31.8 Auxiliary Equations with Repeated or Complex Roots

If m is a repeated root of an auxiliary Equation of the differential Equation $a_0 D^2 y + a_1 D y + a_2 y = 0$ then $y = e^{mx}(c_1 + c_2 x)$ is a solution of the differential Equation.

p.966

(6) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

$D^2y - 6Dy + 9y = 0$

$(D^2 - 6D + 9)y = 0$

aux. eq'n is $m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m = 3$ or $m = 3$

Repeated Roots

General Solution is

$y = e^{3x}(c_1 + c_2 x)$

$= c_1 e^{3x} + c_2 x e^{3x}$

(14) $4D^2y = 12Dy - 9y$

$4D^2y - 12Dy + 9y = 0$

$(4D^2 - 12D + 9)y = 0$

Aux. Eq'n is $4m^2 - 12m + 9 = 0$

$(2m-3)(2m-3) = 0$

or $(2m-3)^2 = 0$

Repeated of $m = \frac{3}{2}$

General Solution is

$y = e^{\frac{3}{2}x}(c_1 + c_2 x)$

$= c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}$

Aux Eq'n $(m+2)^3 = 0$

$m = -2, -2, -2$

General Solution of D.E is

$y = e^{-2x}(c_1 + c_2 x + c_3 x^2)$

or $c_1 e^{-2x} + c_2 x e^{-2x} + c_3 x^2 e^{-2x}$

If a differential equation $a_0 D^2y + a_1 Dy + a_2 y = 0$ has an auxiliary equation with complex roots $\alpha \pm j\beta$ then the solution of the differential equation is

$$y = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

(2) $\frac{d^2y}{dx^2} + 25y = 0$

$D^2y + 25y = 0$

$(D^2 + 25)y = 0$

Aux. Eq'n is $m^2 + 25 = 0$

$m^2 = -25$

$m = \pm 5j$

$\alpha = 0$ & $\beta = 5$

General Solution is

$y = e^{0x}(c_1 \sin 5x + c_2 \cos 5x)$

$= c_1 \sin 5x + c_2 \cos 5x$

(12) $y'' + 4y = 2y'$

$D^2y + 4y = 2Dy$

$D^2y - 2Dy + 4y = 0$

$(D^2 - 2D + 4)y = 0$

Aux. Eq'n is $m^2 - 2m + 4 = 0$

$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$

$= \frac{2 \pm \sqrt{-12}}{2}$

$= \frac{2 \pm 2j\sqrt{3}}{2} = 1 \pm j\sqrt{3}$

$\alpha = 1 \quad \beta = \sqrt{3}$

General Solution is

$y = e^{x}(c_1 \sin \sqrt{3}\, x + c_2 \cos \sqrt{3}\, x)$

(36) $y'''' + 3y''' + 2y'' = 0$ $\quad y=0,\ y'=4,\ y''=-8,\ y'''=16$ when $x=0$

$D^4y + 3D^3y + 2D^2y = 0$ $\quad y=0,\ Dy=4,\ D^2y=-8,\ D^3y=16$ when $x=0$

$(D^4+3D^3+2D^2)y = 0$

Aux. Eq'n is $m^4+3m^3+2m^2=0$

$m^2(m^2+3m+2)=0$

$m^2(m+1)(m+2)=0$

roots $m=0,0,-1,$ or -2.

General Solution is

$y = c_1e^{0x}+c_2xe^{0x}+c_3e^{-x}+c_4e^{-2x}$

$y = c_1+c_2x+c_3e^{-x}+c_4e^{-2x}$

$y' = c_2-c_3e^{-x}-2c_4e^{-2x}$

$y'' = c_3e^{-x}+4c_4e^{-2x}$

$y''' = -c_3e^{-x}-8c_4e^{-2x}$

$x=0$ & $y=0 \Rightarrow 0 = c_1+0+c_3+c_4$

$x=0$ & $y'=4 \Rightarrow 4 = c_2-c_3-2c_4$

$x=0$ & $y''=-8 \Rightarrow -8 = c_3+4c_4$

$x=0$ & $y'''=16 \Rightarrow 16 = -c_3-8c_4$

$8=-4c_4$

$c_4=-2$

$-8 = c_3+4(-2)$

$-8 = c_3+-8 \Rightarrow c_3=0$

$4 = c_2-0-2(-2)$

$4 = c_2+4 \Rightarrow c_2=0$

$0 = c_1+0+(-2) \Rightarrow c_1=2$

Particular Solution is

$y = 2-2e^{-2x}$

(34) $9y''+16y=0$ $\quad y'=0$ & $y=2$ when $x=\frac{\pi}{2}$

$9D^2y+16y=0$ $\quad Dy=0$ & $y=2$ when $x=\frac{\pi}{2}$

$(9D^2+16)y=0$

Aux. Eq'n is $9m^2+16=0$

$m=\pm\frac{4}{3}j$

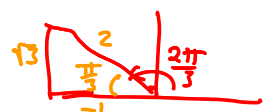
$\alpha=0$ & $\beta=\frac{4}{3}$

General Solution is

$y=e^{0x}(c_1\sin\frac{4}{3}x+c_2\cos\frac{4}{3}x)$

$= c_1\sin\frac{4}{3}x+c_2\cos\frac{4}{3}x$

$y' = \frac{4}{3}c_1\cos\frac{4}{3}x-\frac{4}{3}c_2\sin\frac{4}{3}x$



$y=2$ & $x=\frac{\pi}{2} \Rightarrow 2 = c_1\sin\frac{2\pi}{3}+c_2\cos\frac{2\pi}{3}$

$2 = c_1\frac{\sqrt{3}}{2}-c_2\frac{1}{2}$

$\Rightarrow c_2=\sqrt{3}c_1-4$

$y'=0$ & $x=\frac{\pi}{2} \Rightarrow 0=\frac{4}{3}c_1(-\frac{1}{2})-\frac{4}{3}c_2\frac{\sqrt{3}}{2}$

$0=\frac{4}{3}c_1(-\frac{1}{2})-\frac{4}{3}(\sqrt{3}c_1-4)\frac{\sqrt{3}}{2}$

$0=-\frac{2}{3}c_1-2c_1+\frac{8\sqrt{3}}{3}$

$\frac{8}{3}c_1=\frac{8\sqrt{3}}{3} \Rightarrow c_1=\sqrt{3}$

$c_2=\sqrt{3}(\sqrt{3})-4=3-4=-1$

Particular Solution is

$y=\sqrt{3}\sin\frac{4}{3}x-\cos\frac{4}{3}x$