Optimization of the minimum eigenvalue for a class of second order differential operators

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Abstract. We consider the problem of how the least eigenvalue of a Sturm-Liouville problem changes as the coefficients are varied. For a certain class of such problems, it is proved that the least eigenvalue can be maximized by placing constraints on the coefficients. Applications are made to various types of column problems. As a preliminary, we develop the spectral theory for a one term operator which has application to a variety of Sturm-Liouville problems.

1. Introduction

A basic problem in design is to optimize an eigenvalue while subjecting the coefficients of the operator to certain constraints. A common type of constraint is to keep the total mass fixed and attempt to maximize the first eigenvalue which is the same as maximizing the lowest frequency of vibration. Some recent motivation for a spectral analysis of these problems comes from a classic problem considered by Euler.

Euler posed and solved the problem of buckling of a prismatic column under self-weight. This problem was then extended by Keller and Niordson [14] by asking what height the column could reach if one allowed the column to taper while holding the volume constant. This eigenvalue problem is, after a scaling to normalize certain parameters,

\begin{equation}
-a(x)^2u'' = \lambda \left( \int_x^1 a(\tau) d\tau \right) u, \quad 0 < x < 1, \quad u(x) = \lim_{x \to 1} a(x)^2u'(x) = 0.
\end{equation}

The function $a(x)$ is the scaled cross sectional area of the column, and it is desired to maximize the first eigenvalue over those coefficients $a(x) > 0$ such that $\int_0^1 a(x) dx = 1$.

This problem, as well as many others in the literature, is singular over certain classes of coefficients. In [14], a candidate $\tilde{a}$ was found from perturbation theory.
of the form
\[
\tilde{a}(x) = \begin{cases} 
O(1), & x \to 0, \\
 c(1 - x)^3 + O((1 - x)^4), & x \to 1.
\end{cases}
\]
For \(\tilde{a}\) of the form (1.2), the problem is singular at 1 and has a continuous or essential spectrum \([25c, \infty)\). Because of the existence of a continuous spectrum, it was pointed out by Cox and McCarthy \([3, 17]\) that the perturbation methods of \([14]\) might not be applicable. By further restriction of the class of admissible \(a(x)\), Cox and McCarthy established the existence of a purely discrete spectrum, and proved the existence of a maximizer within this class.

Another classic problem considered by Krein \([16]\) is to construct a string of length \(L\) with a fixed mass \(M\) so as to minimize or maximize the lowest frequency. The eigenvalue problem is:
\[
-y'' = \lambda \rho(x)y, \quad y(0) = y(L) = 0,
\]
and the constraint is \(\int_0^L \rho(x) \, dx = M\). The lowest frequency is achieved by a point mass at the center, and there is no maximum of the fundamental frequency. In both cases we see there is no density \(\rho(x)\) within the class of \(L(0, L)\) functions. If however, the class of admissible \(\rho(x)\) is restricted by requiring \(0 < h \leq \rho(x) \leq H\), \(hL \leq M \leq HL\), then both minimizing and maximizing densities exist. An extension of this problem may be found in Reid \([25]\). Thus we see that in optimizing the least eigenvalue of a Sturm-Liouville problem, the class of admissible coefficients is an important consideration.

The text by Gajewski and Zyczkowski \([9]\) is devoted to a variety of optimal design problems including problems of vibration. The spectral theory of a tapered rod with equation of the form
\[
-(A(x)u')' = \lambda u, \quad 0 < x \leq 1, \quad u(1) = 0, \quad \lim_{s \to 0} A(s)u'(s) = 0,
\]
has been considered by Stuart \([26]\). In particular, the spectral analysis of (1.3) is investigated under the assumption that \(A(s)/s^p \to L\) exists as \(s \to 0\). The range of \(p\) considered is \(0 < p \leq 2\).

Since the number of boundary conditions depends on the deficiency index of the differential operator, and the spectrum may not be purely discrete, it is desirable to pose these problems in a Hilbert space where the analysis of selfadjoint operators can be applied. For the most part these problems have not been so formulated. In the case of \(\tilde{a}\) of the form (1.2), the problem (1.1) is in the limit point case at 1, and the boundary condition \(a(x)^2 u'(x) \to 0\) as \(x \to 1\) is automatic for all functions in the domain of the maximal operator (see section 2).

Thus a proper formulation of these singular problems first requires computation of the deficiency index at the singular point to know how many boundary conditions a selfadjoint formulation requires. A critical question considered in \([3]\) is whether or not there exists an eigenvalue below the essential spectrum. A useful test for finding the lowest point \(\sigma_e\) of the essential spectrum is given in section 2 (Friedrich’s test \([7]\) is used in \([3]\)). Once \(\sigma_e\) is known for a selfadjoint operator \(T\), then there is an eigenvalue below \(\sigma_e\) if there is an element \(\phi\) in the domain of \(T\) so that \(\langle T\phi, \phi \rangle < \sigma_e \langle \phi, \phi \rangle\), where \(\langle \cdot, \cdot \rangle\) is the inner product; in fact from spectral theory,
\[
\sigma_{\text{min}}(T) := \inf \text{ spectrum } T = \inf \{ \langle T\phi, \phi \rangle : \phi \in \text{domain } T, \|\phi\| = 1 \}.
\]
In case the spectrum of $T$ is purely discrete, $\sigma_{\text{min}}(T)$ is the least eigenvalue of $T$. The form of $\langle T\phi, \phi \rangle$ is usually obtained by integration by parts so as to use a Rayleigh quotient minimization principle to compute eigenvalues. Only in certain situations can the endpoint data at the singular point be neglected. This is discussed in section 2.

Indeed, the problem of maximizing the lowest eigenvalue of an operator with purely discrete spectrum may be generalized to the problem of maximizing $\sigma_{\text{min}}(T)$ for an operator with essential spectrum.

All of these questions for a problem with two singular endpoints can be resolved by considering each endpoint separately [27]. For this reason we consider an operator with one singular endpoint $b$ and restrict our attention to

$$L[y] = -\frac{1}{w} (py')', \quad -\infty < a < x < b \leq \infty.$$ 

The coefficients $p(x)$, $w(x)$ satisfy

$$p(x) > 0, \quad w(x) > 0; \quad w, \frac{1}{p} \in L_{\text{loc}}(a,b)$$

where $L_{\text{loc}}(a,b)$ is the space of functions Lebesgue integrable on compact subsets of $[a,b]$. The space of Lebesgue integrable functions on $[a,b]$ is denoted by $L[a,b]$.

While the operator (1.5) is not as general as $-(1/w)[(py')' + qy]$, it covers many eigenvalue applications and we can give a rather complete analysis of it.

In the process of our analysis, we consider three other variations of the Keller-Niordson problem. In these problems we assume the cross sections of the columns are annular. The first is to consider a column where the outer radius is fixed and the column is hollow with an inner radius of $r(x)$. In (1.1) we have $a(x) = r^2(x)$ with circular cross sections. In this hollow column problem, we have after scaling to give an outer radius of one and height of one, a case of (1.5) where

$$p(x) = 1 - r^4(x), \quad w(x) = \int_x^1 [1 - r^2(t)] dt, \quad 0 < r(x) < 1,$$

and the optimization problem is to require the ratio of volume of the column to volume of enclosing cylinder to be constant. This translates to $\int_0^1 r^2(t) dt = \theta, \quad 0 < \theta < 1$.

A second problem is to consider a hollow column with inner radius fixed and outer radius $r(x)$ variable. Again scaling so that the inner radius is one and the height is one, this leads to (1.5) where

$$p(x) = r^4(x) - 1, \quad w(x) = \int_x^1 [r^2(t) - 1] dt, \quad 1 < r(x),$$

and the optimization problem is to require the ratio of the volume of the column to the volume of column plus internal cylinder to be constant. This requires, $\int_0^1 r^2(t) dt = \theta, \quad 1 < \theta$.

A final problem, which has been analyzed by McCarthy [18], is to consider a column which is a hollow tube of constant thickness $t$. This leads to (1.5) where

$$p(x) = (t + r(x))^4 - r^4(x), \quad w(x) = \int_x^1 [t^2 + 2tr(s)] ds, \quad 0 < r(x),$$

and the constraint is constant volume $w(0)$. 
In sections 2 and 3 we develop the spectral theory of (1.5). In section 4 we give an alternate representation of \( \sigma_{\text{min}}(T) \). In section 5 we prove by rearrangement theorems some monotonicity properties of \( \sigma_{\text{min}}(T) \). In section 6 we give a continuity theorem for the least eigenvalue and then apply this theorem to establish the existence of a maximum least eigenvalue for certain classes of problems. It is an open problem to devise general and stable numerical algorithms to compute this maximum least eigenvalue and the corresponding optimizing coefficients. There has been some success in the special cases where there is an explicit relationship between the optimal coefficients, and the relevant optimality condition can be used to remove the coefficients from the Sturm-Liouville problem, see for example [14, 17].

2. The spectral theory of \( L \)

2.1. Limit Circle & Limit Point classification. The operator \( L \) given by (1.5)–(1.6) acts in the Hilbert space \( L^2_w(a, b) \) of (equivalence classes) complex-valued functions \( f \) satisfying

\[
\int_a^b w(x)|f(x)|^2 \, dx < \infty.
\]

The maximal operator \( L_1 \) associated with \( L \) is the restriction of \( L \) to the domain

\[
D(L_1) = \{y \in L^2_w(a, b) : y, py' \in AC_{\text{loc}}[a, b], L[y] \in L^2_w(a, b)\},
\]

where \( AC_{\text{loc}}[a, b] \) is the set of functions that are absolutely continuous on compact subintervals of \( [a, b) \). The unclosed minimal operator \( L'_0 \) associated with \( L \) is defined as the restriction of \( L_1 \) to functions with compact support in \( (a, b) \), and the minimal operator \( L_0 \) is defined as the closure of \( L'_0 \). The relations \( L_0^* = L_1 \) and \( L_1^* = L_0 \) hold where \( * \) is the adjoint and all selfadjoint operators \( A \) generated by \( L \) satisfy \( L_0 \subseteq A \subseteq L_1 \).

The point \( a \) is a regular point for \( L \), and we take \( b \) to be a singular point, i.e., either \( b = \infty \) or one of the two integrals \( \int_a^b w \) or \( \int_a^b 1/p \) is infinite. To define a self-adjoint operator associated with \( L \), we must assign boundary conditions. The number of such boundary conditions is two if \( L \) is in the limit-circle (LC) case at \( b \), i.e., when all solutions of \( L[y] = 0 \) are in \( L^2_w(a, b) \); otherwise \( L \) is said to be in the limit-point (LP) case at \( b \). The terminology arises from the geometric method of assigning boundary conditions at the singular point \( b \). In the limit point case all self-adjoint operators generated by \( L \) are obtained by restricting \( L_1 \) to a domain of the form,

\[
D_\alpha(L) = \{y \in D(L_1) : y(a) \cos \alpha + (py')(a)\sin \alpha = 0\}.
\]

where \( 0 \leq \alpha < \pi \).

In the limit-circle case, the domain of \( D(L_1) \) must be further restricted by imposing a self-adjoint boundary condition at \( b \) (coupled boundary conditions are also possible). Since \( L[u] = 0 \) has two linearly independent solutions

\[
y_1(x) = 1, \quad y_2(x) = \int_a^x \frac{ds}{p(s)},
\]

a criterion for the limit circle case is immediate.

**Theorem 2.1.** The limit-circle case holds for \( L \) at \( b \) if and only if

\[
\int_a^b w(s) \left( \int_a^s \frac{d\tau}{p(\tau)} \right)^2 \, ds < \infty.
\]

We consider now the application of Theorem 2.1 to the column problems when power tapers are used.
EXAMPLE 2.2. For (1.1) with \( a(x) = c(1 - x)^q, \ q > 0, c > 0 \), application of Theorem 2.1 yields that
\[
q > \frac{4}{3} \Rightarrow \text{LP}, \quad \frac{1}{2} \leq q < \frac{4}{3} \Rightarrow \text{LC},
\]
and \( 0 < q < 1/2 \) implies \( b \) is a regular point. For (1.7) with \( 1 - r^2(x) = c(1-x)^q, \ q > 0, c > 0 \), application of Theorem 2.1 yields that
\[
q \geq 4 \Rightarrow \text{LP}, \quad 1 \leq q < 4 \Rightarrow \text{LC},
\]
and \( 0 < q < 1 \) implies \( b \) is a regular point. For (1.8) with \( r^2(x) - 1 = c(1-x)^q, \ q > 0, c > 0 \), application of Theorem 2.1 we obtain the same results as for (1.7). For (1.9), the point \( b \) is a regular point since \( \int_a^1 1/p < \infty, \int_a^1 w < \infty \).

The Lagrange bilinear form \([\cdot, \cdot]\) associated with \( L \) is
\[
[u, v] = -(pu')v + u(pv'),
\]
and since \([u, v]' = wL[u]v - wuL[v],\) it is the case that
\[
(2.2) \quad [u, v](b) := \lim_{x \to b} [u, v]
\]
ests and is finite for all \( u, v \in \mathcal{D}(L_1) \); further this limit is zero if and only if \( L \) is in the limit-point case at \( b \) [20, 27].

Note that in the LP case, when \( \int_a^b w < \infty, \) we have \( y_1 \in \mathcal{D}(L_1), \) and from (2.2) by taking \( v(x) = y_1(x) = 1, \) it follows that for all \( u \in \mathcal{D}(L_1), \)
\[
(2.3) \quad \lim_{x \to b} p(x)u'(x) = 0.
\]
It should be noted that (2.3) does not always hold in the LP case. If it is added as a boundary condition, then it is a restriction on the domain of a self-adjoint operator with domain given by (2.1), and it may have the effect of producing a proper restriction of the self-adjoint operator with domain (2.1). If it does not restrict the self-adjoint operator it will in any case be redundant. Note also that if \( \int_a^b w = \infty, \) then \( L \) is in the LP case; further if \( \int_a^b w < \infty \) and \( \int_a^b 1/p < \infty, \) then \( L \) is in the LC case (\( b \) is a regular point).

In the LC case (2.3) may be taken as a boundary condition to define a selfadjoint operator. Parameterization of limit-circle boundary conditions may be found in Fulton [8] and Hinton and Shaw [11] which we now describe. In terms of the solutions \( y_1, y_2 \) defined above,
\[
(2.4) \quad [y, y_2](b) = \lim_{x \to b} [-(py')(x) \int_a^x \frac{d\tau}{p(\tau)} + y(x)],
\]
\[
(2.5) \quad [y, y_1](b) = \lim_{x \to b} [-(py')(x)],
\]
which exists for all \( y \in \mathcal{D}(L_1) \) since \( y_1, y_2 \in \mathcal{D}(L_1) \) in the LC case. Using (2.4) and (2.5), the self-adjoint operators determined by \( L \) in the case of separated boundary conditions in the LC case are given by restricting \( L_1 \) to a domain of the form
\[
(2.6) \quad \mathcal{D}_{\alpha, \beta} = \{ y \in \mathcal{D}(L_1) : \cos \alpha y(a) + \sin \alpha (py')(a) = 0, \cos \beta [y, y_1](b) + \sin \beta [y, y_2](b) = 0 \},
\]
where \( 0 \leq \alpha < \pi, \ 0 < \beta \leq \pi.\)
A complete discussion of the maximal, minimal operators, deficiency index and construction of boundary conditions for singular differential expressions may be found in the books of Niamark [20] and Weidmann [27].

2.2. Strong Limit Point Criterion. An eigenvalue \( \lambda \) for a selfadjoint operator \( T \) satisfies \( T \phi = \lambda \phi \) for an eigenvector \( \phi \) and hence \( \lambda = \langle T \phi, \phi \rangle / \langle \phi, \phi \rangle \) where \( \langle \cdot, \cdot \rangle \) denotes the inner product. In case \( T \) is a differential expression, \( \langle T \phi, \phi \rangle \) is usually expressed as a quadratic form by integration by parts; for a selfadjoint operator \( T \) generated by \( L \) we have

\[
\langle T \phi, \phi \rangle = \int_a^b w(s) \left[ -\frac{1}{w} (p \phi')' \right] \bar{\phi} \, ds
\]

(2.7)

Hence for the equation

\[
\lambda = \frac{\int_a^b p(s) |\phi'(s)|^2 \, ds}{\int_a^b w(s) |\phi(s)|^2 \, ds}
\]

(2.8)

to hold, one must have \( p \phi' \phi_a^b = 0 \). For example with \( \phi(a) = 0 \), one must have \( \lim p(x) \phi'(x) \phi(x) = 0 \) as \( x \to b \). In the LP case, this condition is not automatic; when \( \lim p(x)y'(x)y(x) = 0 \) as \( x \to b \) for all \( y \in D_1(L) \), it is called the strong limit-point (SLP) condition, and a differential expression can be LP without being SLP. Criteria for strong limit-point have been given by a number of authors, e.g. Everitt, Gertz, and Weidmann [4], Everitt, Gierz, and McLeod [5], Everitt [6], Hinton [10], and Race [24]. For our applications we state a special case of Theorem 3 of [10] as applied to (1.5).

Theorem 2.3. The operator \( L \) in (1.5) is in the SLP case at \( b \) provided there is a function \( \eta(x) > 0 \) on \([a, b)\) such that the following conditions hold.

(i) \( \eta \) and \( p \eta' \) are in \( AC_{loc} \).

(ii) \( \int_a^b w(x) \eta^2(x) \, dx = \infty \) and \( \int_a^b \frac{1}{p(x) \eta^2(x)} \, dx < \infty \).

(iii) \( 0 \leq -(p(x) \eta')'(x) \).

(iv) \( \left| p(x) \eta'(x) \eta(x) \int_x^b \frac{1}{p \eta^2} \right| = O(1) \) as \( x \to b \).

Two examples applying Theorem 2.3 are given below.

Example 2.4. We consider a generic example with a finite singular point and power weights. We take the interval to be \([0, 1)\) with 1 as the singular point. Let in (1.5),

\[ p(x) = (1 - x)^\gamma, \quad w(x) = (1 - x)^\delta. \]

Application of Theorem 2.1 yields that (1.5) is LC at 1 if and only if

\[
\delta > \max \{-1, 2\gamma - 3\}.
\]

Application of Theorem 2.3 with \( w(x) \eta^2(x) = (1 - x)^{-1} \) yields that (1.5) is SLP at 1 if

\[
\delta > \gamma - 2 \quad \text{and} \quad (1 + \delta)(\delta + 3 - 2\gamma) \leq 0.
\]
For $\delta = \gamma - 2$, the choice $w(x)\eta^2(x) = (1 - x)^{-1}\ln^2(1 - x)$ yields after some calculation that (1.5) is SLP at 1. One obtains the same results as (2.9) and (2.10) with the singularity at 0, and $p(x) = x^\gamma$, $w(x) = x^\delta$.

Example 2.5. Consider the column examples discussed in Example 2.2. Using the same choice of $a(x)$ or $r(x)$ as before, we have by choosing $\eta$ as above that

$$4 \leq q < 3 \Rightarrow \text{SLP for (1.1)}$$

$$4 \leq q \Rightarrow \text{SLP for (1.7) and (1.8)}$$

For $q = 3$ in (1.1), the second choice of $\eta$ above shows the SLP condition also holds.

2.3. Criterion for Discrete Spectrum. If $T$ is a closed linear operator with dense domain in a Hilbert space, then the essential spectrum of $T$ is the set of all $\lambda \in \mathbb{C}$ such that the range of $T - \lambda I$ is not closed. There are various definitions of essential spectrum in the literature. The definition here is appropriate for ordinary differential operators since the null space is finite dimensional and when the range is closed, it has finite co-dimension. Finite dimensional extensions leave the essential spectrum invariant; hence all selfadjoint extensions of a minimal ordinary differential equation operator have the same essential spectrum as the minimal and maximal operator. There is a close connection between oscillation and essential spectrum which we now state for the Sturm-Liouville operator

$$M[y] = \frac{1}{w}(-py')' + qy, \quad a \leq x < b \leq \infty,$$

with real coefficients $w(x), p(x) > 0, w, 1/p, q \in L_{\text{loc}}(a, b)$. The Sturm comparison theorem yields for the equation $M[y] = \lambda y$ that there is a number $\mu, -\infty \leq \mu \leq \infty$, such that for $\lambda > \mu$, all solutions of $M[y] = \lambda y$ are oscillatory, i.e., have infinitely many zeros on $[a, b)$, and for $\lambda < \mu$, no nontrivial solution of $M[y] = \lambda y$ has infinitely many zeros on $[a, b)$. The number $\mu$ is called the oscillation constant for $M$ and it satisfies [27, p. 220],

$$\mu = \sigma_e(M) := \inf \{\lambda : \lambda \in \text{essential spectrum of } M_1\},$$

where $M_1$ denotes the maximal operator of $M$.

We recall now results from Hinton [12] as applied to (1.5). Note that the spectrum of $L_{a} := L_1|D_{a}(L)$ or of $L_{a, \beta} := L_1|D_{a, \beta}(L)$ is purely discrete when $\sigma_e(L) = \infty$. Further, from [12],

$$\int_{a}^{b} w(x) \, dx = \int_{a}^{b} \frac{dx}{p(x)} = \infty \Rightarrow \sigma_e(L) = 0,$$

(2.11)

$$\int_{a}^{b} w(x) \, dx < \infty, \quad \int_{a}^{b} \frac{dx}{p(x)} < \infty \Rightarrow \sigma_e(L) = \infty;$$

(2.12)

otherwise define

$$g(x) = \begin{cases} \int_{a}^{x} \frac{1}{p} \int_{x}^{b} w \, dx & \text{if } \int_{a}^{b} \frac{1}{p} = \infty, \quad \int_{a}^{b} w = < \infty, \\ \int_{a}^{b} \frac{1}{p} \int_{a}^{x} w \, dx & \text{if } \int_{a}^{b} \frac{1}{p} < \infty, \quad \int_{a}^{b} w = \infty. \end{cases}$$

Then

$$\frac{1}{4g^*} \leq \sigma_e(L) \leq \frac{1}{4g_*},$$

(2.13)

(2.14)
where

\[(2.15) \quad g_* := \liminf_{x \to b} g(x), \quad g^* := \limsup_{x \to b} g(x).\]

Moreover combining the results of [12] and those of Nehari [19], we have

\[(2.16) \quad \sigma_e(L) = 0 \iff g^* = \infty, \quad \sigma_e(L) = \infty \iff g^* = 0.\]

We return to our examples.

**Example 2.6.** With \(w\) and \(p\) as in Example 2.4, we find that

\[
\int_0^1 w(x) \, dx = \infty \iff \delta \leq -1, \quad \int_0^1 \frac{1}{p(x)} \, dx = \infty \iff \gamma \geq 1.
\]

Thus by (2.11) \(\delta \leq -1, \gamma \geq 1\) implies \(\sigma_e(L) = 0\), while by (2.12) \(\delta > -1, \gamma < 1\) implies \(\sigma_e(L) = \infty\). If \(\delta = \gamma = 2\), then a computation shows that \(g_* = g^* = 1/(\gamma-1)^2\), so that \(\sigma_e(L) = 4(\gamma-1)^2\). If \(\gamma \geq 1, \delta > -1\), then further computation gives \(g^* = 0\) for \(\delta > \gamma - 2\), and \(g^* = \infty\) for \(\delta < \gamma - 2\). Finally, if \(\gamma < 1, \delta \leq -1\), then \(g^* = 0\) for \(\delta > \gamma - 2\), and \(g^* = \infty\) for \(\delta < \gamma - 2\). Combining these results yields that

\[(2.17) \quad \sigma_e(L) = \infty \text{ for } \delta > \gamma - 2,\]

\[(2.18) \quad \sigma_e(L) = 0 \text{ for } \delta < \gamma - 2,\]

\[(2.19) \quad \sigma_e(L) = 4(\gamma - 1)^2 \text{ for } \delta = \gamma - 2.\]

**Example 2.7.** Suppose in (1.7) that \(r\) is increasing and \(\int_0^1 1/p = \infty\). Then

\[
g(x) = \int_0^x \frac{ds}{1 - r^2(s)} \int_x^1 \frac{1 - r^2(s)}{1 - r^2(x)} \, ds
\]

Simplification of \(g(x)\) using \((1 + r^2(s)) \geq 1\) shows that \(g(x) \leq r(1 - x)\) and hence \(g^* = 0\) giving \(\sigma_e(L) = \infty\). Similarly in (1.8), \(r\) decreasing shows that \(g^* = 0\). Thus if \(r\) is increasing in (1.7), or decreasing in (1.8) then the spectrum associated with each Sturm-Liouville problem is discrete.

2.3.1. *A second criterion for* (2.8). We give here another condition, in addition to the SLP criterion, for

\[(2.20) \quad \lim_{x \to b} p(x)\phi'(x)\phi(x) = 0 \text{ for all } \phi \in D_\alpha(L) \text{ or for all } \phi \in D_{\alpha,\beta}(L).\]

Assume that \(g_* > 0, \int_a^b w < \infty\), and we have either LP case or we have the LC case with \(\beta = \pi/2\) in (2.6). Then by (2.3) in the LP case, and a direct calculation in the LC case, we have for \(\phi \in D(L_{\alpha,\beta})\) or \(\phi \in D(L_\alpha),\)

\[(2.21) \quad \lim_{x \to b} (p\phi')(x) = 0.\]

If \(\int_a^b 1/p < \infty\), then \(\phi\) is bounded on \([a, b]\) since

\[
|\phi(x) - \phi(a)| \leq \int_a^b |p(x)\phi'(x)| \frac{dx}{p(x)} < \infty
\]

and thus \((p\phi')(x) \to 0\) as \(x \to b\) so that (2.20) holds.
If $\int_a^b 1/p = \infty$, then L’Hospital’s rule gives

$$\lim_{x \to b} \frac{\phi^2(x)}{\int_a^x ds/p(s)} = \lim_{x \to b} 2(p\phi'(x)) =: k \leq \infty$$

which exists by applying (2.7). Suppose $k \neq 0$. Then for some $c > 0$, $\phi^2(x) \geq c \int_a^x ds/p(s)$ on some $[a', b)$, $a' > a$. Also $g_\ast > 0$ gives for some $d > 0$, $\int_a^b ds/p(s) \int_x^b w(s)ds \geq d$ on $[a', b)$. Thus

$$\int_{a'}^x \phi^2(s) ds \geq \int_{a'}^x w(s) \left( \int_s^b w(t) dt \right)^{-1} ds \geq cd \ln \left( \int_s^b w(t) dt \right)^{-1} \bigg|_{a'}^x.$$  

This inequality implies $\int_a^b w\phi^2 = \infty$ which is contrary to $\phi \in D(L_{\alpha,\beta})$ or $\phi \in D(L_{\alpha})$. Thus $k = 0$ and (2.20) holds.

2.4. Assumption on $\sigma_e(L)$. We will frequently use the following assumption

$$0 < \sigma_e(L) \leq \infty.$$  

Note that if 0 is not an eigenvalue then (2.23) implies 0 is in the resolvent set of $L_\alpha$ (LP case) or of $L_{\alpha,\beta}$ (LC case).

Note also that the LP case implies $y_2 \notin L^2_w(a, b)$ since $y_2 \in L^2_w(a, b)$ implies $y_1 \in L^2_w(a, b)$. Further $\int_a^b w(x) dx < \infty \iff y_1 \in L^2_w(a, b)$, and by (2.11),

$$\int_a^b w(x) dx = \infty, \quad \sigma_e(L) > 0 \implies \int_a^b \frac{dx}{p(x)} < \infty.$$  

When $\int_a^b 1/p < \infty$, we define

$$y_3(x) := \int_x^b \frac{1}{p(t)} dt.$$  

If (2.24) holds, we also have $y_3 \in L^2_w(a, b)$ as we now show. Since $g^* < \infty$ by (2.16), there is a number $M$ so that

$$\int_s^b \frac{dt}{p(t)} \leq M \left( \int_s^b w(t) dt \right)^{-1}.$$  

Hence

$$\int_a^b w(s) \left( \int_s^b \frac{dt}{p(t)} \right)^2 ds \leq M^2 \int_a^b w(s) ds \left( \int_s^a w(t) dt \right)^{-2} = M^2 \left( \int_a^{a+1} w(t) dt \right)^{-1}$$  

which implies $y_3 \in L^2_w(a, b)$.

3. The Resolvent Operator

Throughout this section we assume (2.23) holds and that zero is not an eigenvalue of $L_\alpha$ in the limit point case and not an eigenvalue of $L_{\alpha,\beta}$ in the limit circle.
case. Then for $f \in L^2_w(a, b)$, the equation $L_\alpha(y) = f$, or the equation $L_{\alpha, \beta}(y) = f$, has a unique solution which is given by

$$
y(x) = \frac{1}{g(x, t)} w(f(t)) dt,
$$

where

$$
g(x, t) = \begin{cases} z_1(t)z_2(x), & t < x, \\ z_1(x)z_2(t), & t > x, \end{cases}
$$

and $z_1, z_2$ are solutions of $L(z) = 0$ which we now define. The function $z_1$ must satisfy the boundary condition

$$
\cos \alpha z_1(a) + \sin \alpha (pz_1')(a) = 0,
$$

so we take $z_1$ to be of the form

$$z_1(x) = k[\sin \alpha - \cos \alpha \int_a^x \frac{1}{p(t)} dt].$$

The function $z_2$ must satisfy boundary condition at $b$, which is $z_2 \in L^2_w(a, b)$ in the LP case, so we take in the LP case, using (2.24),

$$z_2(x) = \begin{cases} 1 & \text{if } \int_a^b w(x) dx < \infty, \\
y_3(x) = \int_a^x ds/p(s) & \text{if } \int_a^b w(x) dx = \infty\end{cases}
$$

In the LC case we require that

$$z_2(x) = c_1 + c_2 y_2(x)
$$

satisfy the boundary condition

$$
\cos \beta [z_2, y_1](b) + \sin \beta [z_2, y_2](b) = c_1 \cos \beta - c_2 \sin \beta = 0.
$$

The constants $k, c_1, c_2$ are determined from the requirement on the Wronskian of $z_1, z_2$ that

$$z_1(x)(pz_2')(x) - z_2(x)(pz_1')(x) \equiv -1.
$$

In the LP case, only $k$ is to be determined. If $\int_a^b w < \infty$, then (3.5) yields that $k \cos \alpha = -1$. The assumption that 0 is not an eigenvalue implies $\alpha \neq \pi/2$ for otherwise $z_2$ would be an eigenfunction. If $\int_a^b w = \infty$, then (3.5) yields

$$k[-\sin \alpha + \cos \alpha \int_a^b 1/p(x) dx] = -1.
$$

If $-\sin \alpha + \cos \alpha \int_a^b 1/p(x) dx = 0$, then $z_2$ is an eigenfunction as before. Hence $k$ is determined by these two equations in the LP case.

In the LC case, we first normalize by taking $k = 1$. Then (3.5) yields

$$c_1 \cos \alpha + c_2 \sin \alpha = -1.
$$

Equations (3.4) and (3.6) have a unique solution if

$$\cos \beta \sin \alpha + \cos \alpha \sin \beta = \sin(\alpha + \beta) \neq 0.
$$

Since $0 \leq \alpha < \pi$, $0 < \beta \leq \pi$, the only possibility for $\sin(\alpha + \beta) = 0$ is for $\alpha + \beta = \pi$.

However, $\alpha = \beta = \pi/2$ yields that $y_1(x) = 1$ is an eigenfunction for $\lambda = 0$. For $\beta \neq \pi/2, \alpha + \beta = \pi$, the function $y(x) = \tan \beta + \int_a^x 1/p$ is an eigenfunction for
\[ \lambda = 0. \] Hence the constants \( c_1, c_2 \) and thus \( z_2 \) are completely determined by (3.4) and (3.6).

It will be useful to have formulae which yield the cases when the kernel \( g \) defines a Hilbert-Schmidt operator since this is another case where the spectrum is purely discrete. We also define the Green’s function \( G \) by

\[ \text{(3.7)} \quad G(x,t) = \frac{1}{2} w(x) \frac{1}{p} w(t) g(x,t) \]

Thus

\[ \text{(3.8)} \quad \|G\|^2 := \int_a^b \int_a^b G^2(x,t) dt dx = \int_a^b \int_a^b w(x) w(t) g^2(x,t) dt dx \]

\[ = 2 \int_a^b \int_a^b w(x) w(t) g^2(x,t) dt dx \]

\[ = 2 \int_a^b \int_a^b w(x) w(t) z_1^2(t) z_2^2(x) dt dx \]

where we have used the symmetry of \( g, g(x,t) = g(t,x) \).

For later estimates, we summarize three cases.

Case I. \( \int_a^b w = \infty \) (hence LP case holds and \( \int_a^b 1/p < \infty \) since we assume \( \sigma_e(L) > 0 \)). Then

\[ \text{(3.9)} \quad z_1(x) = k [\sin \alpha - \cos \alpha \int_a^x \frac{1}{p(s)} ds], \quad z_2(x) = \int_a^b \frac{ds}{p(s)} \]

where

\[ k = \left[ \sin \alpha - \cos \alpha \int_a^b \frac{1}{p(s)} ds \right]^{-1} \]

in (3.9).

Case II. LP and \( \int_a^b w < \infty \). Then

\[ \text{(3.10)} \quad z_1(x) = -\tan \alpha + \int_a^x \frac{1}{p(s)} ds, \quad z_2(x) = 1 \]

Case III. LC (hence \( \int_a^b w < \infty \)). Then

\[ \text{(3.11)} \quad z_1(x) = \sin \alpha - \cos \alpha \int_a^x \frac{1}{p(s)} ds, \quad z_2(x) = c_1 + c_2 \int_a^x \frac{1}{p(s)} ds, \]

where \( c_1 \) and \( c_2 \) in (3.11) are determined by (3.4) and (3.6).

**Example 3.1.** We consider the column examples again and take \( a(x) \) or \( r(x) \) as in Example 2.2. Note that \( \int_0^1 w < \infty \) in all cases. For the LP case or LC case, we calculate that

\[ 0 < q < 3 \Rightarrow \|G\|^2 < \infty \quad \text{for (1.1)} \]

\[ 0 < q \Rightarrow \|G\|^2 < \infty \quad \text{for (1.7) and (1.8)} \]

It is to be noted that the formulae for Green’s function in the LC case is also valid in the regular case in which case, (3.4) can be replaced by

\[ [z_2(b) - (p \frac{z_2'}{2} ) (b) \int_a^b \frac{ds}{p(s)}] \cos \beta - (p \frac{z_2'}{2} ) (b) \sin \beta = 0. \]
4. A special characterization of $\sigma_{\min}(T)$

We assume throughout this section that $\sigma_{\min}(T) > 0$ where $T = L_\alpha$ in the LP case and $T = L_{\alpha,\beta}$ in the LC case. Let $G$ be as in (3.7), and where $g$ is given in Section 3. Then if $T(y) = f$ and $\phi = w^{1/2}f$, it is readily verified that

\[ < Ty, y >_{L^2(a,b)} = < G^0\phi, \phi >_{L^2(a,b)}, \]

where, with $G$ as in (3.7),

\[ G^0\phi(x) = \int_a^b G(x,t)\phi(t) \, dt. \]

It follows immediately from (4.1) and the characterization (1.4) of $\sigma_{\min}(T)$ that the following lemma holds.

**Lemma 4.1.** If $\sigma_{\min}(T) > 0$, then

\[ \frac{1}{\sigma_{\min}(T)} = \max_{\phi \in L^2(a,b)} \frac{< G^0\phi, \phi >_{L^2(a,b)}}{< \phi, \phi >_{L^2(a,b)}}. \]

We also use the notation

\[ \sigma_{\min}(T) = \sigma_{\min}(p, w) \]

to show the dependence on the coefficients $p, w$. The dependence on the boundary conditions is suppressed in this notation.

We now give some cases in which $< G^0\phi, \phi >_{L^2(a,b)}$ takes a special form. This form will be useful in deriving a monotone property of eigenvalues.

Case I. Assume $\int_a^b w = \infty$ and $\alpha = \pi/2 \Leftrightarrow y'(a) = 0$ in (2.1). Then $z_1(x) = 1$ and $z_2(x) = \int_a^b 1/p$. For $\phi \in L^2(a,b)$, and $\Phi(x) = \int_a^x w^{1/2} \phi$, (3.7), and integration by parts yields,

\[ < G^0\phi, \phi >_{L^2(a,b)} = \int_a^b \left( \int_a^b \frac{\Phi(t)}{p(t)} \, dt \right) \Phi'(x) \, dx \]

where we have used $g^* < \infty$ ( from (2.16) and $\sigma_e(L) > 0$ ) implies

\[ z_2(t)|\Phi(t)| \leq z_2(t)^{1/2} \left( z_2(t) \int_a^t w(s) \, ds \right)^{1/2} \left( \int_a^t \Phi^2(s) \, ds \right)^{1/2} \to 0 \text{ as } x \to b. \]
Case II. Here we assume that $\alpha = 0 \iff y(a) = 0$, and further that either the LC case holds with $\beta = \pi/2$, or that the LP case holds (This case contains cases II and III of section 3). We define here $\Phi(x) = \int_a^b w^2 \phi$. In the LP case, $z_1(x) = \int_a^x 1/p$, $z_2(x) = 1$ while in the LC case, $z_1(x) = -\int_a^x 1/p$, $z_2(x) = -1$.

An analysis similar to Case I also shows that

$$< G^0 \phi, \phi >_{L^2(a,b)} = \int_a^b \int_a^x G(x,t)\phi(x)\phi(t)dt dx + \int_a^b \int_x^b G(x,t)\phi(x)\phi(t)dt dx$$

$$= \int_a^b \int_a^x \left( \int_a^t \frac{ds}{p(s)} \right) \Phi'(x)\Phi'(t)dt dx +$$

$$\int_a^b \int_x^b \left( \int_x^b \frac{ds}{p(s)} \right) \Phi'(x)\Phi'(t)dt dx.$$

Integration by parts on the inner integrals and simplifying yields

$$< G^0 \phi, \phi >_{L^2(a,b)} = \int_a^b \frac{\Phi^2(x)}{p(x)} dx$$

$$= \int_a^b \left( \int_x^b \frac{\Phi(t)}{p(t)} dt \right) \Phi'(x) dx$$

$$= \int_a^b \frac{\Phi^2(x)}{p(x)} dx,$$

as before.

5. A rearrangement inequality

In this section we use rearrangement inequalities to show that under certain conditions, the value of $\sigma_{\text{min}}(p,w)$ will increase under rearrangement of the coefficients. Suppose that

$$\mu_f(t) = |\{x \in (a,b) : f(x) > t\}|$$

is the measure of the set on which $f$ exceeds $t$. For a nonnegative function $f$, we define

$$f^\sharp(x) := \sup\{t > 0 : \mu_f(t) > x\}$$

to be the decreasing rearrangement of $f$ on $(a,b)$. The increasing rearrangement of $f$ is $f^\flat(x) = f^\sharp(b+a-x)$. It can be shown that

$$\int_a^b f(x) dx = \int_a^b f^\flat(x) dx = \int_a^b f^\sharp(x) dx.$$

In the examples below we need a property of the first eigenfunction of a Sturm-Liouville problem; see Theorem 14.10 of [27]. For separated boundary conditions in a singular or regular self-adjoint Sturm-Liouville problem defined on an interval $J$, with an increasing sequence of eigenvalues below the essential spectrum, the $n$-th eigenfunction has $n-1$ zeros in the interior of $J$. Note the eigenvalues are simple since the boundary conditions are separated. Our applications use the first eigenvalue, and their corresponding eigenfunctions have no zeros. We denote the first eigenfunction by $\phi_0$ and assume it has norm one. We also assume that the first eigenvalue $\lambda_0 > 0$ for cases I and II of section 4. To have $\lambda_0 > 0$, we see by (2.8) that it is sufficient that (2.20) holds, $\phi(a)\phi'(a) \geq 0$ and $\phi$ is not a constant.
Consider the first eigenfunction \( \Phi(0) \) for arbitrary \( p, w \). It follows from (2.1) that \( \Phi'(0) = 0 \), and we take \( \Phi(0) > 0 \) without loss of generality, so that

\[
(p \Phi'(0))(x) = -\lambda_0 \int_a^x w(s)\Phi(0)(s) \, ds
\]

from which we conclude that \( \Phi \) is decreasing on \( [a, b] \) since \( \Phi' \) is positive on \( [a, b] \).

When \( \Phi \in \mathcal{L}^2(a, b) \) is decreasing, \( \int_a^x f(x)\Phi(x) \, dx \geq \int_a^x f_2(x)\Phi(x) \, dx \) by [23, p.153]. Also (p.153). Also \( (\sqrt{w})_\lambda = \sqrt{w} \), by [2, p. 345]. Therefore,

\[
(\int_a^b \Phi^2(x) \, dx) / p(x) \int_a^b \left( \int_a^x w^\frac{p}{2}(s)\Phi(s) \right)^2 \, dx / p(x)
\]

Whenever \( \chi \) is increasing, \( \int_a^b f(x)\chi(x) \, dx \geq \int_a^b f_2(x)\chi(x) \, dx \) by [23, p.153]. Also (1/p)_\lambda = 1/p_2, by [2, p. 345]. Since \( \int_a^b (w^2(s))\chi^2 \Phi(s) \, ds \) is an increasing function of \( x \),

\[
\int_a^b \left( \int_a^x (w^2(s))\chi^2 \Phi(s) \right)^2 \, dx / p(x) \geq \int_a^b \left( \int_a^x (w^2(s))\chi^2 \Phi(s) \right)^2 \left( \frac{1}{p(x)} \right)^2 \, dx
\]

\[
= \int_a^b \left( \int_a^x (w^2(s))\chi^2 \Phi(s) \right)^2 \, dx / p_2(x).
\]

We now apply this inequality to the least eigenvalue \( \lambda_0 \). Since \( \alpha = \frac{\pi}{2} \) for this case, the first eigenfunction \( \Phi_0 \), for the coefficient functions \( w_2 \) and \( p_2 \), with corresponding \( \Phi_0 \), is a decreasing function. By (4.2) and (4.3),

\[
\frac{1}{\sigma_{\min}(p, w)} \geq \int_a^b \Phi_0^2(x) / p(x) \, dx.
\]

By (5.2) and (5.3), we have

\[
\int_a^b \Phi_0^2(x) / p(x) \, dx = \int_a^b \left( \int_a^x (w^2(s))\chi^2 \Phi_0(s) \right)^2 \, dx / p_2(x) = \frac{1}{\sigma_{\min}(p_2, w_2)}
\]

since the last equality holds by (4.3) and the fact that \( \Phi_0 \) is an eigenfunction. This yields the inequality

\[
\sigma_{\min}(p_2, w_2) \geq \sigma_{\min}(p, w).
\]

**Theorem 5.2.** Assume \( \lambda_0 > 0 \) and the conditions of case II of section 4, i.e., \( \int_a^b w < \infty, \alpha = 0 \), either the LC case holds with \( \beta = \pi/2 \), or the LP case holds, and \( \Phi(x) = \int_a^x w^\frac{p}{2} \Phi(x) \) for \( \phi \in \mathcal{L}^2(a, b) \). Then

\[
\sigma_{\min}(p^2, w^2) \geq \sigma_{\min}(p, w).
\]
Consider first the first eigenfunction $\phi_0$ with arbitrary $p, w$. We have $\phi_0(a) = 0$, and we take $\phi'_0(a) > 0$ without loss of generality. We make the Prüfer transformation
\begin{equation}
\phi_0(t) = \rho(t) \sin \theta(t), \quad (p\phi'_0)(t) = \rho(t) \cos \theta(t).
\end{equation}

Then as is well known,
\begin{equation}
\theta' = \frac{1}{p} \cos^2 \theta + \lambda_0 w \sin^2 \theta, \quad \rho' = \frac{\rho}{2} \left[ \frac{1}{p} - \lambda_0 w \right] \sin(2\theta).
\end{equation}

The boundary condition $\phi_0(a) = 0$ gives $\theta(a) = 0$, and (5.6) implies $\theta$ is increasing. Since $\theta$ has no zeros on $(a, b)$, we obtain that $\theta(b-) = \lim \theta(x)$ exists as $x$ tends to $b$ and $\theta(b-) \leq \pi$. We want to show $\theta(b-) \leq \pi/2$. Assume then $\theta(b-) > \pi/2$. If now $\int_a^b 1/p = \infty$, then (5.6) implies $\lim \theta(x) = \infty$ which is a contradiction. Thus $\int_a^b 1/p < \infty$ and since also $\int_a^b w < \infty$, we are in the LC case. By (5.6) we conclude that $\rho(b-) = \lim \rho(x)$ exists as $x$ tends to $b$. Returning to (5.5), we obtain that
\[
\lim_{x \to b} (p\phi'_0)(x) = \rho(b-) \cos \theta(b-) \neq 0,
\]
which is a contradiction in case II since $p\phi'_0$ tends to zero as $x$ tends to $b$. Thus $\theta(b-) \leq \pi/2$ and $\phi_0$ is increasing on $[a, b]$.

When $\phi \in L^2(a, b)$ is increasing, $\int_a^b f(x)\phi(x)dx \geq \int_a^b f^2(x)\phi(x)dx$ by [23, p.153]. Also $(\sqrt{w})^\eta = \sqrt{(w_\eta)}$, by [2, p. 345]. Therefore, with $\Phi(x) = \int_a^x 1/p$,
\begin{equation}
\int_a^b \frac{\Phi^2(x)}{p(x)} dx = \int_a^b \left( \int_a^b w^{\frac{\eta}{2}}(s)\phi(s) \right)^2 \frac{dx}{p(x)} 
\geq \int_a^b \left( \int_a^b w^{\frac{\eta}{2}}(s)\phi(s) \right)^2 \frac{dx}{p(x)} 
= \int_a^b \left( \int_a^b w^{\eta}(s)\phi(s) \right)^2 \frac{dx}{p(x)}.
\end{equation}

Whenever $\eta$ is decreasing, $\int_a^b f(x)\eta(x)dx \geq \int_a^b f_\eta(x)\eta(x)dx$ by [23, p.153]. Also $(1/p)^\eta = 1/p^\eta$, by [2, p. 345]. Since $\int_a^b (w^{\eta}(s))^\eta \phi(s)ds$ is a decreasing function of $x$,
\begin{equation}
\int_a^b \left( \int_a^b (w^{\eta}(s))^\eta \phi(s) \right)^2 \frac{dx}{p(x)} \geq \int_a^b \left( \int_a^b (w^{\eta}(s))^\eta \phi(s) \right)^2 \left( \frac{1}{p(x)} \right) \frac{dx}{p^\eta(x)} 
= \int_a^b \left( \int_a^b (w^{\eta}(s))^\eta \phi(s) \right)^2 \frac{dx}{p(x)}.
\end{equation}

We now apply this inequality to the least eigenvalue $\lambda_0$. Since $\alpha = 0$, the first eigenfunction $\phi_0$ for the coefficient functions $w^\phi$ and $p^\phi$, with corresponding $\Phi_0$, is an increasing function. By (4.2) and (4.4),
\[
\frac{1}{\sigma_{\min}(p, w)} \geq \int_a^b \frac{\Phi_0^2(x)}{p(x)} dx.
\]
By (5.7) and (5.8), we have,
\[
\int_a^b \Phi_0^2(x) \frac{dx}{p(x)} = \int_a^b \left( \int_a^b (w^2(s))^{\frac{1}{2}} \phi_0(s) \right)^2 \frac{dx}{p^2(x)} = \frac{1}{\sigma_{\text{min}}(p^2, w^2)},
\]
since the last equality holds by (4.4) and the fact that \( \phi_0 \) is an eigenfunction. This yields the inequality
\[
\sigma_{\text{min}}(p^2, w^2) \geq \sigma_{\text{min}}(p, w).
\]

6. Continuity of the least eigenvalue

We consider now a sequence of eigenvalue problems with coefficients \( p_0, \{p_n\}^\infty_{n=1}, \)
\( w_0, \{w_n\}^\infty_{n=1} \) with the same boundary conditions. The corresponding Green’s functions are denoted by \( g_0, \{g_n\}^\infty_{n=1} \) and \( G_0, \{G_n\}^\infty_{n=1} \) where \( g_n \) and \( G_n \) are related as
in (3.7). In this section we have discrete spectrum by reason of the Green’s function being Hilbert-Schmidt.

**Theorem 6.1.** Suppose the following conditions hold.
\[
\lim_{n \to \infty} G_n(x, t) = G_0(x, t) \text{ a.e. in } [a, b] \times [a, b]
\]
\[
0 < \sigma_{\text{min}}(p_n, w_n) \text{ for } n = 0, 1, ...
\]
\[
|G_n(x, t)| \leq H(x, t) \text{ for an } H \text{ satisfying } \int_a^b \int_a^b H(x, t)^2 dt dx < \infty
\]
Then
\[
\left| \frac{1}{\sigma_{\text{min}}(p_0, w_0)} - \frac{1}{\sigma_{\text{min}}(p_n, w_n)} \right| \leq \|G_0 - G_n\|.
\]
and \( \|G_0 - G_n\| \to 0 \) as \( n \to \infty \).

**Proof.** Let \( \phi \in L^2(a, b) \) satisfy \( ||\phi|| = 1 \), then by application of (4.2) and the Cauchy-Schwarz inequality, we obtain
\[
< G_0^0 \phi, \phi >_{L^2(a,b)} = < G_n^0 \phi, \phi >_{L^2(a,b)} + < G_0^0 \phi - G_n^0 \phi, \phi >_{L^2(a,b)} \leq \frac{1}{\sigma_{\text{min}}(p_n, w_n)} + \|G_0 - G_n\|.
\]
Applying (4.2) again we have that
\[
\frac{1}{\sigma_{\text{min}}(p, w)} \leq \frac{1}{\sigma_{\text{min}}(p_n, w_n)} + \|G_0 - G_n\|.
\]
From (6.6) and repeating the argument with \( G_n \) and \( G_0 \) switched, (6.4) follows.

**Example 6.2.** We now apply Theorem 6.1 to (1.7). Let \( B(x), 0 < B(x) < 1 \), be a increasing function on \( [0, 1] \) with \( B(x) \to 1 \) as \( x \to 1 \). The function \( B \) is used to control the rate of taper of the column. Let \( 0 < \theta < 1 \), and let \( S(r, B) \) be the set of all increasing functions \( r(x) \) on \( [0, 1] \) such that \( 0 < r(x) \leq B(x) \), and
\[ \int_0^1 r^2(x) \, dx = \theta. \] For physical reasons and because of the results of Section 5, it is reasonable to consider only increasing \( r \). We have the bounds

\[ p(x) = 1 - r^4(x) \geq 1 - r^2(x), \quad w(x) = \int_x^1 [1 - r^2(s)] \, ds \leq [1 - x][1 - r^2(x)], \]

and

\[ \int_0^t \frac{ds}{p(s)} \leq \int_0^t \frac{ds}{1 - r^2(s)} \leq \frac{t}{1 - r^2(t)}. \]

For boundary conditions, we use \( \alpha = 0 \) at \( x = 0 \) and if the equation (1.7) is in the LC or regular case we use \( \beta = \pi/2 \). Note that \( \int_0^1 w < \infty \). For \( 0 \leq t \leq x < 1 \), the formulas for Green’s functions given by (3.10) or (3.11) yield

\[ G(x, t) = w(x)w(t) \left( \int_0^t \frac{ds}{p(s)} \right)^2 \]

(6.7) \[ \leq (1 - x)(1 - t)[1 - r^2(x)][1 - r^2(t)] \left( \frac{t}{1 - r^2(t)} \right)^2 \]

(6.8) \[ \leq (1 - x)(1 - t)t^2 := H(x, t). \]

We extend \( H \) to \( t > x \) by making \( H \) symmetric; hence \( G(x, t) \leq H(x, t) \) on \([0, 1) \times [0, 1)\). Clearly \( H^2 \) is integrable on \([0, 1) \times [0, 1)\). Hence \( \| \| G \|^2 < \infty \) and the spectrum is purely discrete for \( r \in S(r, B) \). The formula for \( G \) above shows \( G(x, t) \geq 0 \), and if \( \lambda = 0 \) is not an eigenvalue, then we will have \( \sigma_{\min}(p, w) > 0 \). If \( \lambda = 0 \) is an eigenvalue, then \( \phi(x) = \int_0^x 1/p \) is an eigenfunction since \( \alpha = 0 \). If we are in the LP case, then this is a contradiction as \( g_1(x) = 1 \in L^2_\| w \| (0, 1) \Rightarrow \phi \not\in L^2_\| w \| (0, 1) \). If we are in the LC case we must have \( (p\phi')(x) \to 0 \) as \( x \to \) b since \( \beta = \pi/2 \). Clearly this does not hold for \( \phi(x) = \int_0^x 1/p \).

Let now \( s := \sup\{ \sigma_{\min}(p, w) : r \in S(r, B) \} \)

and let \( r_n \) be a sequence in \( S(r) \) such that \( \sigma_{\min}(p_n, w_n) \to s \) as \( n \to \infty \). Since the \( r_n \) are increasing and bounded, we have by Helly’s theorem that \( \{ r_n \} \) has a pointwise convergence subsequence which we take to be \( \{ r_n \} \), say \( r_n(x) \to \tilde{r}(x) \) on \( 0 \leq x < 1 \) as \( n \to \infty \). Let \( \tilde{p}, \tilde{w}, \tilde{G} \) be the functions corresponding to \( \tilde{r} \). For \( 0 \leq t \leq x < 1 \),

\[ \frac{1}{p_n(t)} \leq \frac{1}{p_n(x)} \leq \frac{1}{1 - B^2(x)} \]

since \( p_n \) is a decreasing function. Thus by Lebesgue’s theorem,

\[ \lim_{n \to \infty} \int_0^t \frac{ds}{p_n(s)} = \int_0^t \frac{ds}{\tilde{p}(s)} \quad \lim_{n \to \infty} w_n(t) = w(t), \]

and hence \( G_n(x, t) \to G(x, t) \) as \( n \to \infty \) a.e. on \([0, 1) \times [0, 1)\). Also

\[ \lim_{n \to \infty} \int_0^1 r_n(s) \, ds = \int_0^1 \tilde{r}(s) \, ds = \theta. \]

Thus \( \tilde{r} \in S(r, B) \) and Theorem 6.1 gives that \( s = \sigma_{\min}(\tilde{p}, \tilde{w}) \) so that \( \tilde{r} \) minimizes the spectrum in \( S(r, B) \). This example is a special case of Case II of section 4.

We can prove a similar result for (1.8), but we need two additional conditions which are reasonable physical ones.
EXAMPLE 6.3. We now apply Theorem 6.1 to (1.8). Let \( B(x) > 1 \) be a decreasing function on \([0, 1)\) with \( B(x) \to 1 \) as \( x \to 1 \). The function \( B \) is used to control the rate of taper of the column. Let \( 1 < \theta, K > 0 \), and let \( S(r, K, B) \) be the set of all decreasing functions \( r(x) \) on \([0, 1)\) such that \( B(x) < r(x), r(0) \leq K \) and \( \int_0^1 r^2(x) \, dx = \theta \). For physical reasons and because of the results of Section 5, it is again reasonable to consider only decreasing \( r \). We have the bounds

\[
p(x) = r^4(x) - 1 \geq r^2(x) - 1, \quad w(x) = \int_x^1 |r^2(s) - 1| \, ds \leq |1 - x| [r^2(x) - 1],
\]

and

\[
\int_0^t \frac{ds}{p(s)} \leq \int_0^t \frac{ds}{r^2(s) - 1} \leq \frac{t}{r^2(t) - 1}
\]

For boundary conditions, we use \( \alpha = 0 \) at \( x = 0 \) and if the equation (1.8) is in the LC or regular case we use \( \beta = \pi/2 \). The function \( G \) is bounded by \( H \) as in (6.7) which implies a discrete spectrum, and for the same reason as in Example 6.2, \( \sigma_{\min}(p, w) > 0 \).

Let now

\[
s := \sup\{\sigma_{\min}(p, w) : r \in S(r, K, B)\}
\]

and let \( r_n \) be a sequence in \( S(r, K) \) such that \( \sigma_{\min}(p_n, w_n) \to s \) as \( n \to \infty \). Since the \( r_n \) are decreasing and bounded, we have by Helly’s theorem that \( \{r_n\} \) has a pointwise convergence subsequence which we take to be \( \{r_\alpha\} \), say \( r_\alpha(x) \to \tilde{r}(x) \) on \( 0 \leq x < 1 \) as \( n \to \infty \). Let \( \tilde{p}, \tilde{w}, \tilde{G} \) be the functions corresponding to \( \tilde{r} \). For

\[
0 \leq t \leq x < 1 \Rightarrow 1/p_n(t) \leq 1/p_n(x) \leq \frac{1}{B^4(x) - 1}
\]

since \( p_n \) is a decreasing function. Thus by Lebesgue’s theorem,

\[
\lim_{n \to \infty} \int_0^t \frac{ds}{p_n(s)} = \int_0^t \frac{ds}{\tilde{p}(s)}, \quad \lim_{n \to \infty} w_n(t) = w(t);
\]

hence \( G_n(x, t) \to G(x, t) \) as \( n \to \infty \) a.e. on \([0, 1) \times [0, 1)\). Also

\[
\lim_{n \to \infty} \int_0^1 r_n(s) \, ds = \int_0^1 \tilde{r}(s) \, ds = \theta
\]

Thus \( \tilde{r} \in S(r, K, B) \) and Theorem 6.1 gives that \( s = \sigma_{\min}(\tilde{p}, \tilde{w}) \) so that \( \tilde{r} \) minimizes the spectrum on \( S(r, K, B) \). This is another example of Case II of section 4.

As a last application of Theorem 6.1 we prove an existence theorem for maximizing the least eigenvalue for two coefficients. For boundary conditions we take:

\[
(6.10) \quad \alpha = 0, \text{ LP with } \int_a^b w(x) \, dx < \infty, \text{ or } LC \text{ with } \alpha = 0, \beta = \frac{\pi}{2}.
\]

Recall that \( \int_a^b w(x) \, dx < \infty \) also in the LC case. From (3.10) and (3.11) we see that

\[
\alpha = 0, \text{ LP, } \int_a^b w(x) \, dx < \infty \Rightarrow z_1(x) = \int_a^x \frac{dt}{p(t)}, \quad z_2(x) = 1,
\]

and

\[
\alpha = 0, \text{ LC, } \beta = \frac{\pi}{2} \Rightarrow z_1(x) = -\int_a^x \frac{dt}{p(t)}, \quad z_2(x) = -1,
\]
With the Green's function $G$ given by (3.7), we have from (3.8) then that

\begin{equation}
\|G\|^2 = 2 \int_a^b \int_a^x w(x)w(t) \left( \int_a^t \frac{ds}{p(s)} \right)^2 \, dx dt.
\end{equation}

Assume we have decreasing coefficients $\tilde{p}, \tilde{w}$ so that (6.10) holds for $\tilde{p}, \tilde{w}$ (see Theorem 2.1 for the LP condition), and the corresponding $\tilde{G}$ satisfies, $\|\tilde{G}\| < \infty$ where $\|\tilde{G}\|$ is computed by (6.11). Define a class of functions $C(\tilde{p}, \tilde{w}, K)$ by

$$C(\tilde{p}, \tilde{w}, K) := \{p, w : K \geq p(0), p(x) \geq \tilde{p}(x), w(x) \leq \tilde{w}(x), a \leq x < b\}.$$

**Theorem 6.4.** Assume $\tilde{p}, \tilde{w}$ satisfy (6.10). Define

$$s := \sup\{\sigma_{\min}(p, w), p, w \in C(\tilde{p}, \tilde{w}, K) : (6.10) \text{ holds}\}$$

Then there exists $p_0, w_0 \in C(\tilde{p}, \tilde{w}, K)$ such that $s = \sigma_{\min}(p_0, w_0)$.

**Proof.** For the same reason as in Examples 6.2 and 6.3, we have $\sigma_{\min}(p, w) > 0$. Let $p_n, w_n \in C(\tilde{p}, \tilde{w}, K)$ be such that $\sigma_{\min}(p, w) \to s$ as $n \to \infty$. By (5.9), and using $f(x) \leq g(x)$ implies $f^2(x) \leq g^2(x)$, we see that it is sufficient to suppose that $p_n, w_n$ are decreasing on $[a, b)$. Since the sequences $p_n, w_n$ are uniformly bounded on every $[a, x], x < b$, we can apply Helly’s theorem to obtain subsequences which converge pointwise on $[a, b)$. Without loss of generality we take these to be $\{p_n\}, \{w_n\}$, say as $n \to \infty, a \leq x < b$,

$$p_n(x) \to p_0(x), \quad w_n(x) \to w_0(x).$$

Applying Theorem 6.1 with $H = \tilde{G}$ completes the proof. \hfill \Box

For regular Sturm-Liouville problems, continuity of the eigenvalues with respect to the coefficients has been proved by Kong, Wu, and Zettl [15]. By continuity the metric of $\mathcal{L}(a,b)$ is assigned to the coefficients. With the pointwise convergence of the coefficients used above, this is a weaker notion of convergence. For regular problems, similar notions of this weaker convergence have been considered by Battle [1], but again for the regular case. For singular problems, the Green’s function approach used above allows the proof of continuity of eigenvalues by means of the norms of the Green’s functions.

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