29.4 Double Integrals

If $z = f(x,y)$ the Double Integral of function over x and y is

$$\int_a^b \left[\int_{g(x)}^{G(x)} f(x,y)\,dy\right]dx \quad \text{or} \quad \int_a^b \int_{g(x)}^{G(x)} f(x,y)\,dy\,dx$$

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(3) $$\int_2^4 \boxed{\int_0^1 xy^2\,dx}\,dy = \int_2^4 y^2 \int_0^1 x\,dx\,dy$$

$$= \int_2^4 y^2 \left[\frac{x^2}{2}\right]_0^1 dy = \int_2^4 y^2\left(\frac{1}{2} - 0\right)dy = \int_2^4 \frac{1}{2}y^2\,dy$$

$$= \frac{1}{2}\,\frac{y^3}{3}\Big|_2^4 = \frac{1}{6}4^3 - \frac{1}{6}2^3 = \frac{64}{6} - \frac{8}{6} = \frac{56}{6} = \frac{28}{3}$$

(6) $$\int_0^4 \underbrace{\int_1^{\sqrt{y}} x - y\,dx}\,dy = \int_0^4 \frac{x^2}{2} - yx\Big|_{x=1}^{x=\sqrt{y}}\,dy$$

$$= \int_0^4 \left(\frac{y}{2} - y\sqrt{y}\right) - \left(\frac{1}{2} - y\right)dy$$

$y\sqrt{y} = y \cdot y^{1/2} = y^{3/2}$

$$= \int_0^4 \frac{3y}{2} - y^{3/2} - \frac{1}{2}\,dy$$

$$= \frac{3y^2}{4} - \frac{y^{5/2}}{5/2} - \frac{1}{2}y\Big|_0^4$$

$$= \left(\frac{3}{4}(4)^2 - \frac{2}{5}4^{5/2} - \frac{1}{2}4\right) - (0)$$

$$= 12 - \frac{64}{5} - 2 = -\frac{14}{5}$$

⑧ $\displaystyle \int_4^9 \underbrace{\int_0^x \sqrt{x-y}\;dy}\,dx = \int_4^9 -\int_0^x \underbrace{(x-y)^{1/2}}_{u^{1/2}}\;\underbrace{-dy}_{du}\;dx$

$u = x-y$

$\dfrac{du}{dy} = -1$

$du = -dy$

$$= -\int_4^9 \frac{u^{3/2}}{3/2}\Bigg|_{y=0}^{y=x} dx$$

$$= -\int_4^9 \frac{2}{3}(x-y)^{3/2}\Bigg|_{y=0}^{y=x} dx$$

$$= -\frac{2}{3}\int_4^9 0 - x^{3/2}\,dx = \frac{2}{3}\int_4^9 x^{3/2}\,dx$$

$$= \frac{2}{3}\,\frac{2}{5}\,x^{5/2}\Bigg|_4^9 = \frac{4}{15}\left(9^{5/2} - 4^{5/2}\right)$$

$$= \frac{4}{15}(243-32) = \frac{844}{15}$$

⑫ $\displaystyle \int_{-1}^1 \underbrace{\int_1^{e^x} \frac{1}{xy}\,dy}\,dx = \int_{-1}^1 \frac{1}{x}\int_1^{e^x} \frac{1}{y}\,dy\;dx$

$$= \int_{-1}^1 \frac{1}{x}\;\ln y\,\Bigg|_{y=1}^{y=e^x} dx$$

$$= \int_{-1}^1 \frac{1}{x}\left[\underbrace{\ln e^x}_{=x} - \underbrace{\ln 1}_{=0}\right] dx$$

$$= \int_{-1}^1 \frac{1}{\cancel{x}}(\cancel{x}-0)\,dx = \int_{-1}^1 dx$$

$$= x\,\Big|_{-1}^{1}$$

$$= 1-(-1) = 2$$