$ and $v = \alpha_1 + 2\alpha_2$.

Then $B(u, v) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1/4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7/4 \end{bmatrix} = -5/4$

For $i \in I_n$, define a linear transformation $S_i : V \to V$ by the rule

$$S_i(v) = v - 2B(\alpha_i, v)\alpha_i$$

Proposition 1. For all $i \in I_n$, $S_i^2 = Id|_V$. Hence $S_i \in GL(V)$.

2. For all $i \neq j$ in I_n, $S_i S_j$ has order m_{ij} as an element of the group $GL(V)$.

Proof: The proof of 1 is straightforward:

$$S_i^2(v) = S_i(S_i(v)) = S_i(v - 2B(\alpha_i, v)\alpha_i)$$
$$= S_i(v) - 2B(\alpha_i, v) S_i(\alpha_i)$$
$$= v - 2B(\alpha_i, v)\alpha_i - 2B(\alpha_i, v)(\alpha_i - 2B(\alpha_i, \alpha_i)\alpha_i)$$
$$= v - 2B(\alpha_i, v)\alpha_i - 2B(\alpha_i, v)(-\alpha_i)$$
$$= v - 2B(\alpha_i, v)\alpha_i + 2B(\alpha_i, v)\alpha_i = v.$$

Since $S_i^2 = Id|_V$, then $S_i \in GL(V)$ with $S_i^{-1} = S_i$.

For the proof of part 2, take $i \neq j$ in I_n. Some notation:

$V_{ij} := \mathrm{span}_{\mathbb{R}}\{\alpha_i, \alpha_j\}$, a two-dimensional subspace of V

$\mathcal{B}_{ij} = \{\alpha_i, \alpha_j\}$, a basis for V_{ij}

$\boxed{\text{Exercise}}$ Prove that $S_i(V_{ij}) \subseteq V_{ij}$ and $S_j(V_{ij}) \subseteq V_{ij}$.

$\boxed{\text{Exercise}}$ Let $X_i := \left[S_i|_{V_{ij}}\right]_{\mathcal{B}_{ij}}$ and $X_j := \left[S_j|_{V_{ij}}\right]_{\mathcal{B}_{ij}}$

Show that $X_i = \begin{bmatrix} -1 & -a_{ij} \\ 0 & 1 \end{bmatrix}$ and $X_j = \begin{bmatrix} 1 & 0 \\ -a_{ji} & -1 \end{bmatrix}$,

and hence that $X_i X_j = \begin{bmatrix} -1 + a_{ij}a_{ji} & a_{ij} \\ -a_{ji} & -1 \end{bmatrix}$.

Case 0 Suppose $a_{ij}a_{ji} = 4$.

$\boxed{\text{Exercise}}$ Show that $X_i X_j = \frac{1}{p}\underbrace{\begin{bmatrix} p & p \\ 2 & 1 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{Y}\underbrace{\begin{bmatrix} -1 & p \\ 2 & -p \end{bmatrix}}_{P^{-1}}$. Why does it follow that $(X_i X_j)^m \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for all positive integers m? Why does it follow that $(S_i S_j)|_{V_{ij}}$ has order ∞ in $GL(V_{ij})$ and that $S_i S_j$ has order ∞ in $GL(V)$?

Case 1 Suppose $a_{ij}a_{ji} > 4$.

$\boxed{\text{Exercise}}$ Show that in this case $X_i X_j$ has two distinct real eigenvalues $c_1 > 1$ and $c_2 \leq 1$. Conclude that $(S_i S_j)|_{V_{ij}}$ has order ∞ as an element of $GL(V_{ij})$. Then say why $S_i S_j$ has order ∞ as an element of $GL(V)$.

Case 2 Suppose $a_{ij}a_{ji} < 4$. First we argue that $(S_i S_j)|_{V_{ij}}$ has order m_{ij} as an element of the group $GL(V_{ij})$.

→ Take $a_{ij}a_{ji} = 0$. Then, $X_i X_j = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then, $(S_i S_j)|_{V_{ij}}$ has order 2 as an element of $GL(V_{ij})$.

→ Now take $a_{ij}a_{ji} = 4\cos^2(\pi/m_{ij})$ for $m_{ij} \in \{3, 4, 5, \dots\}$.

$\boxed{\text{Exercise}}$ Show that $X_i X_j$ has two distinct complex eigenvalues $e^{2\pi \mathbf{i}/m_{ij}}$ and $e^{-2\pi \mathbf{i}/m_{ij}}$. Conclude that $(S_i S_j)|_{V_{ij}}$ has order m_{ij} as an element of $GL(V_{ij})$.

Next we argue that $s_i s_j$ has order m_{ij} as an element of $GL(V)$.

With $a_{ij} \cdot a_{ji} < 4$, let $V_{ij}' := \{ v \in V \mid B(\alpha_i, v) = 0 = B(\alpha_j, v) \}$

$\boxed{\text{Exercise}}$ Show that V_{ij}' is a subspace of V.

Show that $V_{ij} \cap V_{ij}' = \{0\}$.

Show that $s_i \big|_{V_{ij}'} = Id \big|_{V_{ij}'}$ and $s_j \big|_{V_{ij}'} = Id \big|_{V_{ij}'}$.

$\boxed{\text{Exercise}}$ Use the "Rank + Nullity Theorem" from matrix/linear algebra to show that $\dim V_{ij}' \geq n-2$

Now let $\mathcal{B}_{ij}' := \{e_1, \dots, e_k\}$ be a basis for V_{ij}', so $k \geq n-2$.

Suppose $c_1 e_1 + \dots + c_k e_k + a\alpha_i + b\alpha_j = 0$ for some scalars c_i, a, b.

If $a \neq 0$, then $\alpha_i + \frac{b}{a}\alpha_j \neq 0$. But then

$$\alpha_i + \frac{b}{a}\alpha_j = -\frac{c_1}{a}e_1 - \dots - \frac{c_k}{a}e_k$$

is a nonzero vector in $V_{ij} \cap V_{ij}'$. So we must have $a = 0$.

Similarly argue that $b = 0$.

So now $c_1 e_1 + \dots + c_k e_k = 0$, and since $\{e_1, \dots, e_k\}$ is a basis for V_{ij}', then each $c_i = 0$.

Then $\{e_1, \ldots, e_k, \alpha_i', \alpha_j'\}$ is a linearly independent set of $k+2 \geq n$ vectors in V. But since $\dim V = n$, then $k+2 = n$, and

$\mathcal{B}' = \{e_1, \ldots, e_{n-2}, \alpha_i', \alpha_j'\}$ is a basis for V.

Then
$$[S_iS_j]_{\mathcal{B}'} = \left[\begin{array}{c|c} [S_iS_j|_{V_{ij}'}]_{\mathcal{B}_{ij}'} & 0 \\ \hline 0 & [S_iS_j|_{V_{ij}}]_{\mathcal{B}_{ij}'} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \mathcal{E}_{(n-2)\times(n-2)} & 0 \\ \hline 0 & X_i X_j' \end{array}\right].$$

Clearly this matrix has order m_{ij} as an element of $GL(n, \mathbb{R})$.

So, S_iS_j has order m_{ij} as an element of $GL(V)$. // Proposition

- <u>A "geometric" representation of W</u>

Define $\sigma : W \longrightarrow GL(V)$ to be the unique homomorphism $W \to GL(V)$ which extends the set mapping $s_i \mapsto S_i$.

<u>Picture</u>

$$\begin{tikzcd} \mathcal{S} = \{s_i\}_{i \in I_n} \arrow[r] \arrow[dr] & W \arrow[d, dashed, "\sigma"] \\ & GL(V) \end{tikzcd}$$

$s_i \mapsto S_i$

This can be done by our Proposition above...

What's "geometric" about this representation?

- → When B is symmetric and positive definite, then s_i is a reflection in the hyperplane orthogonal to α_i
- → When $a_{ij} = a_{ji}$ and $a_{ij}a_{ji} \leq 4$ for all i, j, the representation σ above is called the "standard" geometric representation of W.
- → We'll view our possibly asymmetric, possibly degenerate bilinear form as encoding some kind of geometry on V.

$\boxed{\text{Exercise}}$ We have $B(w.u\ ,\ w.v) = B(u,v)$

for all $w \in W$ and $u, v \in V$ if and only if A is symmetric.

<u>Question 1</u> Is σ one-to-one? (<u>Answer 1</u> Yes ... as we shall see shortly.)

- <u>The root system</u>

Let $\Phi := \{ w.\alpha_i \}_{i \in I_n,\ w \in W}$. <u>NOTE</u>: $0 \notin \Phi$. (Why?)

So, Φ is just a collection of vectors. Clearly if W is finite, then so is Φ.

<u>Question 2</u> When is Φ finite? (<u>Answer 2</u> As we shall see, Φ is finite $\Leftrightarrow$ W is finite.)

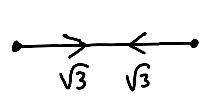
$\boxed{\text{Exercise}}$ Suppose A is symmetric. Show that if $K\alpha \in \Phi$ for some $\alpha \in \Phi$ and scalar K, then $K = \pm 1$.

Hint: Use the previous exercise.

Let $\Phi^+ = \{ \alpha \in \Phi \mid \alpha = c_1\alpha_1 + \cdots + c_n\alpha_n \text{ with each } c_i \geq 0 \}$.

Let $\Phi^- = \{ \alpha \in \Phi \mid \alpha = c_1\alpha_1 + \cdots + c_n\alpha_n \text{ with each } c_i \leq 0 \}$.

Clearly, $\Phi^+ \cap \Phi^- = \phi$. <u>Question 3</u> Does $\Phi = \Phi^+ \cup \Phi^-$? (<u>Answer 3</u> Yes...)

<u>Example</u> Let $\mathcal{G} =$ , so $A = \begin{bmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{bmatrix}$ is symmetric.

Then $[B]_{\mathcal{B}} = \begin{bmatrix} 1 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1 \end{bmatrix}$. Clearly B is symmetric.

It is also positive definite: For $v = a\alpha_1 + b\alpha_2 \neq 0$

$$[a \quad b]\begin{bmatrix} 1 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = [a \quad b]\begin{bmatrix} a - \frac{\sqrt{3}}{2}b \\ -\frac{\sqrt{3}}{2}a + b \end{bmatrix}$$

$$= a^2 - \frac{\sqrt{3}}{2}ab - \frac{\sqrt{3}}{2}ab + b^2$$

$$= a^2 - \sqrt{3}ab + b^2$$

Positive if a and b do not have same sign

$$= a^2 - 2ab + b^2 + (2-\sqrt{3})ab$$

$$= (a-b)^2 + (2-\sqrt{3})ab$$

Positive if a and b have the same sign

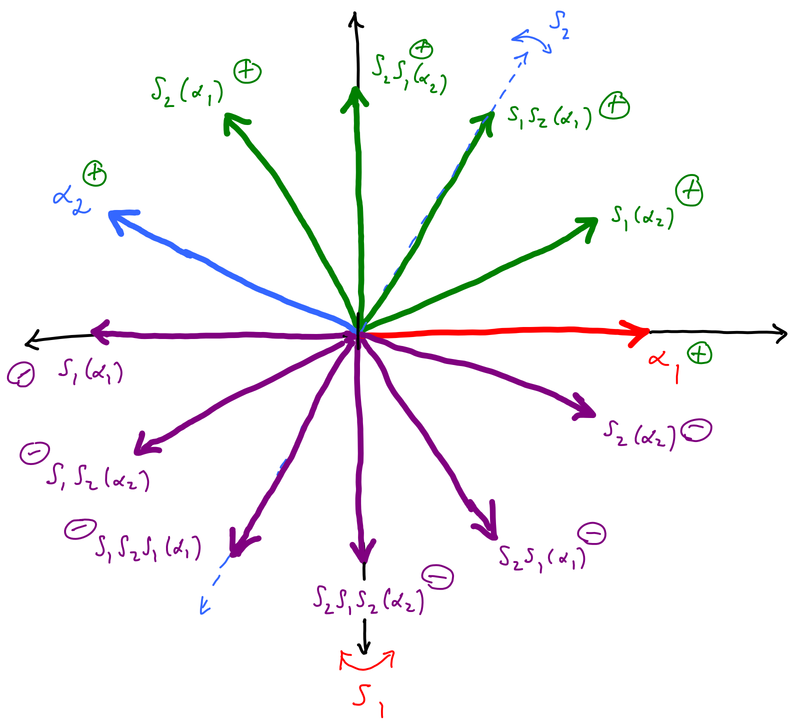
So we can view $V = \mathrm{span}_{\mathbb{R}}\{\alpha_1, \alpha_2\}$ together with B as an inner product space.

Angle between α_1 and α_2:

$$\cos\theta = \frac{B(\alpha_1, \alpha_2)}{\sqrt{B(\alpha_1,\alpha_1)}\sqrt{B(\alpha_2,\alpha_2)}} = \frac{-\sqrt{3}/2}{1 \cdot 1} = \frac{-\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{5\pi}{6}.$$

Picture



$$\Phi^+ = \{ \alpha_1, \alpha_2, s_2(\alpha_1) = \alpha_1 + \sqrt{3}\alpha_2, s_1(\alpha_2) = \alpha_2 + \sqrt{3}\alpha_1,$$
$$s_1 s_2(\alpha_1) = 2\alpha_1 + \sqrt{3}\alpha_2, s_2 s_1(\alpha_2) = \sqrt{3}\alpha_1 + 2\alpha_2 \},$$

and $\Phi^- = -\Phi^+$. We also have $\Phi = \Phi^+ \cup \Phi^-$.

$\boxed{\text{Exercise}}$ Work through the same details as the previous example for

$$\mathcal{G} = \bullet \xrightarrow{\sqrt{2}} \xleftarrow{\sqrt{2}} \bullet$$

$\boxed{\text{Exercise}}$ If $M = \begin{bmatrix} 1 & -c \\ -c & 1 \end{bmatrix}$ for some real number $c \geq 0$,

under what conditions is M positive definite?

- <u>A fundamental theorem for (asymmetric) geometric representations</u>

Theorem Let $w \in W$ and $i \in I_n$.

Reference: See [B1], §4.2

If $\ell(w s_i) > \ell(w)$, then $w.\alpha_i \in \Phi^+$.

If $\ell(w s_i) < \ell(w)$, then $w.\alpha_i \in \Phi^-$.

NOTE: Recall from Unit 2 Part 1 page 11, either $\ell(w s_i) > \ell(w)$ OR $\ell(w s_i) < \ell(w)$

Corollary 1 σ is one-to-one

Proof! Suppose $w \in \ker \sigma$. Then $w.\alpha_i = \alpha_i$ for all $i \in I_n$.

Now there exists some $s_i \in S$ such that $\ell(w s_i) < \ell(w)$

, if and only if $w \neq \varepsilon$. (Why?)

The fact that $w.\alpha_i = \alpha_i$ for all $i \in I_n$ means we must have

$\ell(w s_i) > \ell(w)$ for all $i \in I_n$. Then we must have $w = \varepsilon$.

So $\ker \sigma = \{\varepsilon\}$, and hence σ is one-to-one. //

Corollary 2 $\Phi = \Phi^+ \cup \Phi^-$

$\boxed{\text{Exercise}}$ Prove Corollary 2.

- <u>Root multiples in the root system Φ</u>

→ Nodes γ_i and γ_j in $\mathcal{G} = (\Gamma, A)$ are <u>odd-neighborly</u> if m_{ij} is odd.

If, in addition, $a_{ij} \neq a_{ji}$, then we say that γ_i and γ_j form an <u>odd asymmetry</u>.

Set $K_{ji} := \sqrt{\dfrac{a_{ji}}{a_{ij}}} = \dfrac{-a_{ji}}{2\cos(\pi/m_{ij})}$. Set $v_{ji} := (s_i s_j)^{(m_{ij}-1)/2}$.

<u>Facts</u>: $v_{ji}\cdot\alpha_i = K_{ji}\,\alpha_j$, $v_{ij} = v_{ji}^{-1}$, and $K_{ij}K_{ji} = 1$.

Reference: See [P12]

→ A path of odd neighbors ("ON-path") in $\mathcal{G}$ is a sequence $\mathcal{P} = [\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_p}]$ of nodes from Γ for which consecutive pairs are odd-neighborly. (Say $\mathcal{P}$ has length p, allow $p=0$.)

Set $w_{\mathcal{P}} := v_{i_p,i_{p-1}} \cdots v_{i_2,i_1}\, v_{i_1,i_0} \in W$ and $\Pi_{\mathcal{P}} := K_{i_p,i_{p-1}} \cdots K_{i_2,i_1} K_{i_1,i_0}$.

Then $w_{\mathcal{P}}\cdot\alpha_{i_0} = \Pi_{\mathcal{P}}\,\alpha_{i_p}$.

If for every cyclic ON-path $\mathcal{P} = [\gamma_{i_0}, \ldots, \gamma_{i_p=i_0}]$ in $\mathcal{G}$ it is the case that $\Pi_{\mathcal{P}} = 1$, then we say $\mathcal{G}$ is unital ON-cyclic.

<u>NOTE</u>: A cyclic ON-path $\mathcal{P}$ is unital $\Leftrightarrow$ $\underbrace{a_{i_0,i_1}a_{i_1,i_2}\cdots a_{i_{p-1},i_0}}_{\text{"clockwise"}} = \underbrace{a_{i_0,i_{p-1}}\cdots a_{i_2,i_1}a_{i_1,i_0}}_{\text{"counter-clockwise"}}$

→ <u>Theorem (D.)</u> Let $w \in W$ and $i \in I_n$.

Then $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and $K > 0$ if and only if $w.\alpha_i = w_{\mathcal{P}}.\alpha_i$ for some ON-path $\mathcal{P} = [\gamma_{i_0=i}, \ldots, \gamma_{i_p=x}]$, in which case $K = \Pi_{\mathcal{P}}$.

Reference: See [P12]

Also, $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and $K < 0$ if and only if $w.\alpha_i = (w_{\mathcal{P}} s_i).\alpha_i$ for some ON-path $\mathcal{P} = [\gamma_{i_0=i}, \ldots, \gamma_{i_p=x}]$, in which case $K = -\Pi_{\mathcal{P}}$

→ Corollary For each $\alpha \in \Phi$, let $\mathcal{M}_\alpha := \{ k \in \mathbb{R} \mid k\alpha \in \Phi \}$

Then $\mathcal{M}_\alpha \supseteq \{\pm 1\}$, and $\mathcal{M}_\alpha = \{\pm 1\}$ for all $\alpha \in \Phi$

$\iff \mathcal{G}$ has no odd asymmetries.

Reference: See [P12]

Also, $\mathcal{M}_\alpha$ is finite for all $\alpha \in \Phi$

$\iff \mathcal{G}$ is unital ON-cyclic.

→ Corollary For each $w \in W$, let $N(w) = \{ \alpha \in \Phi^+ \mid w.\alpha \in \Phi^- \}$.

Then $|N(w)| \geq \ell(w)$, with equality for all $w \in W$

$\iff \mathcal{G}$ has no odd asymmetries.

Reference: See [P12]

Also, $N(w)$ is finite for all $w \in W$

$\iff \mathcal{G}$ is unital ON-cyclic.

$\boxed{\text{Exercise}}$ Use the above Corollary to argue that if Φ is finite, then W is finite. HINT: Contrapositive.

→ What to make of these results?

- When $\mathcal{G}$ has no odd asymmetries, some fundamental features of root systems for standard geometric representations are preserved.
- When $\mathcal{G}$ is unital ON-cyclic, $N(w)$ is finite, a fact which features prominently in several arguments pertaining to root systems / Tits cones.

Notably, the proof by Brink and Howlett that for the standard geometric representation, the set of "elementary" roots is finite. → Reference: See [P4]

- <u>Subsets of generators</u>

→ Let $J \subseteq I_n$. Let W_J = subgroup of W generated by $\{s_i\}_{i \in J}$.

Call W_J a "parabolic" subgroup.

Let $W^J = \{ w \in W \mid \ell(w s_j) > \ell(w) \text{ for all } j \in J \}$

<u>Theorem</u> For all $\bar{w} \in W$, there exist unique elements $\bar{w}_J \in W_J$ and $\bar{w}^J \in W^J$ such that $\bar{w} = \bar{w}^J \bar{w}_J$. In this case, $\ell(\bar{w}) = \ell(\bar{w}^J) + \ell(\bar{w}_J)$, and $\bar{w}^J$ is the unique smallest length element of the coset $\bar{w} W_J$.

Reference: See [B1], §2.4

For this reason, W^J is called the set of minimal coset representatives for W_J.

Exercise Consider $\mathcal{D} = \overset{\gamma_1}{\bullet} \xrightarrow{p} \xleftarrow{q} \overset{\gamma_2}{\bullet}$ with $pq = 4\cos^2(\pi/m)$ for some integer $m \geq 3$. For $W = W(\mathcal{D})$ and $J = \{1\}$, determine W^J.

→ Next are some further finiteness/infiniteness results.

<u>Theorem</u> If $\mathcal{D}$ is connected and W is infinite, then for each proper subset $J \subset I_n$:

1. (Classical) W^J is infinite. Reference: See [P9]
2. (D.) With $\Phi^J := \{ \alpha \in \Phi \mid \alpha \notin \mathrm{span}_{\mathbb{R}} \{\alpha_j\}_{j \in J} \}$, Φ^J is infinite. Reference: See [P12], cf. [P9]

$\boxed{\text{Exercise}}$ Suppose W is finite. Consider the standard geometric representation of W with root system Φ. In this exercise you will show that W has a unique longest element whose length is $|\Phi^+|$. It is denoted w_0.

1) Suppose $w \in W$ is longest, i.e. $\ell(w) \geq \ell(v)$ for all $v \in W$.

   Use the fundamental theorem for the geometric representation to show that $w.\alpha_i \in \Phi^-$ for all $i \in I_n$. Conclude that $w.\alpha \in \Phi^-$ for all $\alpha \in \Phi^+$, and hence that $\ell(w) = |\Phi^+|$.

2) Suppose w_1 and w_2 are both longest. Argue that $w_2^{-1} w_1 . \alpha_i \in \Phi^+$ for each $i \in I_n$. Use the FTGR to conclude that

   $\ell(w_2^{-1} w_1 A_i) > \ell(w_2^{-1} w_1)$ for all $i \in I_n$. Why does this force $w_2^{-1} w_1 = \varepsilon$?

<u>FACT:</u> If W^J is finite for some proper subset $J \subset I_n$, then by above Theorem W is finite. Write:

$$w_0 = (w_0)^J (w_0)_J .$$

Then $(w_0)^J$ is the unique longest element of W^J, and $(w_0)_J$ is longest in W_J. (See Exercises on p. 17.)

→ Eriksson's Reduced Word Theorem

Suppose $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ is a legal firing sequence from some start position λ on the SC-graph $\mathcal{G} = (\Gamma, A)$. Then $s_{i_p} \cdots s_{i_1}$ is a reduced expression in $W = W(\mathcal{G})$.

Reference: See [P16]

First, a lemma.

<u>Lemma</u>: $(\underbrace{\gamma_i, \gamma_j, \gamma_i, \ldots}_{m_{ij}})$ is legal from some position μ

$\iff$ $(\underbrace{\gamma_j, \gamma_i, \gamma_j, \ldots}_{m_{ij}})$ is legal from μ.

<u>Justification</u>: Our analysis of two-node games shows that for a play sequence of length m_{ij} to have been legal from $\overset{\mu_i}{\bullet}\!\!\longrightarrow\!\!\longleftarrow\!\!\overset{\mu_j}{\bullet}$, it must have been the case that both μ_i and μ_j were <u>positive</u>. //

Proof of <u>Theorem</u>: By contrapositive. Suppose $s_{i_p} \cdots s_{i_1}$ is not a reduced expression in W. Let $\pi := (i_1, \ldots, i_p) \in I_n^*$. So by Tits' Theorem we can reduce $s_{i_p} \cdots s_{i_1}$ by applying elementary simplifications to $(i_1, \ldots, i_p)$. Since $s_{i_p} \cdots s_{i_1}$

is not reduced, then at some point we will apply a length-reducing simplification. Let $y := (j_1, \ldots, j_p)$ be an element of I_n^* obtained by applying only braid simplifications to x (which do not reduce length) and such that a length-reducing simplification can be applied to y. So then $(\gamma_{j_1}, \ldots, \gamma_{j_p})$ is not a legal firing sequence, since consecutive firings will happen at the same node. Then by the above Lemma, no braid simplification applied to $(j_1, \ldots, j_p)$ can result in a legal firing sequence from λ. Hence $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ cannot be legal from λ. //

A partial converse (D.)

Let λ be dominant and nontrivial, and let $J = \{ j \in I_n \mid \lambda_j = 0 \}$, so J is a proper subset of I_n. Suppose W_J is finite. Let $s_{i_p} \cdots s_{i_1}$ be a reduced expression for some element of W^J. Then $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ is a legal firing sequence from λ.

Reference: See [P10]

NOTE: The $J = \phi$ version of this converse is due to Eriksson.

NOTE: It is an open question whether the finiteness hypothesis can be relaxed.

→ Some finiteness exercises

$\boxed{\text{Exercise}}$ Assume that $J \subset I_n$ (proper) and that W^J is finite. Then W is also finite — see page 13 — so we can look at the longest element w_0 of W.

0) Argue that $(w_0)_J$ is the longest element of the finite Coxeter subgroup W_J.

1) If λ meets the hypotheses of the above "partial converse", then show that any game sequence played from λ has length $\ell((w_0)^J) = \ell(w_0) - \ell((w_0)_J)$.

2) Show that $(w_0)^J$ is the unique longest element of W^J.

$\boxed{\text{Exercise}}$ Let $J \subset I_n$ (proper), and let $\bar{w} \in W$. Show that $\bar{w}^J$ is the unique shortest element of the coset $\bar{w}W_J$.

You may use the fact from Theorem 1 of p. 13 that for any $u \in W$, there exist unique $u^J \in W^J$ and $u_J \in W_J$ for which $u = u^J u_J$, and in this case $\ell(u) = \ell(u^J) + \ell(u_J)$. But the rest of that theorem is off limits.

→ Positive roots from the numbers game

Theorem

Given an SC-graph $\mathcal{G}$, suppose $W = W(\mathcal{G})$ is finite.

Let $s_{i_p} \cdots s_{i_1}$ be a reduced expression for w_0 (the longest element of the group).

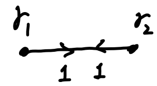
Reference: See [P10]

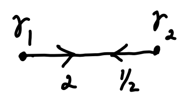
1. Then $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ is a convergent game sequence from any strongly dominant position.

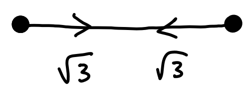
2. For $1 \leq q \leq p$, there exist scalars $c_j^{(q)}$ $(j \in I_n)$ such that for any strongly dominant position $\lambda = (\lambda_j)_{j \in I_n}$,
$$\sum_{j \in I_n} c_j^{(q)} \lambda_j$$
is the number at node γ_{i_q} when this node is fired in the game sequence $(\gamma_{i_1}, \ldots, \gamma_{i_p})$.

3. (D.) $\left\{ \sum_{j \in I_n} c_j^{(q)} \alpha_j \;\middle|\; 1 \leq q \leq p \right\} \subseteq \Phi^+$, with equality
$\iff$ $\mathcal{G}$ has no odd asymmetries.

See the next page for an example

Exercise 1) Use this theorem to generate all positive roots for $\mathcal{G} =$  .

2) Find all positive roots for $\mathcal{G} =$  .

<u>Example:</u> $\mathcal{G}$ =  from pages 8-9 above

Then $W = W(\mathcal{G})$ is the 12-element dihedral group.

Reduced expression for the longest element of W are:

$$w_0 = s_2 s_1 s_2 s_1 s_2 s_1 = s_1 s_2 s_1 s_2 s_1 s_2$$

	a	b
γ_1	$-a$	$\sqrt{3}a+b$
γ_2	$2a+\sqrt{3}b$	$-\sqrt{3}a-b$
γ_1	$-2a-\sqrt{3}b$	$\sqrt{3}a+2b$
γ_2	$a+\sqrt{3}b$	$-\sqrt{3}a-2b$
γ_1	$-a-\sqrt{3}b$	b
γ_2	$-a$	$-b$

The numbers at the fired nodes are:

$$a,\ \sqrt{3}a+b,\ 2a+\sqrt{3}b,\ \sqrt{3}a+2b,\ a+\sqrt{3}b,\ b$$

The positive roots from page 9 above are:

$$\alpha_1,\ \sqrt{3}\alpha_1+\alpha_2,\ 2\alpha_1+\sqrt{3}\alpha_2,\ \sqrt{3}\alpha_1+2\alpha_2,\ \alpha_1+\sqrt{3}\alpha_2,\ \alpha_2$$

Observe the one-to-one correspondence.