31.2 Separation of Variables

A differential equation of the first order and first degree contains the first derivative to the first power. It is more commonly expressed in its differential form,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where $M(x,y)$ and $N(x,y)$ may represent constants, functions of either x ~~and~~ or y, or functions of both x and y. We will be looking at problems that can be rewritten so that the $M(x,y)$ only has x and the $N(x,y)$ only has y:

$$A(x)\,dx + B(y)\,dy = 0 \text{ or } A(x)\,dx = B(y)\,dy$$

This can be solved by integrating each term and adding the constant of integration.

p.945

(4) $y^2\,dy + x^3\,dx = 0$

$$\int y^2\,dy = -\int x^3\,dx$$

$$\left(\frac{1}{3}y^3 = -\frac{1}{4}x^4 + C\right)12$$

$$\boxed{4y^3 = -3x^4 + C}$$

$$y^3 = -\frac{3}{4}x^4 + C$$

$$y = \sqrt[3]{-\frac{3}{4}x^4 + C}$$

(8) $\left(2\overset{y'}{\frac{dy}{dx}}\right)dx = \left(\frac{y(x+1)}{x}\right)dx$

$$\frac{2\,dy}{y} = \frac{y\frac{(x+1)}{x}\,dx}{y}$$

$$\frac{2}{y}\,dy = \frac{x+1}{x}\,dx$$

$$2\int \frac{1}{y}\,dy = \int \frac{\cancel{x}^{1}}{\cancel{x}_{1}} + \frac{1}{x}\,dx$$

$$2\ln|y| = x + \ln|x| + C$$

Assume x & y are positive

$$\boxed{2\ln y = x + \ln x + C}$$

(10) $xyy' + \sqrt{1+y^2} = 0$

$$dx\left(xy\frac{dy}{dx} + \sqrt{1+y^2}\right) = (0)\,dx$$

$$xy\,dy + \sqrt{1+y^2}\,dx = 0$$

$$\frac{\cancel{x}y\,dy}{\cancel{x}(1+y^2)^{1/2}} = \frac{-(1+y^2)^{1/2}\,dx}{x\,(1+y^2)^{1/2}}$$

$$\left(\frac{1}{2}\right)\int (1+y^2)^{-1/2}(2)y\,dy = -\int \frac{1}{x}\,dx$$

$u = 1+y^2$

$du = 2y\,dy$

$$\frac{1}{2}\int u^{-1/2}\,du = -\ln x + c$$

$$\frac{1}{\cancel{2}}\frac{u^{1/2}}{\cancel{1/2}} = -\ln x + c$$

$$\boxed{\sqrt{1+y^2} = -\ln x + c}$$

or

$$\ln x + \sqrt{1+y^2} = c$$

(36) $\dfrac{ds}{dt} = \sec s$ $\quad t = 0$ when $s = 0$

$$\frac{ds}{\sec s} = \frac{\sec s\,dt}{\sec s}$$

$\frac{1}{\sec s} = \cos s$

$$\int \cos s\,ds = \int dt$$

$$\sin s = t + C$$

Find C using $t=0$ & $s=0$

$\sin(0) = 0 + C$

$0 = 0 + C$

$c = 0$

Particular Solution is

$\sin s = t$

(22) $\dfrac{\sin x \sec y\, dx}{\sec y} = \dfrac{dy}{\sec y}$ $\sin x \sec y = y'$

$\sin x\, dx = \cos y\, dy$ $\dfrac{1}{\sec y} = \cos y$

$\int \sin x\, dx = \int \cos y\, dy$

$$\boxed{-\cos x = \sin y + C}$$

(38) $\dfrac{x\, dy}{x \cdot y \ln y} = \dfrac{y \ln y\, dx}{x \cdot y \ln y}$ $x = 2$ when $y = e$

$\int \dfrac{1}{\ln y}\dfrac{1}{y}\, dy = \int \dfrac{1}{x}\, dx$

$u = \ln y \quad du = \frac{1}{y}\, dy$

$\int \frac{1}{u}\, du = \int \frac{1}{x}\, dx$

$\ln u = \ln x + C$

$\ln(\ln y) = \ln x + C$

Find C using $x = 2$ when $y = e$

$\ln(\ln e) = \ln 2 + C$

$\ln 1 = \ln 2 + C$

$0 = \ln 2 + C$

$C = -\ln 2$

particular solution is

$\ln(\ln y) = \ln x - \ln 2.$