Calculus and Analytic Geometry I
Exam 3–Review—Sections 2.7-3.6

§**2.7. Related Rates** A problem gives several rates of changes (derivatives). After reading the problem, you need to sketch a picture or interpret the problem in some way so that you can construct an equation which relates the variables whose rates are mentioned. If this equation has variables besides those whose rates are mentioned, then you'll need to find another equation relating some of the variables so that you can replace one of the variables with the others. Once you have an equation which relates *only* the variables whose rates are mentioned in the problem, then you use implicit differentiation with respect to t and substitute in the given values. The necessary formulas will either be given to you or be easy to find. Two examples follow.

1. A child is walking away from a lamp post at 2 ft/ sec. If the lamp post is 20 feet tall and the child is 4 feet tall, at what rate is the child's shadow (s) lengthening when the child is 16 feet from the lamp post.

2. Water is pouring into an inverted conical vat at the rate of 5 ft^3/min. The vat has a height of 30 feet and a radius of 15 feet. At what rate is the depth of the water in the vat changing when the water is 20 feet deep?

3. A kite is flying at a height of 40 feet. A child is flying it so that it is moving horizontally at a rate of 3 ft/ sec. If the string is taut, at what rate is the string being paid out when the length of the string is 50 feet?

4. A trough is 12 feet long and its ends are in the form of an inverted isosceles triangle having an altitude of 3 feet and a base of 3 feet. Water is flowing into the trough at the rate of 2 ft^3/min. How fast is the water level rising when the water is 1 foot deep?

§**2.8. Differentials** As the value of the input changes, the derivative as a rate of change is used to predict the change in the output and the output of the function at the new input. The following formulas are used, respectively, as the input changes from the original of $x = a$ to the new value of $x = x_{new} = a + \Delta x$ ($\Delta x = x_{new} - a$, and $dx = \Delta x$): $\Delta y \simeq dy = f'(a)dx$ and $f(x_{new}) \simeq L(x_{new}) = f(a) + f'(a)(x_{new} - a)$. The function $L(x) = f(a) + f'(a)(x - a)$ is just the equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$. Two examples follow.

1. Suppose that $f(x) = \sqrt{x}$. We know that $f(256) = 16$. Use differentials to approximate $f(255) = \sqrt{255}$. Also, approximate the change in f as x changes from $x = 256$ to $x = 255$.

2. Use a linear approximation for $f(x) = \ln(x)$ to approximate $\ln(1.1)$. [Recall, $\ln(1) = 0$.] Also, approximate the change in f as x changes from $x = 1$ to $x = 1.1$.

§**3.1. Exponential Functions.** Exponential functions are functions of the form $y = a^x$ for a constant a which satisfies $a > 1$ or $0 < a < 1$. Each of these has a characteristic shape which you should be able to immediately recognize and graph and be able to translate ($f(x-2)$, $f(x+1)$, $f(x)+3$, $f(-x)$, $-f(x)$, etc.). In particular, exponential functions are 1-1. You should also know their limits as $x \to \infty$ and $x \to -\infty$, and be able to use this knowledge to compute limits of functions involving exponentials.

Compute the following limits:

(a) $\displaystyle\lim_{x\to 5^-} e^{\frac{x}{x-5}}$ (b) $\displaystyle\lim_{x\to\infty} \frac{e^{4x} - e^{2x}}{e^{-x} - e^{4x}}$ (c) $\displaystyle\lim_{x\to \pi/2^-} e^{\tan(x)}$

(d) $\displaystyle\lim_{x\to\infty} 2^x$ (e) $\displaystyle\lim_{x\to\infty} 2^{-x}$ (f) $\displaystyle\lim_{x\to\infty} \left(\frac{1}{2}\right)^x$

§**3.2. Inverse Functions and Logarithms.** The graph of a relationship between x and y is the graph of a function if it passes the *vertical* line test. That is, for each x in the domain there is only one y connected to it by the relation. A function is said to be 1-1 if for each y in the range of the function there is just one x in the domain for which $f(x) = y$. The function can be determined to be 1-1 by verifying that its graph passes the **horizontal** line test. An inverse function of a function f will perfectly reverse the action of the function f and is denoted by the notation f^{-1}. The following is the characteristic relationship between functions and their inverse functions: $f(f^{-1}(x)) = x$ and $f^{-1er}(f(x)) = x$. A function has an inverse function if and only if it is a 1-1 function. Since all exponential functions are 1-1 then each exponential function has an inverse function and this inverse function is called a logarithm. Since there are exponential functions of various bases, we denote their inverse functions by logarithms of various bases. In particular, for $f(x) = a^x$, $f^{-1}(x) = \log_a(x)$. [recall $a \neq 1$, and $a > 0$.] Thus the following cancellation laws hold:

$$a^{\log_a(x)} = x \qquad \textit{and} \qquad \log_a(a^x) = x$$

Some bases are more common than others and the bases are suppressed as follows: $\log_e(x) \equiv \ln(x)$ and $\log_{10}(x) \equiv \log(x)$. These are on every scientific calculator and the following conversion formula is used to permit the computation of logarithms of other bases on the calculator: $\log_a(c) = \dfrac{\ln(c)}{\ln(a)}$. Thus, the natural log (ln) and exponential (e) are commonly used in solving exponential equations, along with the corresponding cancellation laws: $e^{\ln(x)} = x$ and $\ln(e^x) = x$. Properties of logarithms and exponentials were reviewed and should be ready to apply whenever necessary. One should be able to tell whether or not a function is 1-1 through its graph, for basic functions one should be able to compute its inverse function, one should be able to solve exponential and logarithmic functions (which will require use of the properties of logarithms and exponentials). You should also know the domain and range of exponential and logarithm functions and be able to compute limits involving them. Finally, one should be able to use a given formula to compute the value of the derivative of an inverse function at a value without actually finding the inverse function itself.

Some good example problems: Page 162-163: 35–38, 43–55, 63–66, and 71–76.

§**3.3. Derivatives of Exponentials and Logarithms; Logarithmic Differentiation** We get the differentiation and integration formulas for exponential functions and logarithmic functions. Logarithmic functions provide us with a new technique of differentiation called Logarithmic Differentiation. Logarithmic Differentiation is called that because one (a) takes the (natural) logarithm of both sides of the equation, (b) applies the following property of logarithms: $\ln(x^r) = r \cdot \ln(x)$, (c) and then takes an implicit derivative of the resulting equation with respect to x. To complete the problem, one must also solve for $\frac{dy}{dx}$ and then replace y by its expression in terms of x. This is essential for taking the derivative of functions which have a variable base and a variable exponent. Be able to differentiate any combination of exponentials of logarithms with other functions, in addition to logarithmic differentiation; I won't generate new ones of these because they are rather generic. However, the first problem below might benefit from logarithmic differentiation and the second and third must be done by logarithmic differentiation.

Find $\frac{dy}{dx}$:

(a) $f(x) = x^2 \ln\left(\frac{\tan(x)}{x^2+1}\right)$ (b) $y = (\cos(x))^x$ (c) $y = (x^2 + e^x)^{\sin(x)}$

§**3.4. Exponential Growth and Decay** These address applications in which a quantity increases or decreases in a manner proportional to its current level. They all can be solved according to the general equation $A = A_o e^{kt}$. There is little variation. For examples, go to other such problems on pages 177–178 and problems 57–60 on page 201.

§**3.5. Inverse Trigonometric Functions and their Derivatives:** None of the trigonometric functions are 1-1 and so (as with obtaining=- $\sqrt{\cdot}$) the domain had to be restricted to a set where the remaining function is 1-1. You should know the restricted domains of sine, cosine, tangent, and secant and their inverses. You should also know how to compute derivatives involving them. As with the function $f(x) = x^2$, [restricted domain: $x \geq 0$ and so with $f^{-1}(x) = \sqrt{x}$ we have $f(f^{-1}(x)) = (\sqrt{x})^2 = x$ always, but $f^{-1}(f(x)) = \sqrt{x^2} = x$ only if $x \geq 0$ (is in the restricted domain)], the inverse trigonometric functions do not perfectly reverse the original function, although the original function always reverses its inverse function. Namely:

$\sin(\sin^{-1}(x)) = x$ for all x in $[-1, 1]$ (the entire range of sine) *but* $\sin^{-1}(\sin(x)) = x$ only for x in $[-\pi/2, \pi/2]$ sine's restricted domain.

$\cos(\cos^{-1}(x)) = x$ for all x in $[-1, 1]$ (the entire range of cosine) *but* $\cos^{-1}(\cos(x)) = x$ only for x in $[0, \pi]$ cosine's restricted domain.

$\tan(\tan^{-1}(x)) = x$ for all x in $(-\infty, \infty)$ (the entire range of tangent) *but* $\tan^{-1}(\tan(x)) = x$ only for x in $(-\pi/2, \pi/2)$ tangent's restricted domain.

$\sec(\sec^{-1}(x)) = x$ for all x in $(-\infty, -1] \cup [1, \infty)$ (the entire range of secant) *but* $\sec^{-1}(\sec(x)) = x$ only for x in $[0, \pi/2) \cup [\pi, 3\pi/2)$ secant's restricted domain.

The tangent inverse function is the only one with domain $(-\infty, \infty)$ and so the only one with a limit as $x \to \infty$ or as $x \to -\infty$.

Most of the problems are generic, so for examples look at pages 183–184: 1–10, 16-28, 30–40.

§**3.6. Hyperbolic Functions, Inverse Hyperbolic Functions, and their Derivatives:** We get 12 new functions and their derivatives; it is a lot all at once so I won't hold your feet to the flame to memorize things about these functions at this time. Too much investigation of this group of functions would detract from the main purpose of the course. You will be given the list of derivative formulas for the exam; you should be able to use that list to find the derivatives of functions involving these 12 new functions. The problems are rather generic, so I would say to just look at the differentiation problems on pages 189–190. The definitions (after the first two) will remind you of the trigonometric functions.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)} \quad \text{sech}(x) = \frac{1}{\cosh(x)} \quad \text{csch}(x) = \frac{1}{\sinh(x)}$$