28.5 Other Trigonometric Forms

① $\int \cos^n x\, dx$ or $\int \sin^n x\, dx$ where n is odd. $n > 1$

Rewrite as $\int \cos^{n-1} x \cos x\, dx$ or $\int \sin^{n-1} x \sin x\, dx$

Then use $\cos^2 x + \sin^2 x = 1$ to change part with even exponent to the other function. Then can be solved using techniques from 28.1

Example 2, p. 854 $\int \cos^5 2x\, dx$

② $\int \cos^n x\, dx$ or $\int \sin^n x\, dx$ where n is even.

Substitute in using Half-Angle Formulas

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2}(1-\cos 2x) \quad \text{or} \quad \cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2}(1+\cos 2x)$$

Ex 3, p. 855 $\int \sin^2 2x\, dx$

$$\int \cos^3 x\, dx = \int \underbrace{\cos^2 x}_{\substack{\cos^2 x + \sin^2 x = 1 \\ \cos^2 x = 1 - \sin^2 x}} \cos x\, dx$$

$u = \sin x$

$du = \cos x\, dx$

$$= \int (1 - \sin^2 x) \cos x\, dx$$

$\int \cos x - \sin^2 x \cos x\, dx$

$\int \cos x\, dx - \int \sin^2 x \cos x\, dx$

$$= \int 1 - u^2\, du$$

$$= u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

$$\int \cos^4 x\,dx = \int \underbrace{\cos^2 x}_{\frac{1}{2}(1+\cos 2x)}\ \underbrace{\cos^2 x}_{\frac{1}{2}(1+\cos 2x)}\,dx \quad \text{or} \int (\cos^2 x)^2\,dx$$

$$= \int \tfrac{1}{2}(1+\cos 2x)\tfrac{1}{2}(1+\cos 2x)$$

$$= \tfrac{1}{4}\int 1 + 2\cos 2x + \underbrace{\cos^2 2x}_{\frac{1}{2}(1+\cos 4x)}\,dx$$

$$= \tfrac{1}{4}\int 1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)\,dx$$

$$= \tfrac{1}{4}\int \tfrac{3}{2} + 2\cos 2x + \tfrac{1}{2}\cos 4x\,dx$$

u=2x
du=2 dx

u= 4x
du= 4 dx

$$= \tfrac{1}{4}\left[\int \tfrac{3}{2}\,dx + \int 2\cos 2x\,dx + \tfrac{1}{4}\int \tfrac{1}{2}\cos 4x\,4\,dx\right]$$

$$= \tfrac{1}{4}\left[\tfrac{3}{2}x + \sin 2x + \tfrac{1}{8}\sin 4x\right] + C$$

$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$

③ $\int \sin^m x \cos^n x\,dx$

a) If m and n are even use half angle formulas

$$\int \cos^2 x \sin^2 x\,dx = \int \underbrace{\tfrac{1}{2}(1+\cos 2x)}_{\cos^2 x}\underbrace{\tfrac{1}{2}(1-\cos 2x)}_{\sin^2 x}\,dx$$

$$= \tfrac{1}{4}\int 1 - \underbrace{\cos^2 2x}_{\frac{1}{2}(1+\cos 4x)}\,dx$$

$$= \tfrac{1}{4}\int 1 - \tfrac{1}{2}(1+\cos 4x)\,dx$$

$$= \tfrac{1}{4}\int \tfrac{1}{2} - \tfrac{1}{2}\cos 4x\,dx$$

u= 4x
du= 4dx

$$= \tfrac{1}{4}\left[\int \tfrac{1}{2}\,dx - \tfrac{1}{2}\cdot\tfrac{1}{4}\int \cos 4x\,4\,dx\right]$$

$$= \tfrac{1}{4}\left[\tfrac{1}{2}x - \tfrac{1}{8}\sin 4x\right] + C$$

$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$$

b) If m is odd then rewrite integral as

$$\int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x \cos^n x \, \sin x \, dx$$

Then express $\sin^{m-1} x$ in terms of $\cos x$ using $\sin^2 x = 1 - \cos^2 x$. Then let $u = \cos x$.

Example 1, p.854

c) If n is odd then rewrite integral as

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x \cos^{n-1} x \, \cos x \, dx$$

Then express $\cos^{n-1} x$ in terms of $\sin x$ using $\cos^2 x = 1 - \sin^2 x$. Then let $u = \sin x$.

$$\int \cos^3 x \sin^4 x \, dx = \int \underbrace{\cos^2 x}_{} \sin^4 x \cos x \, dx$$

$$= \int (1 - \sin^2 x) \sin^4 x \cos x \, dx$$

$u = \sin x$
$du = \cos x \, dx$

$$= \int (\sin^4 x - \sin^6 x) \cos x \, dx$$

$$= \int u^4 - u^6 \, du$$

$$= \frac{1}{5} u^5 - \frac{1}{7} u^7 + C$$

$$= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$$

④ If $\int \tan^m x\,dx$ then rewrite integral as

$$\int \tan^m x\,dx = \int \tan^{m-2} x \tan^2 x\,dx$$

and substitute $\tan^2 x = \sec^2 x - 1$. This will have to be done again if any integrals of $\int \tan^m x\,dx$ with $m \neq 1$ are left.

Example $\int \tan^5 x dx$ (Ex 5, p.855)

⑤ If $\int \sec^n x\,dx$ with n even then rewrite integral as

$$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$$

and substitute $\sec^2 x = 1 + \tan^2 x$.

Note: If n is odd another method will need to be used.

⑥ $\int \tan^m x \sec^n x\,dx$

a) If n is even, then rewrite integral as

$$\int \tan^m x \sec^{n-2} x \sec^2 x\,dx$$

and express $\sec^{n-2} x$ in terms of $\tan x$ using $\sec^2 x = 1 + \tan^2 x$. Then let $u = \tan x$

b) If m is odd, then rewrite integral as

$$\int \tan^{m-1} x \sec^{n-1} x \sec x \tan x\,dx$$

and express $\tan^{m-1} x$ in terms of $\sec x$ using $\tan^2 x = \sec^2 x - 1$. Then let $u = \sec x$.

(Example 4, p.855 $\int \sec^3 t \tan t\,dt$)

c) If n is odd <u>and</u> m is even another method should be used.

$$\int \tan^2 3x \sec^4 3x\,dx = \int \tan^2 3x \underbrace{\sec^2 3x} \sec^2 3x\,dx$$

$$= \int \tan^2 3x \left(1+\tan^2 3x\right) \sec^2 3x\,dx$$

$u = \tan 3x$

$du = 3\sec^2 3x\,dx$

$$= \frac{1}{3}\int \left(\tan^2 3x + \tan^4 3x\right) 3\sec^2 3x\,dx$$

$$= \frac{1}{3}\int u^2 + u^4\,du$$

$$= \frac{1}{3}\left[\frac{u^3}{3} + \frac{u^5}{5}\right] + C$$

$$= \frac{1}{9}\tan^3 3x + \frac{1}{15}\tan^5 3x + C$$

$$\int \tan^3 x \sec^5 x\,dx = \int \underbrace{\tan^2 x} \sec^4 x \sec x \tan x\,dx$$

$$= \int \left(\sec^2 x - 1\right) \sec^4 x \sec x \tan x\,dx$$

$u = \sec x$

$du = \sec x \tan x\,dx$

$$= \int \left(\sec^6 x - \sec^4 x\right) \sec x \tan x\,dx$$

$$= \int u^6 - u^4\,du$$

$$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$$

$\sec^4 2x = (\sec^2 2x)^2$

$$\int \sec^6 2x\, dx = \int \sec^4 2x \ \sec^2 2x\, dx$$

$$= \int (1+\tan^2 2x)^2 \sec^2 2x\, dx$$

$u = \tan 2x$

$du = 2\sec^2 2x\, dx$

$$= \frac{1}{2}\int (1 + 2\tan^2 2x + \tan^4 2x)\, 2\sec^2 2x\, dx$$

$$= \frac{1}{2}\int (1 + 2u^2 + u^4)\, du$$

$$= \frac{1}{2}\left(u + \frac{2u^3}{3} + \frac{u^5}{5}\right) + C$$

$$= \frac{1}{2}\tan 2x + \frac{1}{3}\tan^3 2x + \frac{1}{10}\tan^5 2x + C$$