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Optimal control of insects through sterile insect release and habitat modification

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ABSTRACT

This paper develops an optimal control framework for an ordinary differential equation model to investigate the introduction of sterile mosquitoes to reduce the incidence of mosquito-borne diseases. Existence of a solution given an optimal strategy and the optimal control is determined in association with the negative effects of the disease on the population while minimizing the cost due to this control mechanism. Numerical simulations have shown the importance of effects of the bounds on the release of sterile mosquitoes and the bounds on the likelihood of egg maturation. The optimal strategy is to maximize the use of habitat modification or insecticide. A combination of techniques leads to a more rapid elimination of the wild mosquito population.

1. Introduction

This paper develops a model for the controlled release of sterile insects into an environment where there is an existing population of wild insects. We will also consider the effect of controlling fecundity by altering the environment in such a way that breeding rate is reduced. This activity would take the form of reducing the locations for breeding though removing sources of standing water and of using larvicide or ovacide. We will not consider broad spectrum insecticides because these would also kill our sterilized insects. There has been success in using traps for male insects along with sterile insect release [18], however, we will not consider this third control method in this paper.

The importance of controlling mosquito populations is hard to overstate. It is well known that such diseases as yellow fever, dengue fever, epidemic polyarthritis, Rift Valley fever, Ross River Fever, St. Louis encephalitis, West Nile virus, Japanese encephalitis, LaCross encephalitis, and malaria are carried and transmitted by mosquitoes, [12,26,29,30,34,39,41,42].

This paper considers a model that can applied to many insects, including mosquitoes. Optimal control theory is then applied with a variety of cost functionals to find the best strategy for reducing insect population at minimal cost.

The sterile insect technique was introduced by Knipling [17,18]. The insects are sterilized by irradiation or the application of chemical agents and released to mate with the wild insects. It was used successfully for the screw worm in the late 1950s and early 1960s and great hope was held for using the technique for the control of mosquito populations [19]. Unfortunately, experiments that were carried out with mosquitoes during the same period met with less success. For a discussion of the experimental work in this area see [9,28,38,4].

A number of authors have developed mathematical models of the interaction between sterile and wild mosquitoes, [17,22,3,31]. Some sterile release models have been explicitly connected to particular diseases [7,8,40]. Dumont and Tchuenche [7] consider pulsed sterile release and demonstrate through equilibrium analysis and simulations that frequent small bursts of sterile insects are more effective than larger less frequent releases. Esteva and Yang [8] apply optimal control methods to the rate of introduction of sterile mosquitoes. An approach developed in [40] attempts to control both breeding rates and the rate of introduction of sterile mosquitoes. No bounds have been imposed on the control (s) in any of this work which may be not be realistic biologically.

The use of transgenic insects was developed after the sterile insect technique. Insects carrying a dominant lethal gene are introduced into the population. Alphey et al. [1,2] provide many details of the use of both of these techniques. Models that described the interactions of wild and transgenic mosquitoes include those by Li [23,24] and Diaz et al. [6]. Optimal control methods are applied to the rate of introduction of transgenic mosquitoes by Rafikov et al. [35,36].

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It is our hope that by developing new bounded control models for this technique, we may find strategies that will make it more effective.

1.1. The model

We are particularly interested in Li's model of the release of transgenic mosquito populations [24]. Although our focus is on sterile mosquitoes, we will follow the approach in the referenced paper for the model we develop here because it captures the features we seek to incorporate. We consider a population of wild mosquitoes, u, and a population of sterilized mosquitoes, w. If $b(u,w)$ is the birth rate of the wild mosquitoes and $d_u(u,w)$ and $d_w(u,w)$ are the death rates of the wild population and sterilized population respectively, we obtain

$$\frac{du}{dt} = u(b(u,w) - d_u(u,w))$$
$$\frac{dw}{dt} = -wd_w(u,w) + S(t)$$

where S is the release rate of sterile mosquitoes. We will assume the death rate has a constant component and a component that increases with total population density. Thus we will have

$$d_u(u,w) = M + K(u+w)$$
$$d_w(u,w) = M + K(u+w)$$

where the equality of the constants is an implicit assumption of equal fitness between the wild population and the sterilized population. We now turn our attention to the birthrate, $b(u,w)$.

Continuing to follow the approach in [24], we let $c(u,w,t)$ be the number of matings that occur per unit time. Therefore, we can expect that the number of matings of wild type to wild type will be

$$b(u,w) = c(u,w,t)\frac{u}{u+w}$$

This will give us

$$\frac{du}{dt} = u\left(c(u,w,t)\frac{u}{u+w} - M - K(u+w)\right)$$
$$\frac{dw}{dt} = -w(M + K(u+w)) + S(t)$$

Let us consider a couple of choices for the function $c(u,w,t)$. When the total population is large, we expect that mosquitoes will have no difficulty finding a mate, giving us $c(u,w,t)$ as a function only of time, $A(t)$ which is the product of such factors as the likelihood of a mating producing eggs, the (fixed) proportion of the population that is female, the likelihood that an appropriate place can be found so that when the eggs are laid they will hatch, and so on. $A(t)$ can be reduced through the application of larvicide or insecticide, the clearing of breeding sites, etc. Henceforth, we will generally refer to such habitat modification as the application of insecticide, with the understanding that habitat modification can have other features. The function $A(t)$ will serve as a control as well as S, since we are assuming we can take action to reduce the amount of suitable real estate for successful egg laying. This gives the following model

$$\frac{du}{dt} = u\left(\frac{A(t)u}{u+w} - Mu - K(u+w)\right)$$
$$\frac{dw}{dt} = -w(Mw + K(u+w)) + S(t)$$

When the population is relatively small, we expect the law of mass action to be pertinent with $c(u,w,t) = A(t)(u+w)$ where the function $A(t)$ is similar to the function $A(t)$ described above. This gives us

$$\frac{du}{dt} = u(A(t)u - M - K(u+w))$$
$$\frac{dw}{dt} = -w(M + K(u+w)) + S(t)$$

We are particularly interested in a function that can capture the dynamics of both large and small populations simultaneously. We seek a functional form that will lead to approximately the models above. Once again, we follow the work of Li [24] and choose a Holling-II-type functional response, [15]. Fixing a positive constant $\varepsilon > 0$, we set

$$c(u,w) = A\frac{u+w}{\varepsilon + u + w}$$

giving us

$$\frac{du}{dt} = u\left(\frac{Au}{\varepsilon + u + w} - M - K(u+w)\right)$$
$$\frac{dw}{dt} = -w(M + K(u+w)) + S(t)$$

We now rescale, letting $u = \frac{u}{\varepsilon}$ and $w = \frac{w}{\varepsilon}$. Setting $a = A$, $\mu = M\varepsilon$, $\eta = K\varepsilon$, and $s = \frac{S}{\varepsilon}$ yields our final model,

$$\frac{du}{dt} = u\left(\frac{au}{1+u+w} - \mu - \eta(u+w)\right) \quad (1)$$
$$\frac{dw}{dt} = -w(\mu + \eta(u+w)) + s(t). \quad (2)$$

where the initial conditions are

$$u(0) = u_0, \quad w(0) = w_0 \quad (3)$$

and the controls are bounded with $M_1, M_2, N_1, N_2 \geqslant 0$ such that

$$M_1 \leqslant a(t) \leqslant M_2, \quad N_1 \leqslant s(t) \leqslant N_2. \quad (4)$$

The rest of this paper is organized as follows. In Section 2 we establish basic facts about the ODE model. In Section 3, we obtain the existence of an optimal control pair $(a,s)$for different objective functionals. In Section 4 we implement the forward–backward sweep method for each of our cases to obtain numerical results. Finally, in Section 5, we provide discussion of our results and their implications for the optimal control of mosquito populations.

2. Existence

In this section we will obtain the existence, uniqueness, nonnegativity, and boundedness of solutions to our model in a single theorem.

Theorem 2.1. *For nonnegative initial conditions, the model (1), (2) has a unique solution which exists for all time and is nonnegative in each component.*

Proof. Local existence for the system is standard as in [27]. To obtain the result, we first define supersolutions u_1 and w_1 as in

$$\frac{du_1}{dt} = u_1(a - \eta u_1) \quad \frac{dw_1}{dt} = N_2 - \eta w_1.$$

These supersolutions are bounded on a finite interval. Hence, via a comparison result [33], we have that u and w are bounded above on their interval of existence. Moreover, we can let u_2 and w_2 represent subsolutions of the following system,

$$\frac{du_2}{dt} = -Ku_1 \quad \frac{dw_2}{dt} = -Kw_2.$$

where K is a sufficiently large constant. Therefore, we obtain that u and w are bounded below by zero. Consequently, with the coefficients of our original system (1), (2) being bounded, we obtain that a solution set is nonnegative and bounded. Using a result from Lukes [25], a nonnegative, unique solution exists to the system. □

Since we have the existence for the solution for the mosquito system for all time, we can investigate the optimal control strategy associated with different objective functionals subject to the original system (1), (2).

3. Optimal controls

3.1. Objective functional

We seek to minimize each of the following objective functionals

$$J_0(a,s) = \int_0^T (Ba^2 + s^2 + u)dt \tag{5}$$

and

$$J_1(a,s) = \int_0^T (Ba^2 + s + u)dt \tag{6}$$

over the set of admissible controls

$$V = \{a, s \text{ measurable } | M_1 \leqslant a(t) \leqslant M_2 \text{ and } N_1 \leqslant s(t) \leqslant N_2,\ \forall t \in [0,T]\} \tag{7}$$

Since we have a nonlinear system, we consider a mixture of nonlinear and linear controls. For $J_0(a,s)$, we minimize the nonlinear cost associated with reproduction, $a(t)$, and the nonlinear cost corresponding to the release of sterile mosquitoes, $s(t)$. In $J_0(a,s)$ and $J_1(a,s)$, we also seek to minimize the amount of wild mosquitoes in total. The difference between $J_0(a,s)$ and $J_1(a,s)$ results in analyzing the amount associated with the release of sterile mosquitoes in $J_1(a,s)$ rather than the nonlinear cost of such a process in $J_0(a,s)$. Further, we assume a quadratic cost [8,35,40] since we believe that the effects of the larvicide, adulticide and insecticides represent a a nonlinear function to the system. The quadratic term is multiplied by a coefficient, B, which allows for the relative importance of the term. Essentially, we choose to augment the coefficient of a^2 to analyze the ratio of the importance that one puts on the insecticide versus sterile control factors. In the numerics and discussion sections, we will consider fixing $a = 1$ to allow no insecticide or fixing $s = 0$ to allow no sterile release.

3.2. Existence

We will show the existence of optimal controls for objective functionals (5) and (6). The first case will be shown using a standard theorem from Fleming and Rishel [10].

Theorem 3.1 (*Existence of a quadratic optimal control*). *Given the objective functional (5), subject to the system given by Eqs. (1), (2), with $u(0) = u_0, w(0) = 0$, and the admissible control set (7) then there exists an optimal control $\mathbf{v}^*(t) = (a^*(t), s^*(t))$ such that*

$$\min_{a,s\in V} J_0(a,s) = J_0(a^*, s^*).$$

Proof. In order to obtain the results, we much show that the following conditions are met from Fleming and Rishel, [10]:

1. The class of all initial conditions with a control vector $\vec{v}(t)$ in the admissible control set along with each state equation being satisfied is not empty.
2. The admissible control set V is closed and convex.
3. Each right hand side of the state system is continuous, is bounded above by a sum of the bounded control and the state, and can be written as a linear function of the control vector $\vec{v}(t)$ with coefficients depending on time and the state.
4. The integrand of the objective functional (5) is convex on V and is bounded below.

Note that the supersolutions $\overline{u}, \overline{w}$ of

$$\frac{d\overline{u}}{dt} = \overline{u}a(t)$$
$$\frac{d\overline{w}}{dt} = s(t)$$

are bounded on the finite time interval. Since the solutions are nonnegative, the system (1), (2) is then bounded above and below. As in the Existence Section 2 we know that a solution to the system exists. Hence, condition 1 is fulfilled. The second condition is fulfilled from the definition of the admissible control set V. For the third condition, note that the continuity of the right hand side of system (1), (2) is guaranteed since $u(t)$ and $w(t)$ are nonnegative, for all t in the finite time interval. Next, define

$$\mathbf{f}(\mathbf{v},\mathbf{x},t) = \begin{pmatrix} -\mu u - \eta u^2 - \eta uw \\ -\mu w - \eta uw - \eta w^2 \end{pmatrix} + \begin{pmatrix} \frac{u^2}{1+u+w} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(t) \\ s(t) \end{pmatrix},$$

where $\mathbf{x}$ is the state vector. Using the boundedness of the solutions and the controls,

$$|\mathbf{f}(\mathbf{v},\mathbf{x},t)| \leqslant u_{\max} \left| \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \right| \leqslant \Phi\left(\left|\vec{X}\right| + \left|\vec{V}(t)\right| \right),$$

where u_{max} is an upper bound for $u(t)$ and Φ is dependent upon the coefficients of the system.

For the final condition, define $\hat{a} = (1-p)a_0 + pa_1, \hat{s} = (1-p)s_0 + ps_1$, and $\hat{u} = (1-p)u_0 + pu_1$. It is necessary to show that

$$f(\hat{a},\hat{s},\hat{u},t,T) \leqslant (1-p)f(a_0,s_0,u_0,t,T) + pf(a_1,s_1,u_1,t,T),$$

where f denotes the integrand of objective functional (5), all controls are in the admissible control set (7), and $p \in (0,1)$. Observe that

$$f(\hat{a},\hat{s},\hat{u},t,T) - (1-p)f(a_0,s_0,u_0,t,T) - pf(a_1,s_1,u_1,t,T)$$
$$= B(p^2 - p)(a_0 - a_1)^2 + (p^2 - p)(s_0 - s_1)^2$$

The difference $(p^2 - p)$ must be negative since $p \in (0,1)$. Since $B, (a_0 - a_1)^2$, and $(s_0 - s_1)^2$ are always positive, then $f(\hat{a},\hat{s},\hat{u},t,T) \leqslant (1-p)f(a_0,s_0,u_0,t,T) + pf(a_1,s_1,u_1,t,T)$, as required. Finally, since $u(t)$ is always nonnegative and the controls $a(t)$ and $s(t)$ are bounded below by a nonnegative constant, then the integrand of the objective functional must also be bounded below. □

The existence of an optimal control for objective functional (6) is established using the classical Filippov-Cesari Theorem, [14,37]. We employ this theorem because of the linearity in the sterile control. For application of Fleming and Rishel existence criteria, the appropriate convexity is required in the objective functional for the controls. We do not have that in the second objective functional $J_!$ given by (6). Hence, the Filippov-Cesari Theorem is used. We define

$$\tilde{a}(t) = \frac{a(t)}{M_2} \quad \text{and} \quad \tilde{s}(t) = \frac{s(t)}{N_2},$$

so that $0 \leqslant \tilde{a} \leqslant 1$ and $0 \leqslant \tilde{s} \leqslant 1$. Note also that $\widetilde{V}$ is used to denote the admissible control set that reflects these changes in bounds.

Theorem 3.2 (*Existence of a mixed quadratic/linear optimal control*). *Given the objective functional (6), subject to the system given by Eqs. (1), (2), with $u(0) = u_0, w_0 = 0$, and the admissible control set (7) then there exists an optimal control $\tilde{\mathbf{v}}^*(t) = (\tilde{a}^*(t), \tilde{s}^*(t))$ such that*

$$\min_{\tilde{a},\,\tilde{s}\in \tilde{V}} J_1(\tilde{a},\tilde{s}) = J_1(\tilde{a}^*,\tilde{s}^*),$$

if conditions (1)–(4) of Theorem 7.1 *in the Appendix are met.*

Proof. Applying the notation of Theorem 7.1 to the optimal control problem given above, we have

$$\mathbf{x} = \begin{pmatrix} u \\ w \end{pmatrix}$$

$$\Phi(\mathbf{x},\tilde{\mathbf{v}},t) = \begin{pmatrix} B\tilde{a}^2 + \tilde{s} + u + \gamma \\ \frac{u^2\tilde{a}}{1+u+w} - \mu u - \eta u^2 - \eta uw \\ -\mu w - \eta uw - \eta w^2 + \tilde{s} \end{pmatrix}$$

where $\gamma \leqslant 0$ is a constant defined in the aforementioned theorem in the Appendix.

The first condition is satisfied using an argument similar to that showing the existence of an admissible solution pair using objective functional (5). To show the convexity condition, define

$$w_1(\mathbf{x},\tilde{\mathbf{v}},t) = \begin{pmatrix} B\tilde{a_1}^2 + \tilde{s_1} + u + \gamma_1 \\ \frac{u^2\tilde{a_1}}{1+u+w} - \mu u - \eta u^2 - \eta uw \\ -\mu w - \eta uw - \eta w^2 + \tilde{s_1} \end{pmatrix}$$

and

$$w_2(\mathbf{x},\tilde{\mathbf{v}},t) = \begin{pmatrix} B\tilde{a_2}^2 + \tilde{s_2} + u + \gamma_2 \\ \frac{u^2\tilde{a_2}}{1+u+w} - \mu u - \eta u^2 - \eta uw \\ -\mu w - \eta uw - \eta w^2 + \tilde{s_2} \end{pmatrix}$$

where $\gamma_1, \gamma_2 \leqslant 0$ as in the definition of the constant γ in Φ in the Appendix.

Let $w_3 = \lambda w_1 + (1-\lambda)w_2$, where $\lambda \in [0,1]$. Split the equation into two vectors - one vector containing the modified terms and the other vector the unchanged terms:

$$w_3(\mathbf{x},\tilde{\mathbf{v}},t) = \begin{pmatrix} u \\ -\mu u - \eta u^2 - \eta uw \\ -\mu w - \eta uw - \eta w^2 \end{pmatrix} + \begin{pmatrix} B\bar{a_3} + s_3 + \gamma_3 \\ \left(\frac{u^2}{1+u+w}\right) a_3 \\ s_3 \end{pmatrix},$$

where $a_3 = \lambda\tilde{a_1} + (1-\lambda)\tilde{a_2}$, $s_3 = \lambda\tilde{s_1} + (1-\lambda)\tilde{s_2}$, $\gamma_3 = \lambda\gamma_1 + (1-\lambda)\gamma_2$, and $\bar{a_3} = \lambda\tilde{a_1}^2 + (1-\lambda)\tilde{a_2}^2$. Note that, since $0 \leqslant \tilde{a} \leqslant 1$ and $0 \leqslant \tilde{s} \leqslant 1$, then $a_3, s_3, \bar{a_3} \in \tilde{V}$. Then

$$w_3(\mathbf{x},\tilde{\mathbf{v}},t) = \begin{pmatrix} B\bar{a_3} + s_3 + u + \gamma_3 \\ \left(\frac{u^2}{1+u+w}\right) a_3 - \mu u - \eta u^2 - \eta uw \\ -\mu w - \eta uw - \eta w^2 + s_3 \end{pmatrix},$$

which means that $\lambda w_1 + (1-\lambda)w_2 \in \Phi(\mathbf{x},\tilde{\mathbf{v}},t)$; therefore $\Phi(\mathbf{x},\tilde{\mathbf{v}},t)$ meets the convexity requirement.

For the third condition, a number δ must be found such that $\|x\| \leqslant \delta,\ \forall t \in [0,T]$ and all admissible pairs $(\mathbf{x},\tilde{\mathbf{v}})$. To do this, we must find an upper bound for the state Eqs. (1), (2). However, in the existence proof for objective functional (5), we observed that this system is bounded; therefore, letting this upper bound be δ satisfies this condition.

The fourth condition is satisfied by definition since both controls are bounded.

Since an optimal control $\tilde{\mathbf{v}}^*(t) = (\tilde{a}^*(t),\tilde{s}^*(t))$ exists, then there must also exist an optimal control pair $\mathbf{v}^*(t) = (a^*(t),s^*(t))$. $\square$

3.3. Characterization of the controls

As with existence, we will provide justification for the characterization of the optimal control with only two cases; namely, objective functionals $J_0(a,s)$ in Eq. (5) and $J_1(a,s)$ in Eq. (6).

3.3.1. CASE 1: objective functional $J_0(a,s)$

Theorem 3.3 (*Characterization of the optimal control*). *Given optimal controls $a^*(t), s^*(t)$ and solutions of the corresponding state system, there exist adjoint variables λ_1 and λ_2 satisfying the following:*

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial u} = -1 - \lambda_1\left[\frac{au(2+u+2w)}{(1+u+w)^2} - \mu - \eta(2u+w)\right] + \lambda_2\eta w$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial w} = \lambda_1\left[\frac{au^2}{(1+u+w)^2} + \eta u\right] + \lambda_2[\mu + \eta(u+2w)]$$

where

$\lambda_1(T) = \lambda_2(T) = 0.$

Furthermore, the analytic representation of the optimal control pair (a^*, u^*) is given by

$$a^*(t) = \min\left(\max\left(M_1, -\frac{\lambda_1 u^2}{2B(1+u+w)}\right), M_2\right) \tag{8}$$

$$s^*(t) = \min\left(\max\left(N_1, -\frac{\lambda_2}{2}\right), N_2\right). \tag{9}$$

Proof. Suppose $a^*(t)$ and $s^*(t)$ are optimal controls and that $X = (u,w)$ is a corresponding solution to the system (1), (2). We use standard work in Pontryagin et al. [32] to obtain the result. To find the analytic representation of the optimal controls $a^*(t)$ and $s^*(t)$, begin by forming the Lagrangian. Since the controls are bounded, the Lagrangian is

$$\begin{aligned}\mathcal{L} = H &- W_1(t)(a(t)-M_1) - W_2(t)(M_2 - a(t)) - W_3(t)(s(t)-N_1) \\ &- W_4(t)(N_2 - s(t)),\end{aligned}$$

where H is the Hamiltonian given by

$$\begin{aligned}H = Ba^2 + s^2 + u + \lambda_1 u&\left[\frac{au}{1+u+w} - (\mu + \eta(u+w))\right] \\ &+ \lambda_2[-w(\mu + \eta(u+w)) + s]\end{aligned}$$

and $W_i(t) \geqslant 0$ are penalty multipliers such that

$$\left.\begin{aligned} W_1(t)(a(t)-M_1) = 0 \\ W_2(t)(M_2 - a(t)) = 0 \end{aligned}\right\} \text{ at } a^*(t) \quad \left.\begin{aligned} W_3(t)(s(t)-N_1) = 0 \\ W_4(t)(N_2 - s(t)) = 0 \end{aligned}\right\} \text{ at } s^*(t).$$

To find the analytic representation for $a^*(t)$, analyze the necessary conditions for optimality $\frac{\partial \mathcal{L}}{\partial a} = 0$.

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial H}{\partial a} - W_1 + W_2 = 0 \Rightarrow 2Ba + \frac{\lambda_1 u^2}{1+u+w} - W_1 + W_2 = 0$$

By standard optimality techniques for the characterization for the optimal control $a^*(t)$, we find that

$$a^*(t) = \min\left(\max\left(M_1, -\frac{\lambda_1 u^2}{2B(1+u+w)}\right), M_2\right).$$

Similarly,

$$s^*(t) = \min\left(\max\left(N_1, -\frac{\lambda_2}{2}\right), N_2\right).$$

$\square$

We note that $\lambda_1(t)$ and $\lambda_2(t)$ are bounded above by a constant on the time interval. We compare the differential equations of the adjoint system to a supersolution using that we have bounded solutions u and w from the Existence Section. The adjoint system can be written as

$$\begin{pmatrix} \frac{d\lambda_1}{dt} \\ \frac{d\lambda_2}{dt} \end{pmatrix} = \begin{pmatrix} \left[\frac{-au(2+u+2w)}{(1+u+w)^2} + \mu + \eta(2u+w)\right] & \eta w \\ \left[\frac{au^2}{(1+u+w)^2} + \eta u\right] & \mu + \eta(u+2w) \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Using that the parameters and the solutions to the state equations are nonnegative, we notice that all the terms in the two by two matrix are bounded above by a positive constants, α, β, γ, and δ.

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Using a comparison result [33], we find that $\lambda_1(t) \leqslant K$ and $\lambda_2(t) \leqslant K_1$ where K and K_1 are positive constants. Since the adjoints are bounded, this implies that a^* and s^* are finite.

3.3.2. CASE 2: objective functional $J_1(a, s)$

Theorem 3.4 (*Characterization of the optimal control*). *Given optimal controls $a^*(t), s^*(t)$ and solutions of the corresponding state system, there exist adjoint variables λ_1 and λ_2 satisfying the following:*

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial u} = -1 - \lambda_1 \left[\frac{au(2+u+2w)}{(1+u+w)^2} - \mu - \eta(2u+w)\right] + \lambda_2 \eta w$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial w} = \lambda_1 \left[\frac{au^2}{(1+u+w)^2} + \eta u\right] + \lambda_2[\mu + \eta(u+2w)]$$

where

$\lambda_1(T) = \lambda_2(T) = 0.$

Furthermore, the representation of $a^*(t)$ is given by (8) and the representation of $s^*(t)$, found using the switching function

$$\Psi = \frac{\partial H}{\partial s} = 1 + \lambda_2,$$

is given by

$$s^*(t) = \begin{cases} N_1 & \text{if } \Psi > 0 \\ N_2 & \text{if } \Psi < 0 \\ \text{singular} & \text{if } \Psi = 0. \end{cases} \tag{10}$$

In addition, if $s^*(t)$ is singular on a subinterval (t_1, t_2) of $[0, T]$, then the singularity is of degree one and the representation of the control on the singular interval (t_1, t_2) is given by

$$\begin{aligned} s^*(t) = & -\left(\frac{2B(1+u+w)^4}{3\lambda_1^2 u^4 + 4B\eta\lambda_2(1+u+w)^4}\right) \\ & \times \left\{ \left(\frac{\lambda_1 u^4}{B(1+u+w)^2}\right) \cdot \left[\frac{-\lambda_1^2 u^3[3u + 2(2+u+2w)]}{4B(1+u+w)^4}\right.\right. \\ & + \frac{\lambda_1 u^3}{(1+u+w)^3} - \frac{3\lambda_1[\mu(u+w) + \eta(u^2+w^2) + 2\eta uw]}{2(1+u+w)^2} \\ & \left. + \frac{2[\mu\lambda_1 + \eta\lambda_1 w + 1 - \eta\lambda_2 w] + \eta\lambda_1(2+u+2w)}{2(1+u+w)} - \frac{\eta(\lambda_1+\lambda_2)}{2}\right] \\ & \left. + \eta u[\eta u(\lambda_1 - \lambda_2) - (\mu\lambda_2 + 1)] - 2\eta\lambda_2 w[\mu + \eta(u+w)] \right\}. \end{aligned}$$

Finally, in order for $s^*(t)$ to be minimizing, we must satisfy the Legendre-Clebsch condition [20],

$$\frac{-3\lambda_1^2 u^4}{2B(1+u+w)^4} \geqslant 2\eta\lambda_2.$$

Proof. The Hamiltonian in this case is given by

$$H = Ba^2 + s + u + \lambda_1 u\left[\frac{au}{1+u+w} - (\mu + \eta(u+w))\right] + \lambda_2[-w(\mu + \eta(u+w)) + s].$$

The representation for $a^*(t)$ is the same as in case 1; however, since $s(t)$ is implemented as a linear control, we must examine the switching function

$$\Psi = \frac{\partial H}{\partial s} = 1 + \lambda_2$$

in order to find the analytic representation for $s^*(t)$. Note that, since $\Psi = 0$ on (t_1, t_2), then all time derivatives of Ψ are identically zero; i.e. $\frac{d}{dt}\Psi = 0, \frac{d^2}{d^2t}\Psi = 0$, etc. We will use this idea to find the representation of $s^*(t)$ on a singular region. To begin, notice that

$$\frac{d\Psi}{dt} = \frac{d\lambda_2}{dt} = \lambda_1\left[\frac{au^2}{(1+u+w)^2} + \eta u\right] + \lambda_2[\mu + \eta(u+2w)].$$

Assuming for ease in notational purposes that $a(t) = a^*(t)$ on (t_1, t_2), we substitute (8) and expand to obtain

$$\frac{d\Psi}{dt} = \frac{-\lambda_1^2 u^4}{2B(1+u+w)^3} + \eta\lambda_1 u + \mu\lambda_2 + \eta\lambda_2 u + 2\eta\lambda_2 w.$$

Taking a second time derivative yields

$$\begin{aligned} \frac{d^2\Psi}{dt^2} = & -\frac{1}{2B}\left[\frac{\lambda_1^2 u^4}{(1+u+w)^3}\right]\prime + \eta(\lambda_1 u' + u\lambda_1') + \mu\lambda_2' \\ & + \eta(\lambda_2 u' + u\lambda_2') + 2\eta(\lambda_2 w' + w\lambda_2'). \end{aligned}$$

Since $\frac{d\Psi}{dt} = \frac{d\lambda_2}{dt} = 0$ on (t_1, t_2), we eliminate those terms involving λ_2', resulting in

$$\frac{d^2\Psi}{dt^2} = -\frac{1}{2B}\left[\frac{\lambda_1^2 u^4}{(1+u+w)^3}\right]' + \eta(\lambda_1 u' + u\lambda_1') + \eta\lambda_2 u' + 2\eta\lambda_2 w'.$$

Finding the derivative of the first term yields

$$\begin{aligned} \frac{d^2\Psi}{dt^2} = & \\ & -\frac{1}{2B}\left[\frac{(1+u+w)^3[4\lambda_1^2 u^3 u' + 2\lambda_1 u^4 \lambda_1'] - 3\lambda_1^2 u^4 (1+u+w)^2[u' + w']}{(1+u+w)^6}\right] \\ & + \eta(\lambda_1 u' + u\lambda_1') + \eta\lambda_2 u' + 2\eta\lambda_2 w'. \end{aligned}$$

Finally, after much algebraic simplification, we have that

$$\begin{aligned} \frac{d^2\Psi}{dt^2} = & \left(\frac{\lambda_1 u^4}{B(1+u+w)^2}\right) \cdot \left[\frac{-\lambda_1^2 u^3[3u + 2(2+u+2w)]}{4B(1+u+w)^4}\right. \\ & + \frac{\lambda_1^2 u^3}{B(1+u+w)^3} - \frac{3\lambda_1[\mu u + \eta u^2 + 2\eta uw + \mu w + \eta w^2]}{2(1+u+w)^2} \\ & \left. + \frac{2[\mu\lambda_1 + \eta\lambda_1 w + 1 - \eta\lambda_2 w] + \eta\lambda_1(2+u+2w)}{2(1+u+w)} - \frac{\eta(\lambda_1+\lambda_2)}{2}\right] \\ & + \eta u[\eta u(\lambda_1 - \lambda_2) - (\mu\lambda_2 + 1)] - 2\eta\lambda_2 w[\mu + \eta(u+w)] \\ & + \left(\frac{3\lambda_1^2 u^4}{2B(1+u+w)^4} + 2\eta\lambda_2\right)s(t) \end{aligned}$$

Now, since $\frac{d^2\Psi}{dt^2} = 0$ on (t_1, t_2), we have that

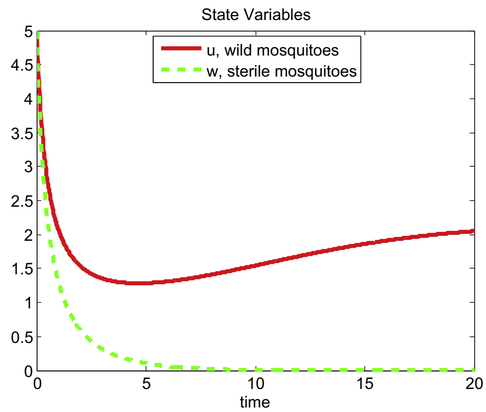

Fig. 1. Objective functional J_0 with $B = 1$, bounds $a(t) = 1$, $0 \leqslant s(t) \leqslant 0$, initial conditions $u_0 = 5$, $w_0 = 5$ over the time interval $[0, 20]$.

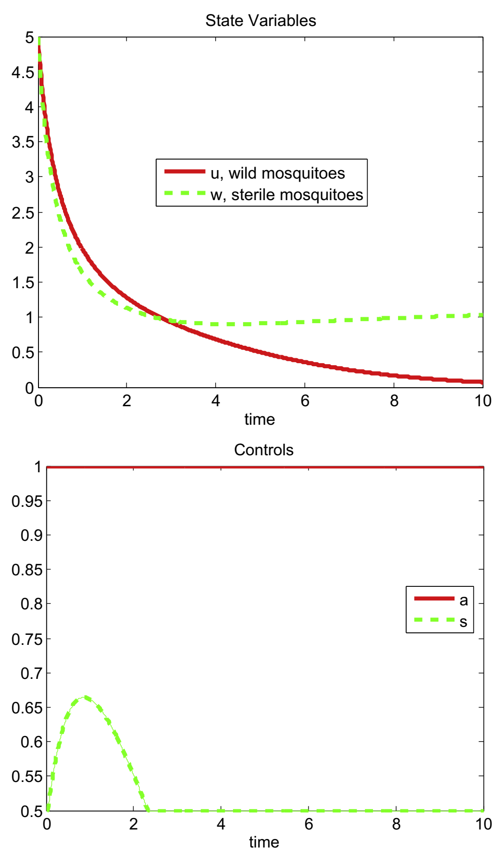

Fig. 2. Objective functional J_0 with $B = 1$, $a(t) = 1$, $0.5 \leqslant s(t) \leqslant 1$, initial conditions $u_0 = 5$, $w_0 = 5$ over the time interval $[0, 10]$.

$$
\begin{aligned}
s(t) = {} & -\left(\frac{2B(1+u+w)^4}{3\lambda_1^2u^4+4B\eta\lambda_2(1+u+w)^4}\right)\left\{\left(\frac{\lambda_1 u^4}{B(1+u+w)^2}\right)\right. \\
& \cdot\left[\frac{-\lambda_1^2u^3[3u+2(2+u+2w)]}{4B(1+u+w)^4}+\frac{\lambda_1^2u^3}{B(1+u+w)^3}\right. \\
& -\frac{3\lambda_1[\mu u+\eta u^2+2\eta uw+\mu w+\eta w^2]}{2(1+u+w)^2} \\
& \left.+\frac{2[\mu\lambda_1+\eta\lambda_1 w+1-\eta\lambda_2 w]+\eta\lambda_1(2+u+2w)}{2(1+u+w)}-\frac{\eta(\lambda_1+\lambda_2)}{2}\right] \\
& \left.+\eta u[\eta u(\lambda_1-\lambda_2)-(\mu\lambda_2+1)]-2\eta\lambda_2 w[\mu+\eta(u+w)]\right\}.
\end{aligned}
$$

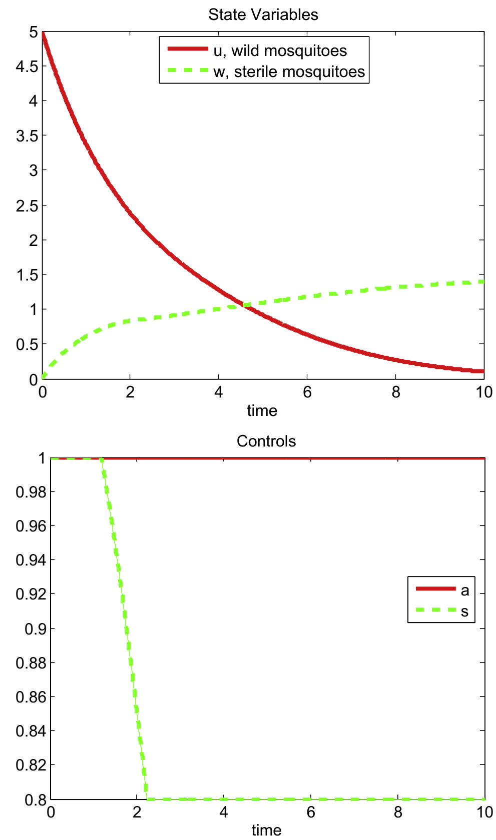

Fig. 3. Objective functional J_0 with $B = 1$, $a(t) = 1$, $0.5 \leqslant s(t) \leqslant 1$, initial conditions $u_0 = 5$, $w_0 = 0$ over the time interval $[0, 10]$.

In order for $s^*(t)$ to be minimizing on (t_1, t_2), the Legendre-Clebsch condition

$$(-1)^q \frac{\partial}{\partial s}\frac{d^{2q}}{dt^{2q}}\frac{\partial H}{\partial s} \geqslant 0$$

must be satisfied. Please note that q is the order of the singularity. In this case, this results in

$$-\frac{3\lambda_1^2u^4}{2B(1+u+w)^4} - 2\eta\lambda_2 \geqslant 0,$$

or

$$\frac{-3\lambda_1^2u^4}{2B(1+u+w)^4} \geqslant 2\eta\lambda_2. \qquad \square$$

There are a few things to recognize. Immediately following the proof of the Characterization Theorem related to $J_0(a, s)$, we found that $\lambda_2(t) \leqslant K_1$. First, we know that s^* must be finite. Moreover, the singular case could contribute to the solution. Lastly, we note that the optimal control representation is unique for small final time. This is due to the opposite orientations of the state and adjoint systems. For discussion of the uniqueness of the optimality system and hence the optimal control, see [21,5].

4. Numerical simulations

At the optimum, we note that the model equations move forward in time from an initial condition, while the adjoint equations move backward in time from a final condition. In some cases, it is possible to use Matlab's **bvp4c** to solve ODE systems with a variety

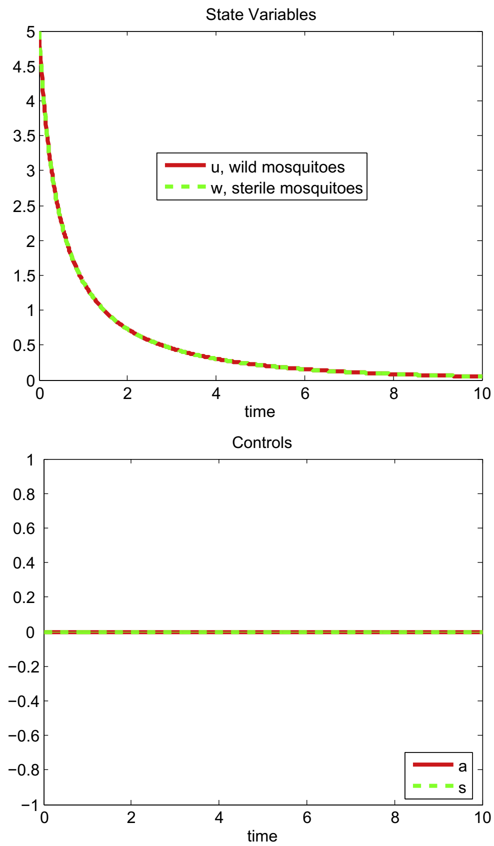

Fig. 4. Objective functional J_0 with $B = 1$, $0 \leqslant a(t) \leqslant 1$, $s(t) = 0$, initial conditions $u_0 = 5$, $w_0 = 5$ over the time interval $[0, 10]$.

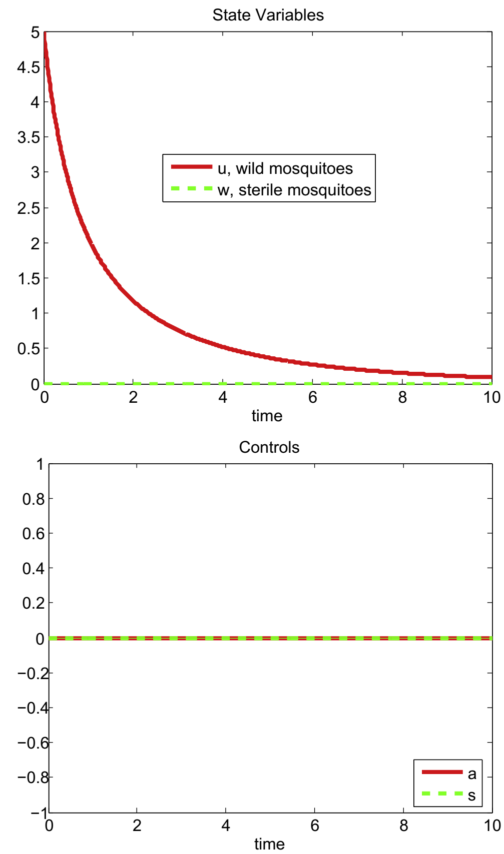

Fig. 5. Objective functional J_0 with $B = 1$, $0 \leqslant a(t) \leqslant 1$, $s(t) = 0$, initial conditions $u_0 = 5$, $w_0 = 0$ over the time interval $[0, 10]$.

of different types of boundary conditions. However, we found convergence problems with this approach. Instead we followed the algorithm developed by Hackbusch [13] and recommended by Lenhart and Workman [21] to solve our optimality system. The scheme is as follows:

1. Initialize the adjoint variables, $\lambda_1^0 = 0, \lambda_2^0 = 0$, and the controls $a^0 = (M_1 + M_2)/2, s^0 = (N_1 + N_2)/2$
2. Use the current adjoint variables $\lambda_1^{j-1}, \lambda_2^{j-1}$ and controls a^{j-1}, s^{j-1} to solve the state equations for the state variables u^j, w^j.
3. Use the current state variables u^j, w^j. to solve the adjoint equations for the adjoint variables λ_1^j, λ_2^j
4. Update the controls a^j, s^j using the control characterizations
5. Repeat steps 2–4 until convergence.

The algorithm was implemented in Matlab, using a Runge–Kutta method to solve the ODEs. The same algorithm was used for both objective functionals and their corresponding optimality systems. Following Li's simulations [24], we set $\mu = 0.25$ and $\eta = 0.2$ for all of the simulations presented below, although other parameters yielded similar results.

4.1. CASE 1: objective functional J_0

With the objective functional

$$J_0(a, s) = \int_0^T (Ba^2 + s^2 + u)dt,$$

we seek to simultaneously reduce the fecundity of the wild mosquitoes by minimizing the nonlinear cost a^2 of the Holling II reproductive term a for the wild mosquitoes, the nonlinear cost of s^2 of the rate s at which the sterile mosquitoes are being introduced, and the total number of wild mosquitoes u present over the time inerval $[0, T]$. We note that a value of $a = 0$ maximizes the impact of the insecticide by eliminating all growth of the wild mosquitoes u. Recall that here, and throughout the paper, we generally refer to habitat modification techniques as the use of insecticide.

Recall that for this objective functional and the bounds $M_1 \leqslant a(t) \leqslant M_2$, $N_1 \leqslant s(t) \leqslant N_2$, the model or state variables u, w and the adjoint variables λ_1, λ_2 satisfy the optimality system

$$\frac{du}{dt} = u\left[\frac{ua(t)}{1 + u + w} - (\mu + \eta(u + w))\right] \tag{11}$$

$$\frac{dw}{dt} = -w(\mu + \eta(u + w)) + s(t) \tag{12}$$

$$\frac{d\lambda_1}{dt} = -1 - \lambda_1\left[\frac{au(2 + u + 2w)}{(1 + u + w)^2} - \mu - \eta(2u + w)\right] + \lambda_2\eta w \tag{13}$$

$$\frac{d\lambda_2}{dt} = \lambda_1\left[\frac{au^2}{(1 + u + w)^2} + \eta u\right] + \lambda_2[\mu + \eta(u + 2w)] \tag{14}$$

with boundary conditions

$$u(0) = u_0, w(0) = w_0, \lambda_1(T) = 0, \lambda_2(T) = 0. \tag{15}$$

In Theorem 3.3, the optimal controls are given by

$$a^*(t) = \min\left(\max\left(M_1, -\frac{\lambda_1 u^2}{2B(1 + u + w)}\right), M_2\right) \tag{16}$$

$$s^*(t) = \min\left(\max\left(N_1, -\frac{\lambda_2}{2}\right), N_2\right). \tag{17}$$

4.1.1. No insecticide with no sterile mosquito release

We begin by considering the interaction of wild and sterile mosquitoes in the absence of insecticide, $a = 1$ and without additional release of sterile insects, $s = 0$. In this case $J_0 = \int_0^T B + u(t)\,dt$ which amounts to minimizing the total number of mosquitoes in the absence of control variables.

In Fig. 1, we see that the wild mosquito population reaches a carrying capacity, while the sterile mosquito population rapidly decays to zero in the absence of additional sterile mosquito release.

4.1.2. No insecticide with sterile mosquito release

We now allow sterile mosquito release in the absence of insecticide, $a = 1$ and $N_1 \leqslant s \leqslant N_2$. In Fig. 2, we see that the wild mosquito population is eliminated by a short release of sterile mosquitoes. In Fig. 3, we see that the wild mosquito population is eliminated by a short constant release of sterile mosquitoes. Once the wild mosquitoes have been eliminated, the introduction of the sterile mosquitoes should be discontinued, allowing the sterile population to die. Further simulations with higher values of N_1 and N_2 yielded a more rapid elimination of the wild mosquitoes but led to a high carrying capacity for the sterile mosquitoes. We note that we encountered convergence issues with our algorithm for small N_1 and that the wild mosquito population was eradicated only when N_1 was sufficiently large.

4.1.3. Insecticide with no sterile mosquito release

We now allow the use of insecticide in the absence sterile mosquito release, $M_1 \leqslant a \leqslant M_2$ and $s = 0$. In Figs. 4 and 5, we see that the wild mosquito population is eliminated through maximal use of insectide.

4.1.4. Mixture of strategies

We now allow sterile mosquito release with the use of insecticide, $M_1 \leqslant a \leqslant M_2$ and $N_1 \leqslant s \leqslant N_2$. In Figs. 6 and 7, we see that both the wild and sterile mosquito populations are eliminated by a moderate release of sterile mosquitoes. We note that in the initial absence of sterile mosquitoes, the sterile population increases before decreasing to zero.

Furthermore, we varied B in this case to allow determine if the insecticide was more (large B) or less (small B) important than the sterile release. An interesting consequence of the formulation of a^* is that, since $M_1 \geqslant 0$, the value of B becomes relevant only when $\lambda_1 < 0$ which was not the case for any parameter set we considered.

4.2. CASE 2: objective functional J_1

With the objective functional

$$J_1(a,s) = \int_0^T (Ba^2 + s + u)dt,$$

we seek to simultaneously reduce the fecundity of the wild mosquitoes by minimizing the nonlinear cost a^2 of the Holling II reproductive term a for the wild mosquitoes, the total number of sterile mosquitoes s and the total number of wild mosquitoes u present over the time inerval $[0, T]$.

Recall that for this objective functional and the bounds $M_1 \leqslant a(t) \leqslant M_2,\ N_1 \leqslant s(t) \leqslant N_2$, the optimal controls are given by

$$a^*(t) = \min\left(\max\left(M_1, -\frac{\lambda_1 u^2}{2B(1+u+w)}\right), M_2\right) \tag{18}$$

and

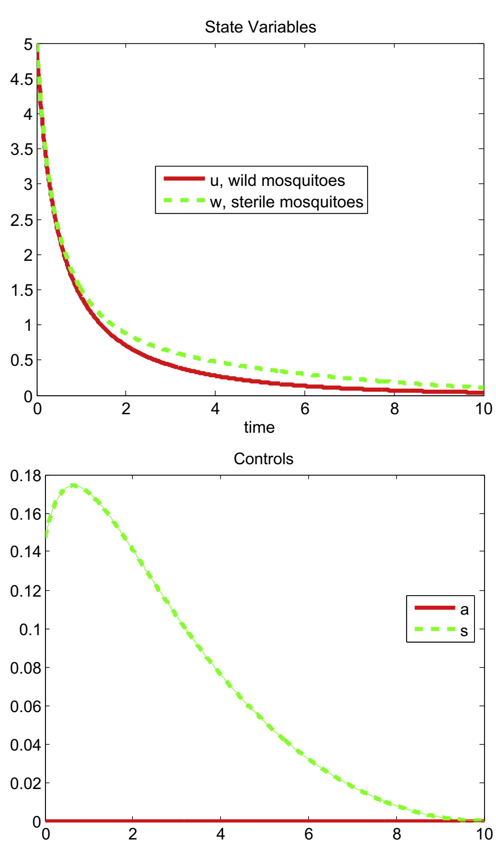

Fig. 6. Objective functional J_0 with $B = 1$, bounds $0 \leqslant a(t) \leqslant 1$, $0 \leqslant s(t) \leqslant 1$, initial conditions $u_0 = 5$, $w_0 = 5$ over the time interval $[0, 10]$.

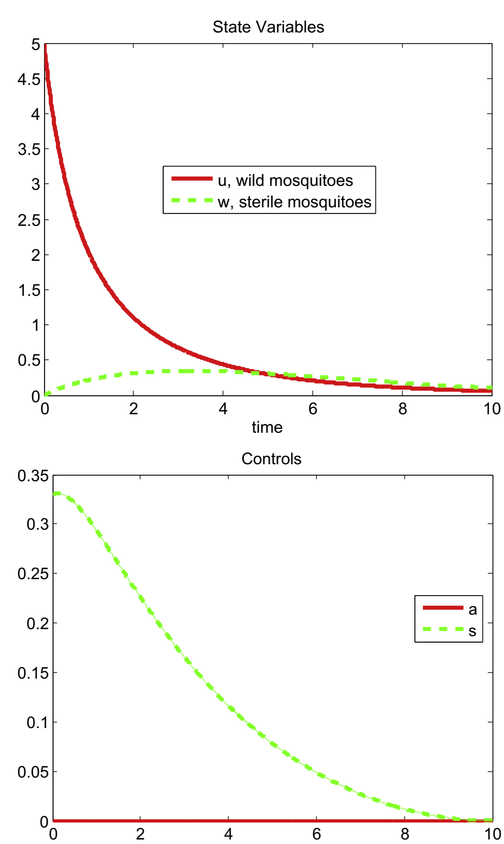

Fig. 7. Objective functional J_0 with $B = 1$, bounds $0 \leqslant a(t) \leqslant 1$, $0 \leqslant s(t) \leqslant 1$, initial conditions $u_0 = 5$, $w_0 = 0$ over the time interval $[0, 10]$.

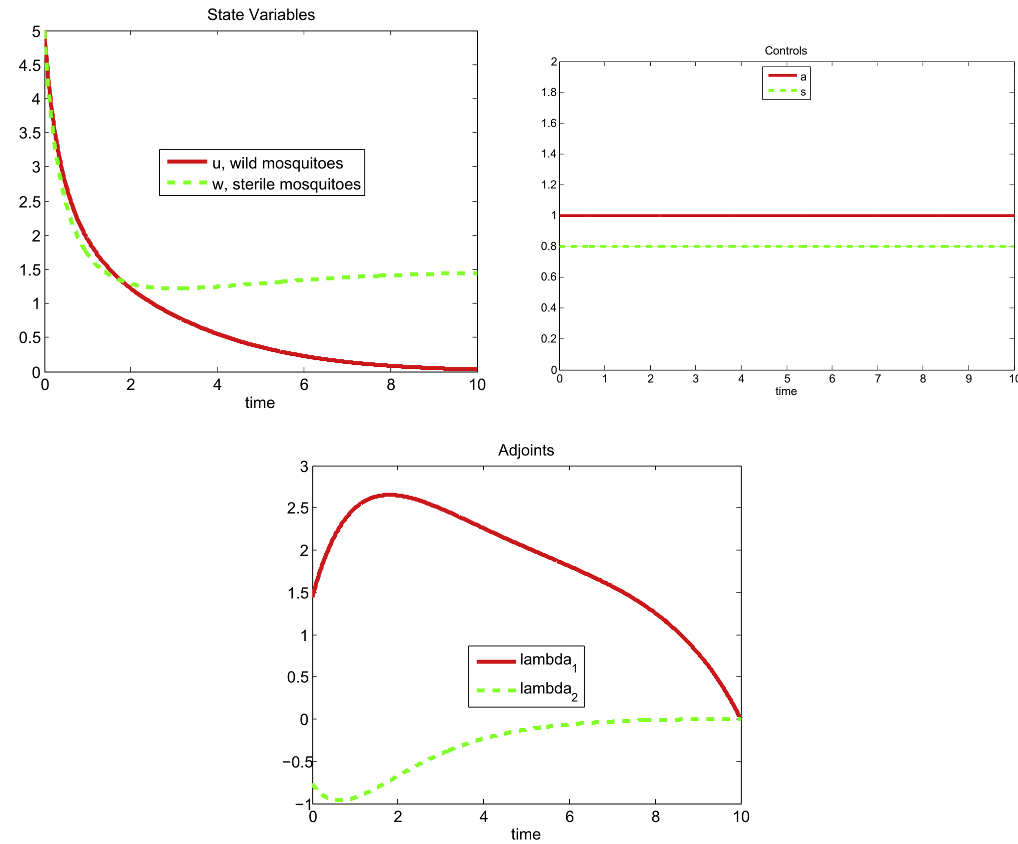

Fig. 8. Objective functional J_1 with $B = 1$, $a(t) = 1$, $0.8 \leqslant s(t) \leqslant 1$, initial conditions $u_0 = 5$, $w_0 = 5$ over the time interval $[0, 10]$.

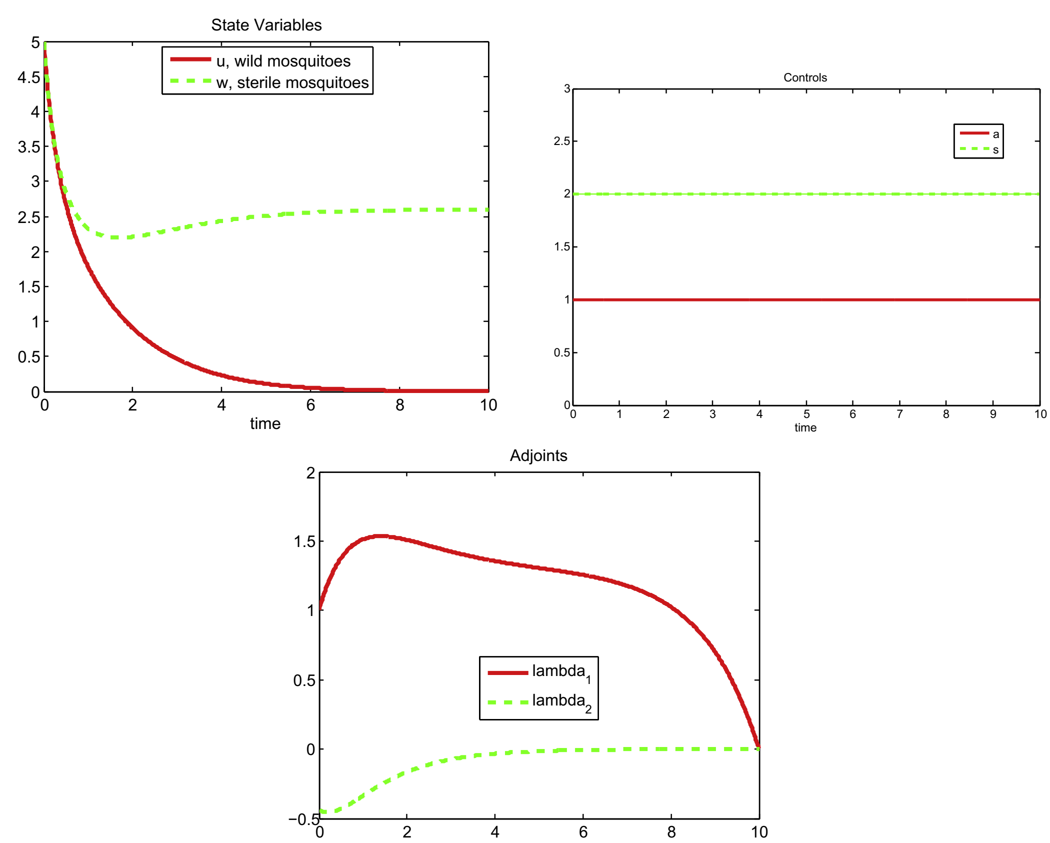

Fig. 9. Objective functional J_1 with $B = 1$, $a(t) = 1$, $2 \leqslant s(t) \leqslant 5$, initial conditions $u_0 = 5$, $w_0 = 5$ over the time interval $[0, 10]$.

$$s^*(t) = \begin{cases} N_1 & \text{if } \Psi > 0 \\ N_2 & \text{if } \Psi < 0 \\ \text{singular} & \text{if } \Psi = 0. \end{cases} \tag{19}$$

where Ψ is the switching function

$$\Psi = \frac{\partial H}{\partial s} = 1 + \lambda_2.$$

The model or state variables u, w and the adjoint variables λ_1, λ_2 satisfy the optimality system (11)–(15).

4.2.1. No insecticide with no sterile mosquito release

We note that for $a = 1$ and $s = 0$, we have $J_1 = \int_0^T B + u(t)\,dt$ which is the same as J_0 for this case. Hence, the behavior of the system is identical to that shown in Fig. 1.

4.2.2. No insecticide with sterile mosquito release

Again, we allow sterile mosquito release in the absence of insecticide, $a = 1$ and $N_1 \leqslant s \leqslant N_2$. We encountered substantial convergence issues for large T in this case. We note that if $1 + \lambda_2 \neq 0$, then the control s is not singular. For most examples considered, this meant that $s^* = N_1$ as in Fig. 8 and 9. In both cases, the wild mosquito population is eliminated by a short release of sterile mosquitoes. The sterile mosquito population is increased and the wild mosquito population is eradicated only when N_1 is sufficiently large.

We present one example in Fig. 10 where $1 + \lambda_2 = 0$. In this case, the value of s^* switched from the upper bound of N_2 to the lower bound of N_1. This is known as a bang-bang control.

4.2.3. Insecticide with or without sterile mosquito release

We now allow the use of insecticide in the presence or absence of sterile mosquito release, $M_1 \leqslant a \leqslant M_2$ and $N_1 \leqslant s \leqslant N_2$ or $s = 0$. Identical results were obtained with or without sterile insect release. In Figs. 11 and 12, we see that the wild mosquito population is eliminated through maximal use of insecticide $a^* = 0$ and minimal use of sterile mosquito release $s^* = 0$.

We note that variation of B is not important for J_1 because the control a^* is the same as for J_0.

5. Discussion

This paper describes an optimal control approach to mosquito population reduction. Our mosquito model is somewhat unusual in that we model the reproductive term using a Holling-II functional response, [24,15]. The advantage of this is that it allows us to simultaneously model the mating dynamics of both small and large populations. Overall, the behavior of the optimal control model is as we expected. We have demonstrated that it is possible, through optimal choices of insecticide use (or habitat modification) and/or sterile insect release, to eliminate the wild mosquito population. We considered several different scenarios combining the use of insecticide with the release of sterile mosquitoes, under two different objective functionals, $J_0(a,s) = \int_0^T (Ba^2 + s^2 + u)dt$ and $J_1(a,s) = \int_0^T (Ba^2 + s + u)dt$. The differences in the objective functional amount to the inclusion of a nonlinear term relating to the release of sterile mosquitoes into the environment. The mathematical effect results in the inclusion of a possible singular control for J_1.

In the absence of insecticide or sterile insect release, the wild mosquitoes will reach a carrying capacity, while the sterile insect population will decay to zero. In the absence of insecticide but in

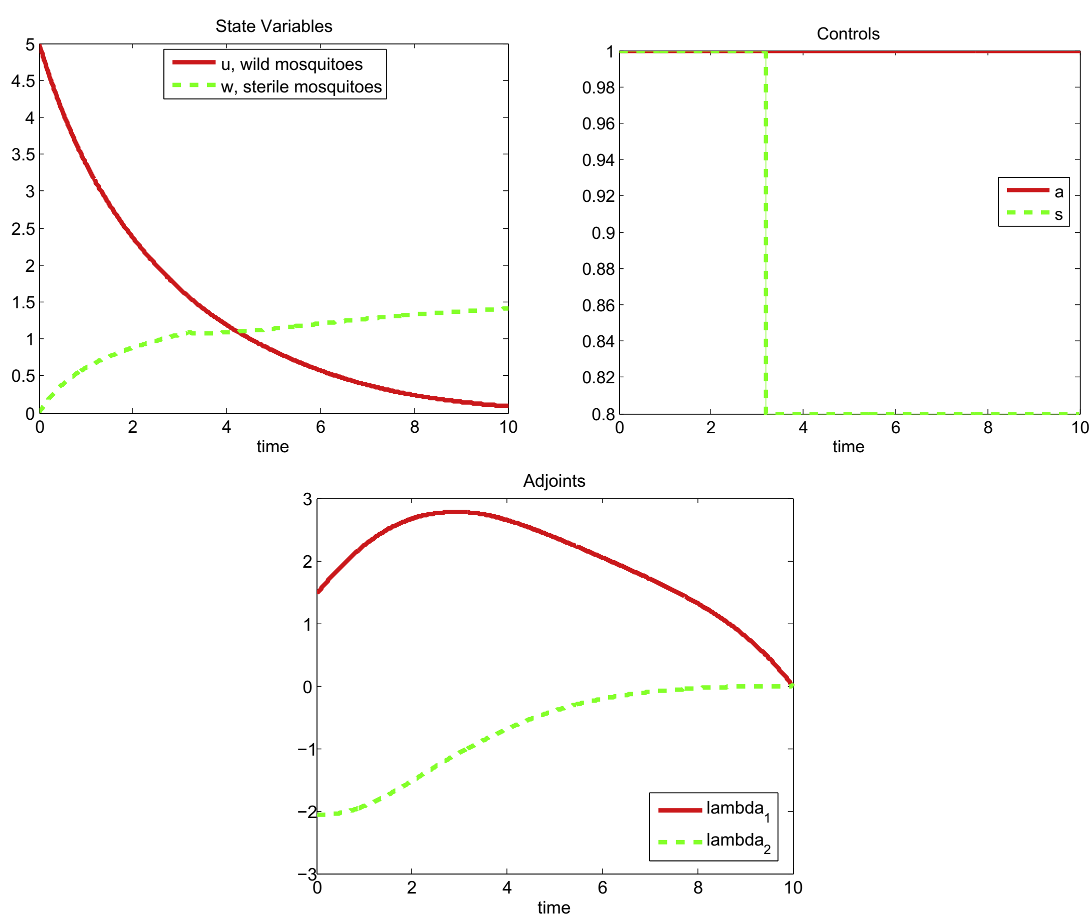

Fig. 10. Objective functional J_1 with $B = 1$, $a(t) = 1$, $0.8 \leqslant s(t) \leqslant 1$, initial conditions $u_0 = 5$, $w_0 = 0$ over the time interval $[0, 10]$.

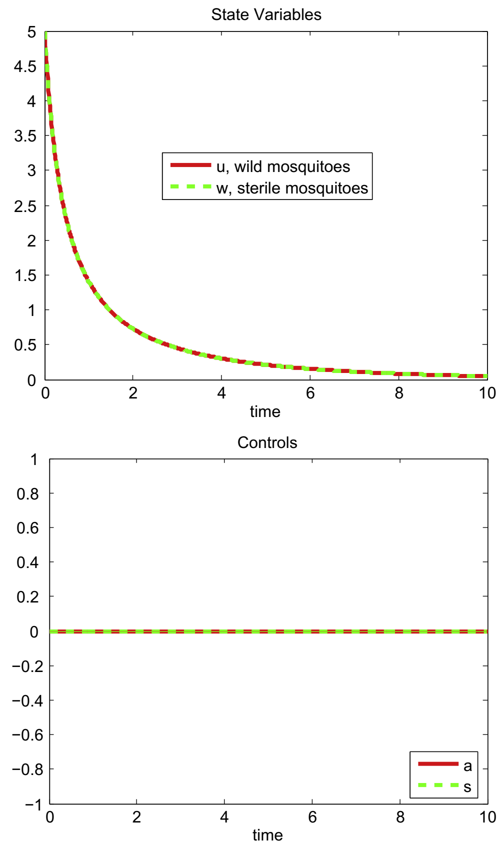

Fig. 11. Objective functional J_0 with $B = 1$, bounds $0 \leqslant a(t) \leqslant 1$, $0 \leqslant s(t) \leqslant 1$, initial conditions $u_0 = 5$, $w_0 = 5$ over the time interval $[0, 10]$.

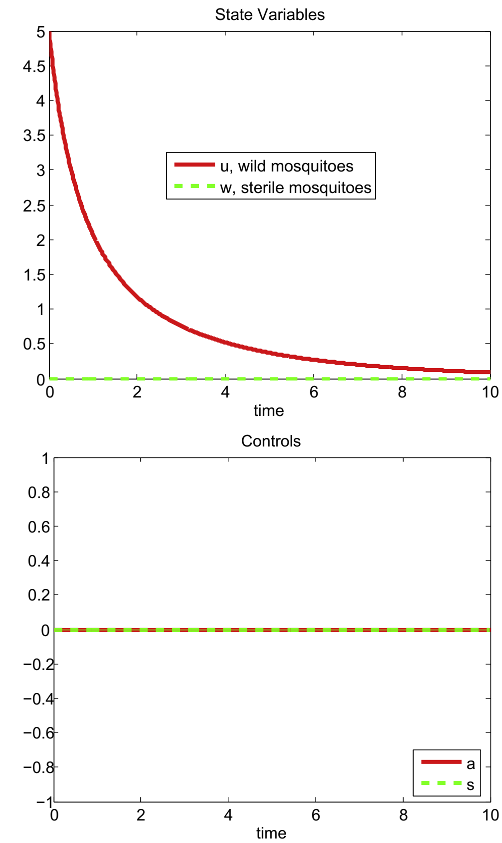

Fig. 12. Objective functional J_0 with $B = 1$, bounds $0 \leqslant a(t) \leqslant 1$, $0 \leqslant s(t) \leqslant 1$, initial conditions $u_0 = 5$, $w_0 = 0$ over the time interval $[0, 10]$.

the presence of sterile insect release, J_0 implies that the optimal approach is to use a short burst of mosquito release to eliminate the wild mosquitoes. For J_1, the optimal approach is to use constant sterile insect release at the minimal level allowed by the lower bound on s. In the absence of sterile insect release but in the presence of insecticide, J_0 implies that the optimal approach is to use maximal insecticide levels $a = 0$.

When both insecticide and sterile insect release is allowed, the optimal approach for both functionals is to force maximal insecticide use. We note that the algorithm was quite sensitive to the bounds on a and s. Our existence and uniqueness results imply that the solutions exist for small T, but convergence was an issue in some cases. For J_0 the sterile release must be longer, albeit at lower levels, than in the absence of insecticide. For J_1, the possibility of a singular control cannot be ignored. We note that if the second adjoint variable satisfies $1 + \lambda_2 = 0$, then the control s is singular. We found only one example where that was the case. In the absence of singular control, the optimal approach is maximize the use of insecticide while minimizing the sterile release.

The effect of the initial condition, w_0 for the sterile mosquitoes cannot be ignored. When $w_0 = 0$, a higher rate of sterile instect release was needed at the beginning of the control period, with an eventual drop to the lower bound.

Overall, our findings are that if insecticide is allowed, it should be used at a maximal level $a = 0$, and that a combination of the techniques leads to a more rapid elimination of the wild mosquito population. Other authors have found similar results using models that incorporated male–female interactions [8,40] or that linked their model to the epidemiology of a particular disease [1,6,7]. Dumont and Tcheuenche [7] found that the R_0 of a disease could be reduced through a combination of periodic sterile insect release and habitat modification. Their work did not incorporate optimal control and to do so in the context of a periodic sterile insect release would require the use of delay differential equations. This open question is one we intend to pursue.

Although some papers apply optimal control, to our knowledge, this has not been done in a bounded context with a more rigorous proof of existence and uniqueness of the optimal control. Furthermore, the Holling II function that we choose to model the reproductive rate of the wild mosquitoes is used only by [24]. This particular function captures the dynamics of both small and large mosquito populations and is crucial as we seek to reduce the wild mosquito population.

Acknowledgements

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Appendix A

Theorem 7.1 (*Filippov–Cesari theorem* [14,37]). *Consider the following optimal control problem:*

$$\min J = \int_0^T F(x(t), u(t), t)dt + S(x(T), T),$$
$$\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \quad g(x(t), u(t), t) \geqslant 0,$$
$$h(x(t), t) \geqslant 0, \quad a(x(T), T) \geqslant 0 \quad b(x(T), T) = 0$$

where T is free on $[0, t_f]$. *Assume that F, f, g, h, S, a, and b are continuous in all their arguments at all points* (x, u, t). *Define the (state-dependent) control region*

$$\Omega(x, t) = \{u \in \mathbb{R}^m |\, g(x, u, t) \geqslant 0\} \subset \mathbb{R}^m$$

and the set

$$\Phi(x, t) = \{(F(x, u, t) + \gamma, f(x, u, t))|\, \gamma \leqslant 0, u \in \Omega(x, t)\} \subset \mathbb{R}^{n+1}$$

where m and n are the number of control and state variables, respectively.

Suppose that the following conditions hold:

1. *There exists an admissible solution pair.*
2. $\Phi(x, t)$ *is convex for all* $(x, t) \in \mathbb{R}^n \times [0, t_f]$.
3. *There exists* $\delta > 0$ *such that* $\|x(t)\| < \delta$ *for all admissible* $\{x(t), u(t)\}$ *and t.*
4. *There exists* $\delta_1 > 0$ *such that* $\|u\| < \delta_1$ *for all* $u \in \Omega(x, t)$ *with* $\|x\| < \delta$.

Then there exists an optimal triple $\{T^*, x^*, u^*\}$ *with* $u^*(\cdot)$ *measurable.*

References

[1] L. Alphey, L. Alphey, M.B. Bonsall, A model framework to estimate impact and cost of genetics-based sterile insect methods for dengue vector control, PLOS One 6 (10) (2011).

[2] L. Alphey, M. Benedict, R. Bellini, G.G. Clark, D.A. Dame, M.W. Service, S.L. Dobson, Sterile-insect methods for control of mosquito-borne diseases: an analysis, Vector-Borne and Zoonotic Diseases 10 (2010) 295.

[3] H. Barclay, Mathematical models for the use of sterile insects, in: V. Dyck, J. Hendrichs, A.S. Robinson (Eds.), Sterile Instect Technique, Springer, 2005, pp. 147–174.

[4] A.C. Bartlett, Insect sterility, insect genetics, and insect control, in: D. Pimentel (Ed.), Handbook of Pest Management in Agriculture, vol. II, CRC Press, Boca Raton, FL, 2, 1990, pp. 279–287.

[5] C. Collins, K.R. Fister, B. Key, M. WIlliams, Blasting neuroblastoma using optimal control of chemotherapy, Math. Biol. Eng. 6 (3) (2009) 451.

[6] H. Diaz, A.A. Ramirez, A. Olarte, C. Clavijo, A model for the control of malaria using genetically modified vectors, J. Theor. Biol. 276 (2011) 57.

[7] Y. Dumont, J.M. Tchuenche, Mathematical studies on the sterile insect technique for the Chikungunya disease and Aedes albopictus, J. Math. Biol. 65 (2012) 809.

[8] L. Esteva, H.M. Yang, Mathematical model to assess the control of Aedes aegypti mosquitoes by the sterile insect technique, Math. Biosci. 198 (2005) 132.

[9] R.W. Fay, E.M. McCray, Jr., J.W. Kirkpatrick, Mosquito News 23 210–214.

[10] Wendell H. Fleming, Raymond W. Rishel, Deterministic and Stochastic Optimal Control, Springer Verlag, 1975.

[12] H. Gaff, D. Hartley, N. Leahy, An epidemiological model of rift valley fever electronic, J. Differen. Equat. (115) (2007) 1.

[13] W.K. Hackbusch, A numerical method for solving parabolic equations with opposite orientations, Computing 20 (1978) 229.

[14] R.F. Hartl, S.P. Sethi, R.G. Vickson, A survey of the maximum principles for optimal control problems with state constraints, SIAM Rev. 37 (2) (1995) 181.

[15] C.S. Holling, The components of predation as revealed by a study of small mammal predation of the European pine sawfly, Canad. Entomol. 91 (1959) 293.

[17] E.F. Knipling, Possibilities of insect control or erradication through the use of sexually sterile males, J. Econ. Entomol. 48 (1955) 459.

[18] E.F. Knipling, The basic principles of insect population suppression and management, Agriculture Handbook 512. US Dept. of Agriculture, Washington DC, 1979.

[19] E.F. Knipling, Sterile insect technique as a screwworm control measure: the concept and its development, in: O.H. Graham (Ed.), Symposium on Eradication of the Screwworm from the United States and Mexico, Misc. Publ. Entomol. Soc. America, vol. 62, 1985, pp. 1–4.

[20] Arthur J. Krener, The high order maximal principle and its application to singular extremals, SIAM J. Contr. Optim. 15 (2) (1977) 256.

[21] S. Lenhart, J.T. Workman, Optimal Control Applied to Biological Models, Chapman & Hall, CRC, 2007.

[22] M.A. Lewis, P. van den Driessche, Waves of extinction from sterile insect release, Math. Biosci. 116 (1993) 221.

[23] J. Li, Simple mathematical models for interacting wild and transgenic mosquito populations, Math. Biosci. 189 (2004) 39.

[24] J. Li, Differential equations models for interacting wild and transgenic mosquito populations, J. Biol. Dyn. 2 (3) (2008) 241.

[25] D.L. Lukes, Differential equations: Classical to controlled, Volume 162, Mathematics in Science and Engineering. Academic Press, 1982.

[26] E. Michael, M. Malecela-Lazaro, B. Maegga, P. Fischer, J. Kazura, Mathematical Models and lymphatic filariasis control: monitoring and evaluating interventions, TRENDS Parasitol. 22 (11) (2006) 529.

[27] R.K. Miller, A.N. Michel, Ordinary Differential Equations, Academic Press, New York, 1982.

[28] H.B. Morlan, E.M. McCray, Jr., and J.W. Kilpatrick, Field tests for with sexually sterile males for cntrol of Aedes aegypti, Mosq. News 22 (3) 295–300.

[29] E. Newton, P. Reiter, A model of the transmission of dengue fever with an evaluation of the impact of ultra-low volume (ULV) insecticide applications on dengue epidemics, Am. J. Trop. Hygiene Med. 47 (6) (1992) 709.

[30] M. Oteroa, H. Solaria, N. Schweigmann, A Stochastic Population Dynamics Model for Aedes Aegypti: Formulation and Application to a City with Temperate Climate, Bulletin of Mathematical Biology 2006 (68) (2006) 1945-1974.

[31] R.E. Plant, M. Mangel, Modeling and simulation in agricultural pest management, SIAM Rev. 29 (2) (1987) 235.

[32] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The Mathematical Theory of Optimal Processes, Gordon and Breach, 1962.

[33] M.H. Potter and H.F. Weinberger. Maximum Principles in Differential Equations. Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

[34] M. Predescu, G. Sirbu, R. Levins, T. Awerbuch-Friedlander, On the dynamics of a deterministic and stochastic model for mosquito control, Appl. Math. Lett. 20 (2007) 919.

[35] M. Rafikov, L. Bevilacqua, A.P.P. Wyse, Optimal control strategy of malaria vector using genetically modified mosquitoes, J. Theoret. Biol. 258 (2009) 418.

[36] M. Rafikov, A.P.P. Wyse, L. Bevilacqua, Controlling the interaction between wild and transgenic mosquitoes, J. Nonlinear Syst. Appl. (2010) 27.

[37] A. Seierstad, K. Sydsaeter, Optimal Control Theory with Economic Applications, North-Holland, Elseveir, Amsterdam, 1987.

[38] C.N. Smith, Possible use of the sterile-male technique for control of Aedes Aegypti, Bull. Wld. Hlth. Org. 36 (1967) 633.

[39] D. Smith, J. Dushoff, F.E. McKenzie, The risk of a mosquito-borne infection in a heterogeneous environment, PLOS Biol. 2 (11) (2007) 1957.

[40] R.C.A. Thome, H.M. Yang, L. Esteva, Optimal control of Aedes aegypti mosquitoes by the sterile insect technique and insecticide, Math. Biosci. 223 (2010) 12.

[41] A. Wyse, A., L. Bevilacqua, M. Rafikov, Simulating malaria model for different treatment intensities in a variable environment, Ecol. Modell. 206 (2007) 322.

[42] L. Yakob, L. Alphey, M. Bonsall, Aedes aegypti control: the concomitant role of competition space and transgenic technologies, J. Appl. Ecol. 45 (2008) 1258.