Calculus and Analytic Geometry I

Applications of the Derivative, *continued*

§**3.4. Exponential Growth and Decay** These address applications in which a quantity increases or decreases in a manner proportional to its current level. They all can be solved according to the general equation $A = A_o e^{kt}$. There is little variation. For examples, go to other such problems on pages 177–178 and problems 57–60 on page 201.

§**3.5. Inverse Trigonometric Functions and their Derivatives:** None of the trigonometric functions are 1-1 and so (as with obtaining=- $\sqrt{\cdot}$) the domain had to be restricted to a set where the remaining function is 1-1. You should know the restricted domains of sine, cosine, tangent, and secant and their inverses. You should also know how to compute derivatives involving them. As with the function $f(x) = x^2$, [restricted domain: $x \geq 0$ and so with $f^{-1}(x) = \sqrt{x}$ we have $f(f^{-1}(x)) = (\sqrt{x})^2 = x$ always, but $f^{-1}(f(x)) = \sqrt{x^2} = x$ only if $x \geq 0$ (is in the restricted domain)], the inverse trigonometric functions do not perfectly reverse the original function, although the original function always reverses its inverse function. Namely:

$\sin(\sin^{-1}(x)) = x$	for all x in $[-1, 1]$ (the entire range of sine)	*but*	$\sin^{-1}(\sin(x)) = x$	only for x in $[-\pi/2, \pi/2]$ sine's restricted domain.
$\cos(\cos^{-1}(x)) = x$	for all x in $[-1, 1]$ (the entire range of cosine)	*but*	$\cos^{-1}(\cos(x)) = x$	only for x in $[0, \pi]$ cosine's restricted domain.
$\tan(\tan^{-1}(x)) = x$	for all x in $(-\infty, \infty)$ (the entire range of tangent)	*but*	$\tan^{-1}(\tan(x)) = x$	only for x in $(-\pi/2, \pi/2)$ tangent's restricted domain.
$\sec(\sec^{-1}(x)) = x$	for all x in $(-\infty, -1] \cup [1, \infty)$ (the entire range of secant)	*but*	$\sec^{-1}(\sec(x)) = x$	only for x in $[0, \pi/2) \cup [\pi, 3\pi/2)$ secant's restricted domain.

The tangent inverse function is the only one with domain $(-\infty, \infty)$ and so the only one with a limit as $x \to \infty$ or as $x \to -\infty$. Most of the problems are generic, so for examples look at pages 183–184: 1–10, 16-28, 30–40.

§**3.7. L'Hôpital's Rule** We looked at limits early in the course and learned how to interpret some forms (for instance, $\frac{c}{0}$, and some $\frac{0}{0}$ forms), but there are 7 forms which we need more machinery to interpret. These are the $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$, 1^∞, ∞^0, and 0^0 forms. L'Hôpital's Rule helps us deal with the first two. For the third form, we physically subtract the two to get a fractional form (likely one of the first two) which we can interpret in some way. The fourth form we turn into a fractional form by noting that $a \cdot b = \frac{b}{1/a}$; this gives us a fractional form which we can likely apply L'Hôpital's Rule to. The last three forms we can deal with by noting that $a^r = e^{r \ln(a)}$. The limit of $r \ln(a)$ gives us a form of type 4, which we can then deal with in the standard way. *I listed examples of these limits on the first part of the Final Exam Review, towards the bottom of the large limit section.*

§**4.1. Maximum and Minimum Values:** This section introduced us to a lot of terminology: Critical Numbers, Absolute Maximum & Local Maximum, Absolute Minimum & Local Minimum, and several important theorems: The Extreme Value Theorem and Fermat's Theorem.

1. Find the critical numbers of the function: $f(x) = \dfrac{x}{4} + \dfrac{1}{x}$.
2. Find the absolute maximum and the absolute minimum of the function $f(x) = \dfrac{x}{4} + \dfrac{1}{x}$ on the interval $[1/2, 4]$.
3. Work other examples of this type. They are all basically the same, just with different functions and derivatives.
4. Also be able to distinguish in a graph the difference between a local extrema and an absolute extrema.

§**4.2: The Mean Value Theorem**

1. Verify that the hypotheses of the Mean Value Theorem are satisfied for $f(x) = x^3 - 4x + 2$ on the interval $[-1, 2]$ and find the value(s) of c provided by its conclusion.
2. Verify that the hypotheses of Rolle's Theorem are satisfied for $f(x) = \sqrt{x} - \frac{x}{2}$ on $[0, 4]$ are satisfied and find the value(s) of c provided by its conclusion.
3. Be able to graphically interpret the MVT as in exercise 7 on page 215.

§**4.3 and 4.4: Derivatives, shapes of graphs, and curve sketching.**

1. Let $f(x) = \dfrac{3}{5}x^5 - 4x^3 + 8$.

   (a) Find the intervals on which the function f is increasing and the intervals on which the function f is decreasing.
   (b) Find any local maxima and local minima of the function f.
   (c) Find the intervals on which the graph of f is concave up and the intervals on which the graph of f is concave down.
   (d) Find any inflection points of the graph of f.
   (e) Use the data you found in parts (a)–(d) to sketch the graph of the function f.

2. Let $f(x) = \left(\frac{x-1}{x+1}\right)^2$; note that $f'(x) = \frac{4(x-1)}{(x+1)^3}$ and $f''(x) = \frac{-8(x-2)}{(x+1)^4}$. Keep me honest by double-checking my derivatives. (a) Find the domain of f; (b) find any vertical asymptotes for f; (c) find any horizontal asymptotes for the function f; (d) find the intervals on which f is increasing, and the intervals on which f is decreasing. (e) find any local extrema; (f) find the intervals on which the graph of f is concave up and the intervals on which the graph of f is concave down; (g) find any inflection points; (h) find the x–intercepts of the function f; (i) find the y–intercept of the function f; (j) Use this information to sketch a graph of this function.

3. Let $f(x) = \left(\frac{x-1}{x+3}\right)^2$; note that $f'(x) = \frac{8(x-1)}{(x+3)^3}$ and $f''(x) = \frac{-16(x-3)}{(x+3)^4}$. Keep me honest by double-checking my derivatives. (a) Find the domain of f; (b) find any vertical asymptotes for f; (c) find any horizontal asymptotes for the function f; (d) find the intervals on which f is increasing, and the intervals on which f is decreasing; (e) find the x–intercepts of the function f; (f) find the y–intercept of the function f; (g) Use this information to sketch a graph of this function.

4. Suppose that $f(x) = \frac{2x-3}{(x-2)}$. (a) Find the domain of f; (b) find any vertical asymptotes for f; (c) find any horizontal asymptotes for the function f; (d) find the intervals on which f is increasing, and the intervals on which f is decreasing; (e) find any local extrema; (f) find the intervals on which the graph of f is concave up and the intervals on which the graph of f is concave down; (g) find any inflection points; (h) find the x–intercepts of the function f; (i) find the y–intercept of the function f; (j) Use this information to sketch a graph of this function.

§**4.5. Optimization.** These sections are applications of the derivative to optimization. The general problem will ask us to optimize (minimize or maximize) some quantity subject to a restriction of some kind. (1) We first locate in the problem statement the quantity to be optimized; (2) we then draw any relevant pictures, assign any necessary variables and develop a rough equation for the quantity to be optimized; (3) If there is more than one independent variable, we look in the problem statement for the restriction and find in this restriction an equation relating the independent variables. We use this equation to substitute until we have the quantity to be optimized expressed in terms of a single independent variable; (4) we look for an interval on which this equation is naturally defined; (5) We find the absolute extrema of this function on the given interval.

3. Robin has 1200 yards of fencing. With it, she intends to make two pens by making a large rectangular enclosure with a fence down the middle parallel to one pair of sides. What are the dimensions of the pen with the maximum area?

4. Now suppose Robin has 1600 yards of fencing. With it, she intends to make three pens by making a large rectangular enclosure with two fences down the middle, parallel to one pair of sides. What are the dimensions of the pen with the maximum area?

5. A sheet of paper is to contain 18 square inches of print. The margins at the top and bottom are each 2 inches. The right and left margin are to be 1 inch each. What are the dimensions of the sheet of paper if the total area of the paper is to be a minimum?

6. A firm which produces mini microwaves has determined that the daily demand function for its microwaves is given by $p = 150 - 0.2x \quad$ for $0 \leq x \leq 750$.

(a) Find the daily revenue function.

(b) Suppose that the daily cost function is given by: $C(x) = 1000 + 50x + 0.05x^2$. Find the daily profit function.

(c) Find the production level which will maximize the daily profit.

§4.7. Antiderivatives. This section holds the concept of reversing the derivative. With the exception of the power rule, each derivative formula is interpreted in reverse to find an antiderivative. The antidifferentiation formula for power functions looks somewhat different: $\frac{d}{dx}[x^n] = \frac{x^{n+1}}{n+1}$, but still reverses the derivative of a power function.

6. (a) Find the most general antiderivative of $\quad f'(x) = 3\sqrt{x} + \frac{1}{x} - 3\sec^2(x) + e^x$

(b). Find the particular $f(x)$ which satisfies (a), if $f(1) = e$.

7. Find the most general antiderivative of the function: $\quad f(x) = x \cdot \sqrt{x} - \cos(x) + \frac{1}{x}$.

8. Find f if $f''(x) = e^x - 3$, $f'(0) = 1$, $f(0) = -1$.

9. Find the most general antiderivative of $f(x) = \frac{x^2 + 3xe^x - 4}{x}$.

10. Find the most general antiderivative of $\quad f'(x) = \frac{x^3 + x + 2}{x^2} - 3\sec(x)\tan(x)$.

11. Find f if $f''(x) = 2e^x + 2x$, $f'(0) = 1$, $f(0) = -1$.

12. Find the most general antiderivative of $f(x) = x(x^3 - 2\frac{e^x}{x} - 2)$.

§**5.1. Areas and Distances:** We begin estimating the area between a curve and the x-axis on an interval, and also to assign an interpretation to that number

13. Approximate the area between the graph of $y = f(x) = 5x - x^2$ and the x–axis on the interval $[0, 4]$ using R_4. Using L_4. Using M_4. Compute R_n. Compute $\lim_{n\to\infty} R_n$.

14. Find R_N, for the function $f(x) = 4x - x^2$ on the interval $[1, 3]$. Compute $\lim_{n\to\infty} R_n$.

15. The velocity of a particle at time t is given by $v(t) = 3t + 2$. Find the area between the graph of this function and the t axis from $t = 0$ and $t = 3$. [Just sketch the picture and find the exact area (since it is a trapezoid); don't bother with an approximation.]

§**5.2. The Definite Integral:** In this section we define the Definite Integral as the limit of Riemann Sums, and investigate its properties.

16. Write the definite integral $\int_{-1}^{3} \sin(x^2)\, dx$ as a limit of Riemann sums.

17. Write the limit of Riemann sums: $\lim_{n\to\infty} \sum_{k=1}^{n} x_k \cos(x_k)\Delta x$ on $[0, \pi]$ as a definite integral.

18. Compute the definite integral $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$ by considering the definition of the symbol itself and interpreting it in terms of areas. Do not compute the limit of any Riemann sums or the FTOC.

19. Compute the definite integral $\int_{0}^{5} 7 - 2x\, dx$ by considering the definition of the symbol itself and interpreting it in terms of areas. Do not compute the limit of any Riemann sums or the FTOC.

20. Express the area between the graph of $f(x) = 9 - x^2$ and the x–axis on the interval $[-3, 3]$ as both a definite integral, and a limit of Riemann sums. Make no attempt to compute either.

21. Suppose that $\int_{-2}^{3} 3x^2 = 35$ and $\int_{-2}^{1} 3x^2\, dx = 26$. Determine $\int_{1}^{3} 3x^2\, dx$.

22. Suppose that $\int_{1}^{5} f(x)\, dx = 7$ and $\int_{-2}^{1} f(x)\, dx = 11$. Determine $\int_{-2}^{5} f(x)\, dx$.

23. Suppose that $\int_{0}^{3} x^2\, dx = 9$, $\int_{0}^{3} x\, dx = 9/2$, and $\int_{0}^{3} 1\, dx = 3$.
(a) Find $\int_0^3 (2x^2 - 3x + 7)\, dx$. (b) Find $\int_3^0 4x^2\, dx$. (c) Find $\int_0^3 (4 - 5x + 3x^3)\, dx$.

§**5.3. Evaluating Definite Integrals:**

24. Compute the following definite integrals [using the Fundamental Theorem of Calculus, Part 1].

(a) $\int_{-2}^{1} 2x - \cos(x)\, dx$

(b) $\int_{1}^{3} 2x(x^2 + 1)dx$

(c) $\int_{1}^{3} \left(\frac{x^3 + 1}{x}\right)\, dx$

(d) $\int_{0}^{1} (\sin(x) + 2\sec^2(x))\, dx$

(e) $\int_0^4 x\sqrt{x}\, dx$

(f) $\int_{0}^{\pi/4} (\sec(x)\tan(x) + 2\sin(x))\, dx$

25. Find the general indefinite integral: (a) $\int x(2x + 3)^2\, dx$ (b) $\int \frac{x^2 + \sqrt{x}}{\sqrt{x}}\, dx$ (c) $\int \frac{\sec(x)}{1 + \tan^2(x)}\, dx$

26. A honeybee population starts with 500 bees and increases at the rate of $p'(t)$ bees per week. What does $500 + \int_0^{15} p(t)\, dt$ represent?

§**5.4. The Fundamental Theorem of Calculus:**

27. Use the Fundamental Theorem of Calculus to compute:

(a) $\frac{d}{dx}\left[\int_1^{x^2} \tan(t^2)\, dt\right]$ (b) $\frac{d}{dx}\left[\int_1^{x} \sin(t^2)\, dt\right]$ (c) $\frac{d}{dx}\left[\int_1^{x^3} t\ln(t)\, dt\right]$

28. Find the average value of the function $f(x) = 2x^3 + 2x$ on the interval $[1, 3]$.

§5.5. Integration by u-substitution & the Chain Rule:

1. Compute the following indefinite integrals.

(a) $\int 2\sec(x)\tan(x)(1+2\sec(x))^3 \ dx$

(b) $\int \frac{6x-9}{(2x^2-6x+7)^4} \ dx$

(c) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} \ dx$

(d) $\int \frac{\cos(\sqrt{x})\sin^3(\sqrt{x})}{\sqrt{x}} \ dx$

(e) $\int \frac{2t^3-3t}{\sqrt[3]{t^4-3t^2+2}} \ dt$

(f) $\int (4x)\sec^3(x^2)\tan(x^2) \ dx$

(g) $\int \cos^2(x)-\sin^2(x) \ dx$

(h) $\int (4x-2)\sec^2(x^2-x) \ dx$

(i) $\int \frac{5x^7}{\sqrt{1-x^8}} \ dx$

(j) $\int \frac{5x^3}{\sqrt{1-x^8}} \ dx$

(k) $\int \frac{3-3\sin(2x)}{2x+\cos(2x)} \ dx$

(l) $\int \frac{3-3\sin(2x)}{2x+\cos(2x)} \ dx$

(m) $\int \frac{2x}{1+x^2} \ dx$

(n) $\int \frac{3-2x}{1+x^2} \ dx$

(o) $\int \frac{3}{1+4x^2} \ dx$

2. Compute the following definite integrals.

(a) $\int_0^5 \sqrt{3x+1} \ dx.$

(b) $\int_0^1 \frac{2x}{1+x^2} \ dx$

(c) $\int_1^{\sqrt{3}} \frac{2}{1+x^2} \ dx$

(d) $\int_0^{\pi/4} (1+\tan^2(x)) \ dx$

(e) $\int_{-2016}^{2016} x\sqrt{1+x^2} \ dx$

(f) $\int_0^{\sqrt{5}} \frac{x}{\sqrt{4+x^2}} \ dx$

(g) $\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\cos(\sqrt{x})}{\sqrt{x}} \ dx$