31.4 The Linear Differential Equation of the First Order

$$dy + Py\,dx = Q\,dx$$

where P & Q are functions of x only.

Multiply both sides by $e^{\int P\,dx}$ ← Integrating Factor I.F.

$$e^{\int P dx}\,\underbrace{dy}\; + \underbrace{e^{\int P dx} P y\, dx}_{d(e^{\int P dx})} = e^{\int P dx} Q\,dx$$

$$d\left(e^{\int P dx} y\right) = e^{\int P dx} Q\,dx$$

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p. 950 (4) $dy + Py\,dx = Q\,dx$ I.F.: $e^{\int P dx} = e^{\int 3 dx} = e^{3x}$

$$dy + \underbrace{3}_{P}y\,dx = \underbrace{e^{-3x}}_{Q}dx$$

Multiply both sides by I.F. of e^{3x}

$$e^{3x}(dy + 3y\,dx) = (e^{-3x}dx)e^{3x}$$

$$e^{3x}dy + e^{3x}3y\,dx = e^{-3x}e^{3x}dx \qquad e^{-3x}e^{3x} = e^{-3x+3x} = e^0 = 1$$

$$\int d(e^{3x}y) = \int 1\,dx$$

$$e^{-3x}(e^{3x}y = x + C)$$

$$y = xe^{-3x} + Ce^{-3x} \quad \text{or} \quad y = e^{-3x}(x + C)$$

(10) $\dfrac{x\,dy}{x} + \dfrac{3y\,dx}{x} = \dfrac{dx}{x}$

I.F. $e^{\int P dx} = e^{\int \frac{3}{x} dx} = e^{3\int \frac{1}{x} dx}$

$= e^{3 \ln x} = e^{\ln x^3} = x^3$

$x^3\left(dy + \underbrace{\frac{3}{x}}_{P} y\,dx\right) = \left(\underbrace{\frac{1}{x}}_{Q} dx\right) x^3$

Multiply both sides by I.F. of x^3

$x^3 dy + x^3 \frac{3}{x} y\,dx = x^3 \frac{1}{x} dx$

$\underbrace{x^3 dy + 3x^2 y\,dx} = x^2 dx$

$\int d(x^3 y) = \int x^2 dx$

$x^3 y = \frac{1}{3}x^3 + C$

$y = \frac{1}{3}\frac{\cancel{x^3}}{\cancel{x^3}} + \frac{C}{x^3}$

$y = \frac{1}{3} + \frac{C}{x^3}$

(22) $y' + y \tan x = -\sin x$

$\frac{dy}{dx} + y \tan x = -\sin x$

I.F. $e^{\int \tan x\, dx} = e^{-\ln \cos x}$

$= e^{\ln(\cos x)^{-1}} = \frac{1}{\cos x} = \sec x$

$\sec x\left(dy + y \underbrace{\tan x}_{P} dx\right) = \left(\underbrace{-\sin x}_{Q} dx\right)\frac{1}{\cos x}$

$\underbrace{\sec x\, dy + y \sec x \tan x\, dx} = -\tan x\, dx$

$\int d(y \sec x) = \int -\tan x\, dx$

$\cos x\,(y \sec x) = \left(-(-\ln \cos x) + C\right)\cos x$

$y = \cos x \ln \cos x + C \cdot \cos x$

(32) $dq - 4q\,du = 2\,du$ $\quad q=2$ when $u=0$

$\underbrace{}_{P}$ $\quad \underbrace{2}_{Q}$

I.F. $e^{\int -4\,du} = e^{-4u}$

$$e^{-4u}(dq - 4q\,du) = (2\,du)e^{-4u}$$

$$\underbrace{e^{-4u}dq - 4e^{-4u}q\,du} = 2e^{-4u}du$$

$w = -4u \Rightarrow dw = -4\,du$

$$\int d(qe^{-4u}) = \left(-\tfrac{1}{4}\right)2\int e^{-4u}(-4)du$$

$$\frac{e^{-4u}q}{e^{-4u}} = \frac{-\frac{1}{2}e^{-4u} + C}{e^{-4u}} \Rightarrow$$

General Solution

$$q = -\frac{1}{2} + Ce^{4u}$$

Find C using $q=2$ when $u=0$

$$2 = -\frac{1}{2} + Ce^{4(0)}$$

$$2 = -\frac{1}{2} + C(1)$$

$$C = 2 + \frac{1}{2} = \frac{5}{2} \text{ or } 2.5$$

Particular solution is $q = -\frac{1}{2} + \frac{5}{2}e^{4u}$