Algebra Review

In this worksheet we will do a broad overview of Algebra. Algebra involves any finite number of additions, subtractions, multiplications, divisions, and roots of variables and constants. Many of the 'rules' of algebra involve simplifying or rewriting algebraic expressions in equivalent but more useful forms, subject to a few conventions and definitions. Here are a few which carry us a long way, but which are sometimes misunderstood.

Convention: *The order of operations.* There is no mathematical reason why $12 - 3 \cdot 2^2 = 0$ is any better than $12 - 3 \cdot 2^2 = 36$; although the first is the only correct interpretation, the expression itself is ambiguous. Similarly, $24 \div 4 \div 2 = 3$ (not 12) and $8 - 3 - 2 = 3$ (not 7) are ambiguous without further guidance. Therefore, everyone agrees to do such computations in a particular order so that this is interpreted the same way by everyone. That is,

(a) parenthetical or grouped expressions are computed first;
(b) exponents are computed next;
(c) multiplications and divisions are computed next, from left to right in order of their appearance;
(d) additions and subtractions are computed next, from left to right in order of their appearance.

This does not say that one will not sometimes get the same answer by doing the computations in a different order (another order is sometimes permitted by other properties of algebra).

Helpful Ideas: (a) It is often useful to think of addition and subtraction as behaving the same. Subtraction is just the addition of a negative number afterall.

(b) It is often useful to think of multiplication and division as behaving the same. Division is just the multiplication of a reciprocal.

Definitions: (a) *Algebraic Expression*: Constants and variables which are connected by a finite number of additions, subtractions, multiplications, divisions, and roots form a *algebraic expression*.

(b) *Term*: Within an algebraic expression, constants and variables which are connected only by multiplication and division operations (no additions or subtractions) form a *term* of that algebraic expression. The individual elements of a term, separated by the multiplication signs, are called *factors* of the term.

(c) *Polynomial*: An algebraic expression in which the variable factors of each term have non-negative integer exponents and appear only in the numerator of that term is called a *polynomial*.

(d) *Rational Expression*: An algebraic expression which is a fraction involving a polynomial in the numerator and a polynomial in the denominator is called a *rational expression*.

(e) *Coefficient*: The coefficient of a variable indicates how many copies of that variable are being added together. Thus, $5x$ means that 5 x's are being added together. That is, $5x = x + x + x + x + x$. This makes it easy to condense lengthy addition problems into a smaller and easy to read form.

(f) *Exponent*: The exponent of a variable indicates how many copies of that variable are being multiplied together. Thus, x^5 means that 5 x's are being multiplied together. That is, $x^5 = x \cdot x \cdot x \cdot x \cdot x$.

Exponents do the same job for multiplication that coefficients do for addition. I know that you already know what coefficients and exponents are, but it is very critical that we know exactly where they come from to understand a few things about how they work.

Reminders for simplifying algebraic expressions: (a) Terms of an algebraic expression which are added together cannot be combined into a single term unless they have identical variable parts (and then the coefficients are added if the terms are both preceded by the same sign and subtracted appropriately otherwise). That is, $3xy + 4xy - 6xy^2 + 3x^2y$ can be simplified as $7xy - 6xy^2 + 3x^2y$, $3xy - 4xy - 6xy^2 + 3x^2y$ can be simplified as $-xy - 6xy^2 + 3x^2y$, and $-3xy - 4xy - 6xy^2 + 3x^2y$ can be simplified as $-7xy - 6xy^2 + 3x^2y$.

(b) Factors of a term cannot be combined into a single factor unless they have the same base (and then their exponents are added when the factors are multiplied [and subtracted when the components are divided]). That is, $5x^2yzx^3$ can be simplified as $5x^5yz$, and $\frac{12x^5y^3z}{4x^3y^6}$ can be simplifies as $\frac{3x^2z}{y^3}$.

(c) Subtractions cancel only additions and divisions cancel only multiplications. Thus, in a rational expression, the division cannot be carried out unless both numerator and the denominator are single term expressions with a common factor. Hence, $\frac{2+6y}{2}$ cannot be reduced in its current form because division by 2 cannot cancel addition of

2 (in the numerator). However, if this expression is rewritten as $\frac{2(x+3y)}{2}$ can be simplified as $x+3y$ because division by 2 cancels the multiplication by 2 in the numerator.

(d) Coefficients speak of the number of like terms being added (or subtracted) and so they distribute across addition or subtractions (the distributive law) but coefficients have nothing to do with products so they do not distribute across multiplication or division signs. Thus, $3(x+2yz)$ can be rewritten as $3x+6yz$, but $3(2y)$ **cannot** be rewritten as $(3\cdot 2)(3\cdot y)$.

(e) Exponents speak of the number of like factors being multiplied (or divided) and so they distribute across multiplications or divisions, but exponents have nothing to do with additions so they do not distribute across addition or subtraction signs. Thus, $(3yz)^2$ can be rewritten as $3^2\cdot y^2\cdot z^2$ and $\left(\frac{x}{3}\right)^2$ can be rewritten as $\frac{x^2}{3^2}$, but $(x+3)^2$ **cannot** be rewritten as x^2+3^2 and $(x-3)^2$ **cannot** be rewritten as x^2-3^2. In fact, $(x+3)^2$ means that there are 2 factors of $(x+3)$ being multiplied together. Hence, $(x+3)^2=(x+3)\cdot(x+3)=x^2+6x+9$ (recall that each term of the factor $(x+3)$ on the left must be multiplied times each term of the factor $(x+3)$ on the right).

(f) Since roots are basically exponents, roots behave in the same way: $\sqrt{3x}=\sqrt{3}\cdot\sqrt{x}$ and $\sqrt{\frac{x}{3}}=\frac{\sqrt{x}}{\sqrt{3}}$, but $\sqrt{x+3}$ **cannot** be rewritten as $\sqrt{x}+\sqrt{3}$ and $\sqrt{x-3}$ **cannot** be rewritten as $\sqrt{x}-\sqrt{3}$. Note that $5=\sqrt{25}=\sqrt{16+9}$, but $\sqrt{16}+\sqrt{9}=4+3=7$ (and $5\neq 7$).

(g) Factors which have the same exponent or root can be combined, even if the base is different: $x^2\cdot y^2$ can be rewritten as $(xy)^2$ because both x and y have the same exponent. Similarly $\sqrt{x}\sqrt{y}$ can be rewritten as $\sqrt{xy}$. The same is true for division: $\frac{x^2}{y^2}$ can be rewritten as $(\frac{x}{y})^2$, and $\frac{\sqrt{x}}{\sqrt{y}}$ can be rewritten as $\sqrt{\frac{x}{y}}$. Recall that multiplication and division behave similarly. Factors that have different exponents or roots cannot be combined if the base is different. Thus x^2y^3 cannot be rewritten in any (appreciably different) way and $5\sqrt{x}$ **cannot** be rewritten as $\sqrt{5x}$. The only (appreciably different) way to write $5\sqrt{x}$ is as $\sqrt{5^2x}$ (this comes from $5\sqrt{x}=\sqrt{5^2}\sqrt{x}=\sqrt{5^2x}$ where the first equality comes from the fact that $5=\sqrt{5^2}$ and the second from the fact that we now have the same root on each factor and so they can be combined [as pointed out above]). Be careful with variables and even roots, however: $\sqrt{x^2}=|x|$, $\sqrt{x^2}\neq x$.

(h) The corresponding notion with addition and subtraction is that two terms with the same coefficient can be combined even though their variable parts are different. Thus, $2xy+2yz$ can be rewritten as $2(xy+yz)$ and $2xy-2yz$ can be rewritten as $2(xy-yz)$. Recall that addition and subtraction behave similarly.

(i) Division problems must be considered carefully. On one hand $\frac{x+y}{2}=\frac{x}{2}+\frac{y}{2}$, but $\frac{2}{x+y}$ **cannot** be rewritten as $\frac{2}{x}+\frac{2}{y}$. The first of these works because division is multiplication by a reciprocal and one can distribute multiplication over addition and subtraction. That is: $\frac{x+y}{2}=\frac{1}{2}(x+y)=\frac{1}{2}x+\frac{1}{2}y=\frac{x}{2}+\frac{y}{2}$. The second of these fails to work because of the order of operations says that any parenthetical or grouped expression must be computed before the division. While there are no parenthesis involved in $\frac{2}{x+y}$, the denominator $x+y$ is a grouped expression since it is the denominator of the rational expression. To write $\frac{2}{x}+\frac{2}{y}$ is to do the division before the addition within the grouped expression.

(j) $(x^m)^n=x^{m\cdot n}$. For example, $(x^3)^2=x^{3\cdot 2}$. This is because, $(x^3)^2$ says that there are 2 x^3 factors being multiplied together, and x^3 says there are 3 x's being multiplied together. Thus, there are 2 sets of 3 x's being multiplied together and 2 sets of 3 is 6, or $2\cdot 3$. Hence, there are $3\cdot 2$ x's being multiplied together.

Algebra Review Problems

Simplify each of the following, if it can be further simplified. Carry out any operations which can be computed.

1. $\dfrac{(3x^2y^{-3})^{-2}}{(6x^{-3}y)^{-2}}$

2. $\dfrac{x^2+2x+1}{x^2-1}$

3. $(2x+3)^2$

4. $(x-2)^3$

5. $\dfrac{10x+5}{5}$

6. $\sqrt{36x^2+9}$

7. $\dfrac{4}{4x-12}$

8. $\dfrac{3(x+h)-2(x+h)^2+1-(3x-2x^2+1)}{h}$

9. $3\sqrt{2xy}-\sqrt{6xy}$

10. $\sqrt{x^2-9}+\sqrt{9}$

11. $\sqrt{a^2}$