Abstract

Given a fixed volume of material and the elastic properties of that material, the optimal design of the tallest tubular column under self-weight is sought. This problem can be formulated as an extremal eigenvalue problem because the height at which the unloaded tubular column will buckle is related to the first eigenvalue of a Sturm-Liouville operator. Considering columns of annular cross-section, the spectrum associated with physical designs is characterized. Existence of an optimal design over a particular class of designs is proven through the use of rearrangements. Necessary conditions of optimality over that class are also established.

Key words: Optimal design, rearrangements, nonsmooth analysis.

1 Introduction

A vast number of optimal design problems arise in the study of mechanical systems. An excellent survey of those concerning column buckling is the book by Gajewski and Zyczkowski [7]. Euler was the first to address such problems when he posed and solved the buckling problem for prismatic columns under self-weight [5], [6]. The case of the tallest solid elastic column of variable cross-section was first addressed by Keller & Niordson [9], and later by Cox & McCarthy [3] and by McCarthy [10]. The optimal design of thin-walled column under self-weight and an external load was considered by Huang and Sheu [8]. We consider here a tubular column under self-weight and seek the height at which the tubular structure will buckle. That height is proportional to the fourth root of the least eigenvalue of a Sturm-Liouville operator, and so we begin by formulating the extremal eigenvalue problem. In section 2, we characterize the nature of the associated spectrum and present some variational characterizations. We apply rearrangement techniques in section 3 to show that replacement of a particular design by its decreasing arrangement increases the column height associated with that design. Existence of an optimal
design is established in section 4. Finally, necessary conditions for optimality are derived in section 5.

Consider an elastic column under self-weight. Let $A(z)$ be the cross-sectional area and $I(z)$ be its second moment at a height $z$. If $y(z)$ is the lateral deflection, from the vertical, of the cross section at $z$, then the Bernoulli-Euler theory yields the following equilibrium equation

$$EI(z)y''(z) = \int_{z}^{H} \rho g A(\tilde{z}) [y(\tilde{z}) - y(z)] \, d\tilde{z}, \quad 0 < z < H,$$

(1)

where $E$ and $\rho$ are the Young’s modulus and density of the material, $g$ is the acceleration due to gravity, and $H$ is the height of the column.

In order to apply such a model to tubular columns, we consider specifically columns whose cross-sections are annular regions of inner radius $R(z)$ and outer radius $R(z) + T$. The cross-sectional area at a height $z$ is $A(z) = \pi \left( (R(z) + T)^2 - R^2(z) \right)$ and its second moment is $I(z) = \pi \left( (R(z) + T)^4 - R^4(z) \right)$. With $A$ and $I$ now written in terms of $R$, the equilibrium equation (1) now takes the form

$$\frac{E \pi}{4} \left( (T + R(z))^4 - R^4(z) \right) y''(z) = \int_{z}^{H} \rho g \pi \left( T^2 + 2TR(\tilde{z}) \right) [y(\tilde{z}) - y(z)] \, d\tilde{z},$$

$$0 < z < H.$$

The column’s base is assumed to be clamped, $y(0) = y'(0) = 0$, while its apex is assumed to be free.

The design problem amounts to fixing $T > 0$, the wall thickness, and the volume of material, $V$, and varying the inner radius of the tube $R(z) \geq 0$ in order to reach the greatest height.

An interesting consequence of the volume constraint and the nonnegativity of the inner radius is that the height of the tube is bounded. The volume constraint is

$$\int_{0}^{H} A(z) \, dz = V.$$

(2)

Integrating $A(z) = \pi \left( (R(z) + T)^2 - R^2(z) \right)$ we find the following relationship between $T$, $V$, $H$, and $R$,

$$\int_{0}^{H} R(z) \, dz = \frac{V - \pi HT^2}{2\pi T}.$$

(3)

Since $R$, the inner radius, is a non-negative quantity, we see that the height is
bounded from above in terms of $V$ and $T$,

$$H \leq \frac{V}{\pi T^2}$$

(4)

with equality occurring when $R(z) = 0$, or in the case of a solid column of fixed radius $T$.

Let us begin by introducing dimensionless variables

$$x = \frac{z}{H}, \quad t = \sqrt{\frac{H}{VT}}, \quad r(x) = \sqrt{\frac{H}{V R(xH)}},$$

$$\eta(x) = \frac{y(xH)}{H}, \quad \lambda = \frac{4 \rho g H^4}{E V}$$

in the equilibrium equation (1) and the conditions on $y$, to arrive at

$$\left((t + r(x))^4 - r^4(x)\right) \eta'' = \lambda \int_{x}^{1} \left(t^2 + 2tr(\bar{x})\right) [\eta(\bar{x}) - \eta(x)] \, d\bar{x}, \quad (5)$$

$$\eta(0) = \eta'(0) = 0 \quad (6)$$

Differentiating (5) with respect to $x$ and calling $u(x) = \eta'(x)$ we obtain

$$-((t + r(x))^4 - r^4(x))u'(x))' = \lambda \left(\int_{x}^{1} \left(t^2 + 2tr(s)\right) \, ds\right) u(x), \quad (7)$$

$$0 < x < 1,$$

$$0 = [(t + r(1))^4 - r^4(1)]u'(1) = 0. \quad (8)$$

We shall refer to this eigenvalue problem as the Tubular Problem and denote by $\lambda_1(r)$ its least eigenvalue. $\lambda_1(r)$ and its corresponding eigenfunction represent the first buckling mode. We are interested in, given a fixed volume of material and a fixed wall thickness, designing the tallest tubular column stable against buckling. Since $\lambda$ is proportional to $H^4$, maximizing height is equivalent to maximizing the least eigenvalue $\lambda_1(r)$.

Using the dimensionless variable in the normalization condition (3) on $R$, we find that

$$\int_{0}^{1} r \, dx = \frac{1 - \pi t^2}{2\pi t}. \quad (9)$$

Note that the non-negativity of $r$ imposes the condition

$$t^2 \leq \frac{1}{\pi}. \quad (10)$$

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with equality occurring when \( r(x) = 0 \). Restricting ourselves to non-negative radii, \( r \), that satisfy the volume constraint, we consider only

\[
\mathcal{a}_t = \left\{ r : r \geq 0, \int_0^1 r \, dx = \left( 1 - \pi t^2 \right) / 2 \pi t \right\}
\]

(10)

and seek

\[
\sup_{r \in \mathcal{a}_t} \lambda_1(r).
\]

(11)

2 The Spectrum and some Variational Characterizations

Since \( T > 0 \) the cross-sectional area \( A(z) \) and its second moment \( I(z) \) never vanish. It follows that \( t > 0 \) and that the coefficients in (8) never vanish. When the Green’s function associated with the Tubular Problem is square integrable the associated Green’s operator is compact on \( L^2((0,1) \times (0,1)) \) and therefore has a discrete spectrum. Fortunately, this is always true for the Tubular Problem provided that we restrict ourselves to nonnegative inner radii, \( r \), satisfying the normalization condition (9).

**Theorem 1** The spectrum of the Tubular Problem is discrete provided that \( r \in \mathcal{a}_t \).

**PROOF.**

Following the usual construction, see Porter and Stirling [12, Example 6.13], the Green’s function for the Tubular Problem is

\[
g(x, y; r) = \sqrt{q(x)} \sqrt{q(y)} \int_0^{x \wedge y} \frac{ds}{p(s)}
\]

(12)

where \( x \wedge y = \min \{x, y\} \), and

\[
p(x; r) = (t + r(x))^4 - r^4(x)
\]

(13)

\[
q(x; r) = \int_x^1 \left( t^2 + 2tr(s) \right) \, ds.
\]

(14)

Using the non-negativity and normalization of \( r \), we see that

\[
g(x, y; r) \leq \frac{1}{\pi t^4}
\]
which implies that \( g \in L^2(0,1) \times (0,1) \). It follows from Porter & Stirling [12, Theorem 3.4] that the Green’s operator
\[
G(r) \phi(x) \equiv \int_0^1 g(x, y; r) \phi(y) \, dy
\]
is a compact operator on \( L^2(0,1) \). This operator is also self-adjoint and positive and so by the Spectral Theorem, [12, Theorem 4.15], its spectrum is a discrete sequence of nonnegative real numbers.

Let us introduce three variational characterizations for the the first eigenvalue \( \lambda_1(r) \) of the Tubular Problem (7-8). The first characterization of \( \lambda_1(r) \) is given by
\[
\lambda_1(r) = \inf_{u \in H^1(0,1)} \mathcal{R}(r, u) \quad \text{where} \quad \mathcal{R}(r, u) = \frac{\int_0^1 p(x; r)(u'(x))^2 \, dx}{\int_0^1 q(x; r) u^2(x) \, dx}. \tag{15}
\]
The infimum of the Rayleigh quotient, \( \mathcal{R}(r, u) \), equals the first eigenvalue because the Green’s function \( G(r) \) is square integrable. The minimum is attained at \( u_1 \) the first positive eigenfunction of the Tubular Problem (7-8) associated with \( r \) for which \( \|u_1\|_q = 1 \).

The second characterization for \( \lambda_1(r) \) can be found in Porter & Stirling [12, Lemma 5.1]:
\[
\frac{1}{\lambda_1(r)} = \max_{\|\phi\| = 1} \langle G(r) \phi, \phi \rangle, \tag{16}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the usual \( L^2(0,1) \) inner product, \( \|\cdot\| \) denotes the associated norm and \( G(r) \) is the Green’s operator. The maximum is attained at \( \phi_1(x) = \sqrt{q(x; r)} u_1(x) \) where \( u_1 \) is as above.

The Rayleigh quotient (15) will be used to establish necessary conditions for optimality of a design. Although the second variational form given by (16) will be useful in establishing existence of an optimal design, another representation will be crucial. This third representation of the first eigenvalue is a consequence of the integral form of \( q \). Later, we will use it along with the properties of rearrangements to establish the fact that the decreasing rearrangement of a design yields a first eigenvalue that is at least as large as the one corresponding to the original design. The proof given here for completeness is more detailed than that given in [3].

**Lemma 2** If \( r \in ad_t \) then the first eigenvalue \( \lambda_1(r) \) of the Tubular Problem (7-8) can be written in the form
\[
\frac{1}{\lambda_1(r)} = \max_{\|\phi\| = 1} \int_0^1 \frac{1}{p(x; r)} \left( \int_x^1 \sqrt{q(y; r)} \phi(y) \, dy \right)^2 \, dx
\]
where
\[ p(x; r) = (t + r(x))^4 - r(x)^4, \quad q(x; r) = \int_x^1 \left( t^2 + 2tr(s) \right) ds. \]

The maximum is attained at \( \phi_1(x) = \sqrt{q(x; r)}u_1(x) \) where \( u_1 \) is the first eigenfunction of (7-8) associated with \( \lambda_1(r) \).

**PROOF.** We begin with the second variational characterization (16) and use the fact that \( x \wedge y = \min \{x, y\} \) to rewrite the integral in two pieces.

\[
\langle G(r) \phi, \phi \rangle = \int_0^1 \int_0^1 g(x, y; r) \phi(x) \phi(y) dy dx \\
= \int_0^1 \int_0^1 \sqrt{q(x; r)}q(y; r) \left( \int_0^{x/y} \frac{dt}{p(t; r)} \right) \phi(x) \phi(y) dy dx \\
= \int_0^1 \sqrt{q(x; r)} \phi(x) \left[ \int_0^x \sqrt{q(y; r)} \phi(y) \left( \int_0^y \frac{dt}{p(t; r)} \right) dy \right] dx \\
+ \int_0^1 \sqrt{q(x; r)} \phi(x) \left[ \int_x^1 \sqrt{q(y; r)} \phi(y) \left( \int_0^x \frac{dt}{p(t; r)} \right) dy \right] dx \tag{17}
\]

We proceed by integrating by parts. Using the substitution
\[
v(x) = - \int_x^1 \sqrt{q(t; r)} \phi(t) dt, \quad u(x) = \int_0^x \frac{dt}{p(t; r)},
\]
we integrate the inner integral of the first term in (17) by parts

\[
\int_0^x \sqrt{q(y; r)} \phi(y) \left( \int_0^y \frac{dt}{p(t; r)} \right) dy = \int_0^x \frac{dt}{p(t; r)} \left( \int_x^1 \sqrt{q(t; r)} \phi(t) dt \right) + \int_0^x \frac{1}{p(y; r)} \left( \int_y^1 \sqrt{q(t; r)} \phi(t) dt \right) dy \tag{18}
\]

Using (18) in (17), we find that

\[
\langle G(r) \phi, \phi \rangle = \int_0^1 \sqrt{q(x; r)} \phi(x) \left( \int_0^x \sqrt{q(t; r)} \phi(t) dt \right) dy \frac{dx}{p(y; r)}
\]

Using the substitution
\[
v(x) = - \int_x^1 \sqrt{q(t; r)} \phi(t) dt, \quad u(x) = \int_0^x \left( \int_t^1 \sqrt{w(s)} \phi(s) ds \right) \frac{dt}{p(t; r)}
\]
we integrate by parts again and arrive at
\[
\langle G(r)\phi, \phi \rangle = \left[ -\int_0^x \left( \int_y^1 \frac{1}{\phi(t)} dt \right) \frac{dy}{p(y;r)} \int_x^1 \frac{1}{\phi(s)} ds \right]_{x=0}^{x=1} \\
+ \int_0^1 \left( \int_x^1 \frac{1}{\phi(t)} dt \right)^2 \frac{dx}{p(x;r)} \\
= \int_0^1 \frac{1}{p(x;r)} \left( \int_x^1 \frac{1}{\phi(t)} dt \right)^2 dx
\]

(19)

Use of the second characterization (16) leads to the desired result.

3 Application of Rearrangements

Physically, it seems reasonable to taper a tube in order to increase its height. Since the wall thickness is fixed, tapering is carried out by varying the inner radius. A tube that has a wide base and gradually narrows would be expected to be taller that one of uniform inner radius. In this section, we prove that the height of the tubular column can be increased via decreasing rearrangements. We begin by recalling a number of definitions and results from the theory of rearrangements.

**Definition 3** The decreasing rearrangement of a nonnegative function, \( f \), on \((0,1)\) is simply
\[
f^*(x) \equiv \sup\{t > 0 : \mu_f(t) > x\},
\]
where \( \mu_f \) is the distribution function of \( f \),
\[
\mu_f(t) = |\{x \in (0,1) : f(x) > t\}| \quad t \geq 0.
\]
The increasing rearrangement of \( f \) is \( f_*(x) \equiv f^*(1-x) \).

**Remark 4** A useful property of rearrangements is that
\[
\int_0^1 f \, dx = \int_0^1 f^* \, dx = \int_0^1 f_* \, dx.
\]

Thus, in our optimal design problem, if we replace a particular design \( r \in ad_t \) by either its increasing or decreasing rearrangements \( r_* \) or \( r^* \) then the new design will still be in \( ad_t \).

**Proposition 5** If \( f \) is decreasing on the range of \( g \) then the composition \((f \circ g)_* = f \circ g^*\).

**PROOF.** This is a special case of Cox [2, Theorem 1]. Since \( f \) is strictly decreasing
\[(f \circ g)_* (x) = (f \circ g)^* (1 - x)\]
\[= \sup \{t : |\{c \in (0, 1) : (f \circ g)(c) > t\}| > 1 - x\}\]
\[= \inf \{t : |\{c \in (0, 1) : (f \circ g)(c) > t\}| < x\}\]
\[= \inf \{t : |\{c \in (0, 1) : g(c) > f^{-1}(t)\}| < x\}\].

For fixed \(x\), let
\[c_1 = \inf \{t : |\{c \in (0, 1) : g(c) > f^{-1}(t)\}| < x\}\]
and
\[f(c_2) = \inf \{f(z) : |\{c \in (0, 1) : g(c) > z\}| < x\}\].
If \(c_1 < f(c_2)\), then \(c_2 < f^{-1}(c_1)\) and \(|\{c \in (0, 1) : g(c) > c_2\}| < x\). However, this contradicts the fact that \(|\{c \in (0, 1) : g(c) > z\}| < 1 - x\) for every \(z < f^{-1}(c_1)\). Similarly if \(c_1 > f(c_2)\), then \(c_2 > f^{-1}(c_1)\) and \(|\{c \in (0, 1) : g(c) > c_2\}| < x\). However, this contradicts the fact that \(|\{c \in (0, 1) : g(c) > z\}| > x\) for every \(z > f(c_2)\). Thus \(c_1 = f(c_2)\) for fixed \(x\), and so
\[(f \circ g)_* (x) = \inf \{f(t) : |\{c \in (0, 1) : g(c) > t\}| < x\}\].

Since \(f\) is strictly decreasing, minimizing \(f\) is the same as maximizing its argument. Therefore
\[\inf \{f(t) : |\{c \in (0, 1) : g(c) > t\}| < x\} = f \circ \sup \{t : |\{c \in (0, 1) : g(c) > t\}| < x\}\]
\[= f \circ g^*(x)\]

**Theorem 6** For \(r \in \text{ad}_t\)
\[\lambda_1(r) \leq \lambda_1(r^*).\]

**PROOF.** Let \(v\) be the first eigenfunction of the Green's operator associated with the Tubular Problem corresponding to \(r^*\). Recall that
\[v(x) = \sqrt{q(x;r^*)}u(x)\]
where \(u(x)\) is the first nonnegative eigenfunction of the tubular column problem with radius \(r^*\). Since \(u\) and hence \(v\) are nonnegative and \(\int_y^1 r(s) \, ds \geq \int_y^1 r^*(s) \, ds\), it follows that
\[q(y;r) = \int_y^1 \left(t^2 + 2tr(s)\right) \, ds \geq \int_y^1 \left(t^2 + 2tr^*(s)\right) \, ds = q(y;r^*).\]
Using the variational form derived in Lemma 2, we find
\[
\frac{1}{\lambda_1(r)} \geq \int_0^1 \frac{1}{p(x;r)} \left( \int_x^1 \sqrt{q(y;r)v(y)} \, dy \right)^2 \, dx \\
\geq \int_0^1 \frac{1}{p(x;r)} \left( \int_x^1 \sqrt{q(y;r^*)v(y)} \, dy \right)^2 \, dx
\]

Notice that the function \( \left( \int_x^1 \sqrt{q(y;r^*)v(y)} \, dy \right)^2 \) is a nonnegative decreasing function of \( x \). A special case of inequalities established in Pólya and Szegő [11, p.153] is that if \( \xi \) and \( \eta \) are nonnegative functions on \([a, b]\), with \( \eta \) decreasing, then \( \int_a^b \xi(x)\eta(x) \, dx \geq \int_a^b \xi(x)\eta(x) \, dx \). Applying this result and using the fact that \( p(x;r) \) is an increasing function of \( r \), we can establish that

\[
\frac{1}{\lambda_1(r)} \geq \int_0^1 \left( \frac{1}{p(x;r)} \right) \left( \int_x^1 \sqrt{q(y;r^*)v(y)} \, dy \right)^2 \, dx \\
\geq \int_0^1 \frac{1}{p(x;r^*)} \left( \int_x^1 \sqrt{q(y;r^*)v(y)} \, dy \right)^2 \, dx \\
= \frac{1}{\lambda_1(r^*)}.
\]

4 Existence of an Optimal Design

In order to establish existence of an optimal design for the tubular column design problem, we will use the rearrangement result of Theorem 6 and Helly’s selection theorem, which is restated here for convenience.

**Theorem 7** (Helly’s selection theorem, Rudin [13] p. 167) If \( \{f_n\}_{n=1}^\infty \) is a sequence of nonnegative nonincreasing functions on \([a, b]\), then there exists a subsequence \( \{f_{n_k}\}_{k=1}^\infty \) and a function \( f \) such that

\[
f(x) = \lim_{k \to \infty} f_{n_k}(x)
\]

for every \( x \) in \([a, b]\).

**Theorem 8** Consider

\[
ad_t = \{ r : r \geq 0, \int_0^1 r(s) \, ds = \frac{1 - \pi t^2}{2\pi t} \}.
\]

The functional \( r \mapsto \lambda_1(r) \) attains its maximum on \( ad_t \).

**Proof.** Firstly, note that the normalization constraint in the definition of \( ad_t \) is equivalent to the original volume constraint (2), and that the nonnega-
tivity of the inner radius $R$ is equivalent to the nonnegativity of $r$. Recall that these led to the bound

$$H \leq \frac{V}{\pi T^2}$$

on the tube height. By definition $\lambda_1 = 4\rho g H^4/E V$ and so $\lambda_1$ is bounded above on $ad_t$. Letting

$$\lambda_1^{(t)} = \sup_{r \in ad_t} \lambda_1(r)$$

we see that there exists a maximizing sequence $\{r_n\} \subset ad_t$ for which $\lambda_1(r_n) \to \lambda_1^{(t)}$.

By Remark 4 and Theorem 6, we may assume that each $r_n$ is nonincreasing and hence, by Helly’s selection theorem, there exists an $\hat{r}$ and a subsequence (that we will not relabel) such that $r_n \to \hat{r}$ pointwise. It follows by the dominated convergence theorem that

$$\int_x^1 r_n(s) \, ds \to \int_x^1 \hat{r}(s) \, ds$$

and

$$\int_0^{x \wedge y} \frac{ds}{(t + r_n(s))^{3/2} - r_n^4(s)} \to \int_0^{x \wedge y} \frac{ds}{(t + \hat{r}(s))^{3/2} - \hat{r}^4(s)}$$

for each $x$ and $y$. In particular,

$$g(x, y; r_n) \to g(x, y; \hat{r}).$$

A further application of the dominated convergence theorem implies that $g(\cdot, \cdot; r_n) \to g(\cdot, \cdot; \hat{r})$ in $L^2((0, 1) \times (0, 1))$.

All that remains is to show that the eigenvalues depend continuously on the Green’s function. Recall the variational characterization (16)

$$\frac{1}{\lambda_1(r)} = \max_{\|\phi\| = 1} \langle G(r)\phi, \phi \rangle.$$  \hfill (20)

When $\|\phi\| = 1$, Hölder’s inequality gives

$$\langle G(r_2)\phi, \phi \rangle - \|g(\cdot, \cdot; r_1) - g(\cdot, \cdot; r_2)\| \leq \langle G(r_1)\phi, \phi \rangle - \|g(\cdot, \cdot; r_1) - g(\cdot, \cdot; r_2)\|.$$  \hfill (20)

Applying (16) throughout gives

$$\frac{1}{\lambda_1(r_2)} - \|g(\cdot, \cdot; r_1) - g(\cdot, \cdot; r_2)\| \leq \frac{1}{\lambda_1(r_1)} \leq \frac{1}{\lambda_1(r_2)} + \|g(\cdot, \cdot; r_1) - g(\cdot, \cdot; r_2)\|.$$  \hfill (20)

This implies that $\lambda_1(r_n) \to \lambda_1(\hat{r})$. But, by construction, $\lambda_1(r_n) \to \lambda_1^{(t)}$, and so we must have $\lambda_1(\hat{r}) = \lambda_1^{(t)}$.  \hfill (20)
5 Necessary conditions for optimality

Since \( p(x; r) = (r(x) + t)^4 - r^4(x) \) and \( q(x; r) = \int_x^1 (t^2 + 2tr(s)) \, ds \), the Rayleigh quotient is

\[
\lambda_1(r) = \inf_{u \in H^1(0,1)} \mathcal{R}(r, u)
\]

where

\[
\mathcal{R}(r, u) = \frac{\int_0^1 (4r^3t + 6r^2t^2 + 4rt^3 + t^4) (u'(x))^2 \, dx}{\int_0^1 \left[ \int_x^1 (t^2 + 2tr) \, ds \right] u^2(x) \, dx}.
\]

In order to establish necessary conditions for optimality of \( \lambda_1(r) \) for \( r \) satisfying the volume constraint (9), we will use a generalized gradient approach. It should be noted that this result can also be established using a Gâteaux derivative technique. Consider a real-valued Lipschitz function \( F \) on a Banach space \( X \). The generalized directional derivative of \( F \) at \( x \) in the direction \( v \) is

\[
F^0(x; v) \equiv \limsup_{y \to x, t \downarrow 0} \frac{F(y + tv) - F(y)}{t}.
\]

Clarke’s generalized gradient [1] of \( F \) at \( x \) is the nonempty, convex, weak* compact set

\[
\partial F(x) \equiv \{ \xi \in X^*; F^0(x; v) \geq \langle \xi, v \rangle, \forall v \in X \}
\]

where \( X^* \) is the dual of \( X \) and \( \langle x^*, x \rangle \) is \( x^*(x) \) when \( x^* \in X^* \) and \( x \in X \). If \( F \) is a function of two variables \( \partial_1 F(x, y) \) denotes Clarke’s generalized gradient with respect to the first variable \( x \).

**Theorem 9** The necessary conditions for optimality of a nonnegative design \( \hat{r} \) satisfying the constraint

\[
\int_0^1 \hat{r}(x) \, dx = \frac{1 - \pi t^2}{2\pi t}
\]

for the Tubular Problem are

\[
\frac{(12r^2 t + 12rt^2 + 4t^3)}{\int_0^1 (\int_0^x v^2 \, ds) (t^2 + 2tr(x)) \, dx} - 2t\lambda \int_0^1 v^2 \, dx + c = 0
\]

where \( c \) is a positive Lagrange multiplier and \( v \) is the first positive eigenfunction associated with \( r \).

**PROOF.** Consider

\[
\mathcal{R}(r, u) = \frac{f_1(r, u)}{f_2(r, u)}
\]

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with
\[ f_1(r, u) = \int_0^1 \left( 4r^3 t + 6r^2 t^2 + 4rt^3 + t^4 \right) (u'(x))^2 \, dx \]
and
\[ f_2(r, u) = \int_0^1 \left[ \int_x^1 (t^2 + 2tr) \, ds \right] u^2(x) \, dx. \]

\( f_1(r, u) \) is Lipschitz in \( r \) and use of Clarke’s Theorem 2.7.5 [1] yields
\[ \partial_1 f_1(r, u) = \int_0^1 \partial \left( 4r^3 t + 6r^2 t^2 + 4rt^3 + t^4 \right) (u')^2 \, dx = \int_0^1 \left( 12r^2 t + 12t^2 + 4 \right) (u')^2 \, dx. \]

Integration of \( f_2(r; u) \) by parts yields
\[ f_2(r, u) = \int_0^1 \left( \int_0^x u^2 \, ds \right) (t^2 + 2tr(x)) \, dx. \]

Note that \( f_2(r, u) \) is also Lipschitz in \( r \), and a second use of Clarke’s Theorem
2.7.5 yields
\[ \partial_1 f_2(r, u) = \int_0^1 \left( \int_0^x u^2 \, ds \right) \partial \left( t^2 + 2tr \, dx \right) = 2t \int_0^1 \left( \int_0^x u^2 \, ds \right) \, dx. \]

Since \( f_1(r, u) \) and \( f_2(r, u) \) are both Lipschitz in \( r \), it follows that \( f_1(r, u)/f_2(r, u) \)
is also Lipschitz in \( r \). The quotient rule for generalized gradients [1, Thm.
2.3.14] is
\[ \partial \left( \frac{f_1}{f_2} \right)(x) \subseteq \frac{f_2(x) \partial f_1(x) - f_1(x) \partial f_2(x)}{f_2^2(x)}. \]
with equality if \( f_1(x) \geq 0, f_2(x) > 0 \) and if \( f_1 \) and \( -f_2 \) are regular at \( x \). This can be applied with \( f_1 \) and \( f_2 \) as above and the fact that \( \lambda = f_1(r, u)/f_2(r, u) \) to give the result
\[ \partial_1 \mathcal{R}(r, u) = \frac{\int_0^1 (12r^2 t + 12t^2 + 4 \lambda^3) (u')^2 \, dx - 2t \lambda \int_0^1 \left( \int_0^x u^2 \, ds \right) \, dx}{\int_0^1 \left( \int_0^x u^2 \, ds \right) (t^2 + 2tr(x)) \, dx}. \]

Let \( v \) be the first positive eigenfunction corresponding to \( \lambda_1(\hat{r}) \) where \( \hat{r} \) is
the optimal design. Since \( r \rightarrow \lambda_1(r) \) is the infimum of a family of Lipschitz
functions, it is also Lipschitz. Using an argument similar to that in Cox and
Overton [4, Thm 4.3], it can be established that its generalized gradient at \( \hat{r} \)
satisfies
\[ \partial \lambda_1(\hat{r}) = \frac{\int_0^1 (12r^2 t + 12t^2 + 4 \lambda^3) (v')^2 \, dx - 2t \lambda \int_0^1 v^2 \, ds \, dx}{\int_0^1 \left( \int_0^x v^2 \, ds \right) (t^2 + 2tr(x)) \, dx}. \]
Next, recall the volume constraint
\[ \int_0^1 r(x) \, dx = \frac{1 - \pi t^2}{2\pi t}. \]

The Lagrange Multiplier Rule, Clarke’s Theorem 6.1.1 [1], gives the existence of a nontrivial pair of constants \( c_1 \geq 0, c_2 \) for which
\[
c_1 \int_0^1 \left[ (12r^2t + 12rt^2 + 4t^3)(\frac{v'}{v}r^2 + 2t\lambda \int_0^x v^2 ds \right] \, dx + c_2 \int_0^1 \, dx = 0
\]
Without loss of generality, we can set \( c_1 = 1 \) and consideration of the integrand gives the desired result.

References


