31.7 Higher Order Differential Equations

General Linear Differential Equation of the nth order

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = b$$

where a's and b are functions of x or constants.

If $b = 0$ then equation is homogeneous.

If $b \neq 0$ it is called nonhomogeneous.

For convenience we can denote the derivatives using operator notation, D. (nth derivative $= D^n$)

$$a_0 D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = b.$$

Second-order Homogeneous Equations with constant coefficients:

$$a_0 \underbrace{D^2 y}_{\frac{d^2y}{dx^2}} + a_1 \underbrace{Dy}_{\frac{dy}{dx}} + a_2 y = 0$$

write auxiliary equation for Diff. Equation.

$$\boxed{a_0 m^2 + a_1 m + a_2 = 0}$$ Quadratic Equation

$m^2 = D^2 y \qquad m = Dy$

If there are two distinct roots $m_1 \,\&\, m_2$ for auxiliary equation then the solution for the differential Equation is

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

$c_1 \,\&\, c_2$ are constants, $m_1 \,\&\, m_2$ are roots of auxiliary eq'n.

p.962

6 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$

$D^2y - 2Dy - 8y = 0$

or

$(D^2 - 2D - 8)y = 0$

Auxiliary Equation is $m^2 - 2m - 8 = 0$

$(m-4)(m+2) = 0$

$m = 4$ or -2

General Solution is

$$\boxed{y = C_1 e^{-2x} + C_2 e^{4x}}$$

4 $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

$D^2y + Dy = 0 \Rightarrow (D^2 + D)y = 0$

Auxiliary Equation is $m^2 + m = 0$

$m(m+1) = 0$

$m = 0$ or $m = -1$

General Solution is

$y = C_1 e^{0x} + C_2 e^{-1x}$

$$\boxed{y = C_1 + C_2 e^{-x}}$$

(12) $4D^2y + 12Dy = 7y$

$4D^2y + 12Dy - 7y = 0 \Rightarrow (4D^2 + 12D - 7)y = 0$

Auxiliary Equation is $4m^2 + 12m - 7 = 0$

$(2m-1)(2m+7) = 0$

$m = \frac{1}{2}$ or $-\frac{7}{2}$

General Solution is

$$y = c_1 e^{-\frac{7}{2}x} + c_2 e^{\frac{1}{2}x}$$

$2y''' + 3y'' - 2y' = 0$

$2D^3y + 3D^2y - 2Dy = 0 \Rightarrow (2D^3 + 3D^2 - 2D)y = 0$

Auxiliary Eq'n is $2m^3 + 3m^2 - 2m = 0$

$m(2m^2 + 3m - 2) = 0$

$m(2m-1)(m+2) = 0$

$m = 0, \frac{1}{2}, -2$

General Solution is $y = c_1 + c_2 e^{\frac{1}{2}x} + c_3 e^{-2x}$

(28) $4y'' - y' = 0 \qquad y' = 2$

$4D^2y - Dy = 0 \qquad Dy = 2$ and $y = 4$ when $x = 0$

Aux. Eq'n is $4m^2 - m = 0$

$m(4m-1) = 0$

$m = 0$ or $m = \frac{1}{4}$

General Solution is

$y = C_1 + C_2 e^{\frac{1}{4}x}$

$y' = \frac{1}{4} C_2 e^{\frac{1}{4}x}$

Find C_1 & C_2

$y' = 2$ when $x = 0$

$2 = \frac{1}{4} C_2 e^{\frac{1}{4}(0)} \Rightarrow 2 = \frac{1}{4} C_2$

$C_2 = 8$

$y = 4$ when $x = 0$

$4 = C_1 + 8e^{\frac{1}{4}(0)} \Rightarrow 4 = C_1 + 8$

$C_1 = -4$

Particular Solution is $y = -4 + 8e^{\frac{1}{4}x}$

(30) $2y'' + 5y' = 0$

$2D^2y + 5Dy = 0 \qquad y = 0$ when $x = 0 \qquad y = 2$ when $x = 1$

Aux. Eq'n is $2m^2 + 5m = 0 \Rightarrow m(2m+5) = 0$

$m = 0$ or $-\frac{5}{2}$

General Solution is $y = C_1 + C_2 e^{-\frac{5}{2}x}$

Find C_1 & C_2

$y = 0$ when $x = 0$: $\quad 0 = C_1 + C_2 e^{-\frac{5}{2}(0)} \Rightarrow 0 = C_1 + C_2$

$C_1 = -C_2$

$y = 2$ when $x = 1$: $\quad 2 = C_1 + C_2 e^{-\frac{5}{2}(1)} \Rightarrow 2 = C_1 + C_2 e^{-\frac{5}{2}}$

$\Rightarrow 2 = -C_2 + C_2 e^{-\frac{5}{2}} \Rightarrow 2 = C_2(-1 + e^{-\frac{5}{2}}) \Rightarrow C_2 = \frac{2}{-1 + e^{-\frac{5}{2}}}$

$C_1 = \frac{2}{1 - e^{-5/2}}$ & $C_2 = \frac{2}{e^{-5/2} - 1}$

Particular solution is $\boxed{y = \frac{2}{1 - e^{-5/2}} + \frac{2}{e^{-5/2} - 1} e^{-5/2 x}}$