Coxeter groups and Combinatorics

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Unit 2: More about Generators and Relations   PART 3

Throughout these notes, "References" come from the annoted references webpage linked to from the site where these notes are posted.

This unit will serve as a reminder/reintroduction to how algebraic objects can somtimes be usefully and succinctly described in terms of generators and relations, and how such descriptions can be very helpful in constructing morphisms between algebraic structures.

- A little more linear algebra

Let V be a real finite-dimensional vector space.

A subset V' of V is a subspace if V' is non-empty and is closed under addition and scalar multiplication.

Example The span of any subset of V is a subspace.

Now suppose $T: V \to V$ is linear. Further, suppose V' is a subspace with $T(V') \subseteq V'$. In this case we say V' is "T-stable".

Suppose $\mathcal{B} = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ is a basis for V such that $V' := \operatorname{span} \mathcal{B}'$, where $\mathcal{B}' = \{e_1, \ldots, e_k\}$, is a T-stable subspace.

Then:

$$[T]_{\mathcal{B}} = \left[\begin{array}{c|c} [T]_{\mathcal{B}'} & X \\ \hline 0 & Y \end{array}\right]$$

"Block triangular"

Further, if the subspace $V'' := \operatorname{span}(\mathcal{B}'')$, where $\mathcal{B}'' = \{e_{k+1}, \ldots, e_n\}$, is also T-stable, then

$$[T]_{\mathcal{B}} = \left[\begin{array}{c|c} [T]_{\mathcal{B}'} & 0 \\ \hline 0 & [T]_{\mathcal{B}''} \end{array}\right]$$

"Block diagonal"

$\boxed{\text{Exercise}}$ If a matrix M is block diagonal $M = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array}\right]$, then $M^k = \left[\begin{array}{c|c} X^k & 0 \\ \hline 0 & Y^k \end{array}\right]$ for all integers $k \geq 1$.

$\boxed{\text{Exercise}}$ Suppose $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a basis of eigenvectors for T, so for each i, $T(e_i) = d_i e_i$ for some scalar d_i. What is $[T]_{\mathcal{B}}$? What is the connection of this question to the T-stable concept?