Calculus and Analytic Geometry I

Applications of the Derivative

(a) A rock is thrown vertically upward from ground level with an initial velocity of 96 ft/sec. The position of the rock t seconds after it is released is given by: $s(t) = -16t^2 + 96t$. What is the velocity of the rock after 3 seconds? When is the rock at its maximum height? What is the maximum height of the rock? When will it hit the ground? With what velocity will it hit the ground?

(b) Air is blown into a spherical balloon. What is the rate of change of the volume of air in the balloon with respect to its radius when $r = 4$ inches? Interpret this.

(c) The daily cost of producing x bicycles is given by: $C(x) = 9000 + 250x + 0.005x^2$. What is the rate of change of cost with respect to daily production level when $x = 20$? Interpret this.

Computation of the Derivative.

Sections 2.3 (atoms, +, -), 2.4 ($\cdot$, $\div$), 2.5 (chain rule), 2.6 (implicit differentiation). Be able to compute any derivative thrown at you. Examples: differentiate the following functions, or answer the posed question.

(a) $f(x) = 5x^3 - 2x\sqrt[3]{x^2} - \frac{7}{x^3}$

(b) $y = \sin(x) - 4\tan(x) + 2\csc(x)$

(c) Find an equation of the tangent line to the curve $y = x^2 - 4x$ which is perpendicular to the line $x + 2y = 7$.

(d) Find the second derivative of the function $y = \sin(x) - 4\tan(x) + 2\csc(x)$.

(e) $f(x) = x^2\sin(x) + \dfrac{\cos(x)}{x^2+1}$

(f) $s = \dfrac{t\sec(t)}{1+\sin(t)}$

(g) Find an equation of the line tangent to the curve $y = \dfrac{\sqrt{x}}{x+1}$ at the point $(4, 2/5)$ on the curve.

(h) $f(x) = \sin(x^2+1)$

(i) $g(x) = \sin^{10}(x^2+1)$

(j) $y = \sin^{10}(x^2\tan(x))$

(k) $y = \sin^{10}\left(\dfrac{\cos(x)}{x^2+1}\right)$

(l) $g(x) = \sec(\sqrt{x}\tan(x))$

(m) $y = \sec(\sqrt{x^2+1})$

(n) $f(x) = \cos(x^2\sqrt{x^2+1})$

(o) $y = \cos^3(x^2\sqrt{x^2+1})$

(p) Compute: $s = \dfrac{5t}{\sqrt[3]{3t+1}}$ and simplify the derivative to get $\frac{ds}{dt} = \frac{5(2t+1)}{(3t+1)^{4/3}}$.

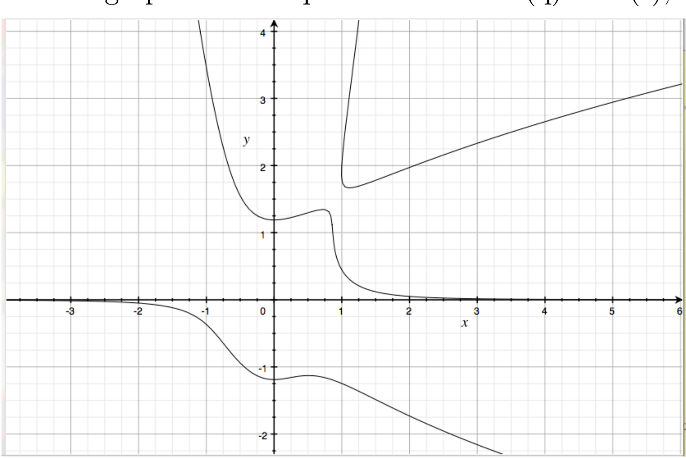
(q) Find $\frac{dy}{dx}$ by implicit differentiation: $y^4 - 3x^2y^3 = 2 - 5x^3y$

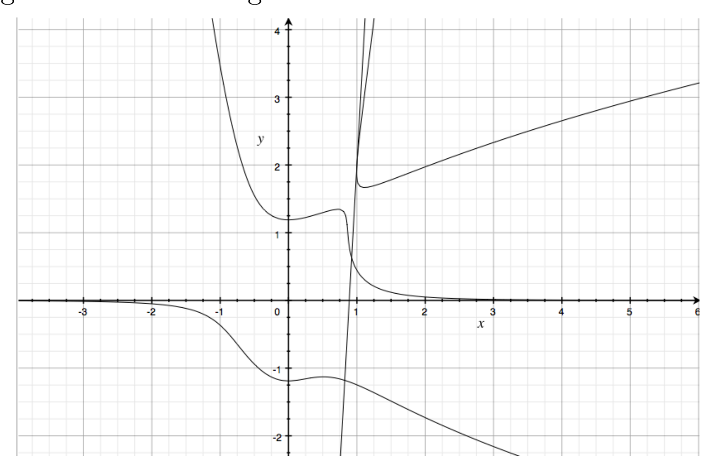
(r) Find $\frac{dy}{dx}$ by implicit differentiation: $\sin(xy) = 1 + x\cos(y)$

(s) Find $\frac{dy}{dx}$ by implicit differentiation: $2x^2y - \sqrt{xy} = 6$.

(t) Verify that the point $(1, 2)$ is on the curve: $y^4 - 3x^2y^3 = 2 - 5x^3y$. Find an equation for the line tangent to the curve $y^4 - 3x^2y^3 = 2 - 5x^3y$ at the point $(1, 2)$. [Note that you have already computed its implicit derivative in (q).]

Below is a graph of the implicit function in (q) and (t), together with the tangent line.





Related Rates

1. A child is walking away from a lamp post at 2 ft/ sec. If the lamp post is 20 feet tall and the child is 4 feet tall, at what rate is the child's shadow (s) lengthening when the child is 16 feet from the lamp post.

2. Water is pouring into an inverted conical vat at the rate of 5 ft^3/min. The vat has a height of 30 feet and a radius of 15 feet. At what rate is the depth of the water in the vat changing when the water is 20 feet deep?

3. A kite is flying at a height of 40 feet. A child is flying it so that it is moving horizontally at a rate of 3 ft/ sec. If the string is taut, at what rate is the string being paid out when the length of the string is 50 feet?

4. A trough is 12 feet long and its ends are in the form of an inverted isosceles triangle having an altitude of 3 feet and a base of 3 feet. Water is flowing into the trough at the rate of 2 ft^3/min. How fast is the water level rising when the water is 1 foot deep?

§**2.8. Differentials** As the value of the input changes, the derivative as a rate of change is used to predict the change in the output and the output of the function at the new input. The following formulas are used, respectively, as the input changes from the original of $x = a$ to the new value of $x = x_{new} = a + \Delta x$ ($\Delta x = x_{new} - a$, and $dx = \Delta x$): $\Delta y \simeq dy = f'(a)dx$ and $f(x_{new}) \simeq L(x_{new}) = f(a) + f'(a)(x_{new} - a)$. The function $L(x) = f(a) + f'(a)(x - a)$ is just the equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$. Two examples follow.

1. Suppose that $f(x) = \sqrt{x}$. We know that $f(256) = 16$. Use differentials to approximate $f(255) = \sqrt{255}$. Also, approximate the change in f as x changes from $x = 256$ to $x = 255$.

2. Use a linear approximation for $f(x) = \ln(x)$ to approximate $\ln(1.1)$. [Recall, $\ln(1) = 0$.] Also, approximate the change in f as x changes from $x = 1$ to $x = 1.1$.

§**3.2. Inverse Functions and Logarithms.** The graph of a relationship between x and y is the graph of a function if it passes the *vertical* line test. That is, for each x in the domain there is only one y connected to it by the relation. A function is said to be 1-1 if for each y in the range of the function there is just one x in the domain for which $f(x) = y$. The function can be determined to be 1-1 by verifying that its graph passes the **horizontal** line test. An inverse function of a function f will perfectly reverse the action of the function f and is denoted by the notation f^{-1}. The following is the characteristic relationship between functions and their inverse functions: $f(f^{-1}(x)) = x$ and $f^{-1er}(f(x)) = x$. A function has an inverse function if and only if it is a 1-1 function. Since all exponential functions are 1-1 then each exponential function has an inverse function and this inverse function is called a logarithm. Since there are exponential functions of various bases, we denote their inverse functions by logarithms of various bases. In particular, for $f(x) = a^x$, $f^{-1}(x) = \log_a(x)$. [recall $a \neq 1$, and $a > 0$.] Thus the following cancellation laws hold:

$$a^{\log_a(x)} = x \qquad \textit{and} \qquad \log_a(a^x) = x$$

Some bases are more common than others and the bases are suppressed as follows: $\log_e(x) \equiv \ln(x)$ and $\log_{10}(x) \equiv \log(x)$. These are on every scientific calculator and the following conversion formula is used to permit the computation of logarithms of other bases on the calculator: $\log_a(c) = \dfrac{\ln(c)}{\ln(a)}$. Thus, the natural log (ln) and exponential (e) are commonly used in solving exponential equations, along with the corresponding cancellation laws: $e^{\ln(x)} = x$ and $\ln(e^x) = x$. You should also know the domain and range of exponential and logarithm functions and be able to compute limits involving them. Finally, one should be able to use a given formula to compute the value of the derivative of an inverse function at a value without actually finding the inverse function itself.

Some good example problems: Page 162-163: 35–38, 43–55, 63–66, and 71–76.

§**3.3. Derivatives of Exponentials and Logarithms; Logarithmic Differentiation** We get the differentiation and integration formulas for exponential functions and logarithmic functions. Logarithmic functions provide us with a new technique of differentiation called Logarithmic Differentiation. Be able to differentiate any combination of exponentials of logarithms with other functions, in addition to logarithmic differentiation; I won't generate new ones of these because they are rather generic. However, the first problem below might benefit from logarithmic differentiation and the second and third must be done by logarithmic differentiation.

Find $\frac{dy}{dx}$:

(a) $f(x) = x^2 \ln\left(\frac{\tan(x)}{x^2+1}\right)$ (b) $y = (\cos(x))^x$ (c) $y = (x^2 + e^x)^{\sin(x)}$

§**3.4. Exponential Growth and Decay** These address applications in which a quantity increases or decreases in a manner proportional to its current level. They all can be solved according to the general equation $A = A_o e^{kt}$. There is little variation. For examples, go to other such problems on pages 177–178 and problems 57–60 on page 201.

§**3.5. Inverse Trigonometric Functions and their Derivatives:** None of the trigonometric functions are 1-1 and so (as with obtaining=- $\sqrt{\cdot}$) the domain had to be restricted to a set where the remaining function is 1-1. You should know the restricted domains of sine, cosine, tangent, and secant and their inverses. You should also know how to compute derivatives involving them. As with the function $f(x) = x^2$, [restricted domain: $x \geq 0$ and so with $f^{-1}(x) = \sqrt{x}$ we have $f(f^{-1}(x)) = (\sqrt{x})^2 = x$ always, but $f^{-1}(f(x)) = \sqrt{x^2} = x$ only if $x \geq 0$ (is in the restricted domain)], the inverse trigonometric functions do not perfectly reverse the original function, although the original function always reverses its inverse function. Namely:

$\sin(\sin^{-1}(x)) = x$	for all x in $[-1, 1]$ (the entire range of sine)	*but*	$\sin^{-1}(\sin(x)) = x$	only for x in $[-\pi/2, \pi/2]$ sine's restricted domain.
$\cos(\cos^{-1}(x)) = x$	for all x in $[-1, 1]$ (the entire range of cosine)	*but*	$\cos^{-1}(\cos(x)) = x$	only for x in $[0, \pi]$ cosine's restricted domain.
$\tan(\tan^{-1}(x)) = x$	for all x in $(-\infty, \infty)$ (the entire range of tangent)	*but*	$\tan^{-1}(\tan(x)) = x$	only for x in $(-\pi/2, \pi/2)$ tangent's restricted domain.
$\sec(\sec^{-1}(x)) = x$	for all x in $(-\infty, -1] \cup [1, \infty)$ (the entire range of secant)	*but*	$\sec^{-1}(\sec(x)) = x$	only for x in $[0, \pi/2) \cup [\pi, 3\pi/2)$ secant's restricted domain.

The tangent inverse function is the only one with domain $(-\infty, \infty)$ and so the only one with a limit as $x \to \infty$ or as $x \to -\infty$.

Most of the problems are generic, so for examples look at pages 183–184: 1–10, 16-28, 30–40.

§**3.7. L'Hôpital's Rule** We looked at limits early in the course and learned how to interpret some forms (for instance, $\frac{c}{0}$, and some $\frac{0}{0}$ forms), but there are 7 forms which we need more machinery to interpret. These are the $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$, 1^∞, ∞^0, and 0^0 forms. L'Hôpital's Rule helps us deal with the first two. For the third form, we physically subtract the two to get a fractional form (likely one of the first two) which we can interpret in some way. The fourth form we turn into a fractional form by noting that $a \cdot b = \frac{b}{1/a}$; this gives us a fractional form which we can likely apply L'Hôpital's Rule to. The last three forms we can deal with by noting that $a^r = e^{r \ln(a)}$. The limit of $r \ln(a)$ gives us a form of type 4, which we can then deal with in the standard way. *I listed examples of these limits on the first part of the Final Exam Review, towards the bottom of the large limit section.*