Calculus and Analytic Geometry I

Big Ideas and/or Often Used Techniques

I've talked in class about being careful to notice and learn a few big ideas, and the smaller things will take care of themselves. Below is a summary of some big ideas.

(A) $a = (a-1)+1$ and/or $a = (a+1)-1$

(B) $\dfrac{a}{b} = \dfrac{a}{b} \cdot \dfrac{c}{c} = \dfrac{a}{bc} \cdot \dfrac{c}{1}$ and/or $\dfrac{a}{b} = \dfrac{a}{b} \cdot \dfrac{c}{c} = \dfrac{ac}{b} \cdot \dfrac{1}{c}$

(C) $b^a = e^{\ln(b^a)} = e^{a \cdot \ln(b)}$ Handy when we want an exponent to turn into a product (legally).

(D) Multiply by the conjugate.
Trying to square the individual terms of $(a-b)$ by squaring $(a-b)$ will also introduce a middle term and will change the size of the original expression $(a-b)$ as well. However

$$(a-b) = \frac{(a-b)}{1} = \frac{(a-b)}{1} \cdot \frac{(a+b)}{(a+b)} = \frac{(a-b)(a+b)}{(a+b)} = \frac{a^2-b^2}{a+b}$$

This is handy when we want to 'get rid of' a square root, and when we see a $(1-\cos(\alpha))$ or $(1-\sin(\alpha))$ or similar expression and we see the chance to use a Pythagorean identity to turn two terms into a single term $\sin^2(\alpha)$ or $\cos^2(\alpha)$. If you've been following the course, you saw this a lot in sections 2.5 and 2.6.

(E) What is the really happening here? What is the main operation/function/idea? For instance:

• When trying to simplify an expression, we often try to decide what to do, and in doing so we look for 'what operation is involving every other part of the expression' or perhaps 'are several operations sharing equal status (a sum and/or difference of several terms).
• If it is a division problem and we want to cancel, we know that we can only cancel a multiplication problem with a division, so we always check "Is the numerator basically involved in a multiplication" and "is the denominator basically involved in a multiplication" so that the division can legally take place.
• If we are differentiating a complicated expression (one that is not an atom) we need to decide if it is basically a constant multiple, a sum, a difference, a product, a quotient, or a composition of other functions. That is, what action is affecting all the other components of the expression.

(F) Terminological or Physical clues. For instance:
• A velocity is a change in distance traveled over a change in time, or a change in position traveled over a change in time $\left(\frac{\Delta s}{\Delta t}\right)$.
• The derivative of a function $f$ at $x = a$ is expressed by the notation $f'(a)$ is the instantaneous rate of change of the function $f$ at $x = a$ is the slope of the line tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ is the slope of the graph of $y = f(x)$ at the point $(a, f(a))$ is the limit $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$ and $f'(a) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$

(G) Substitution. This is more difficult to show without using a specific example. For a specific example, I'll use

$$\lim_{x\to 0} \frac{\sin(3x)}{x}$$

(which would also be a good example for (B) above).

If in the above limit, we let $u = 3x$, with the idea of replacing all the $x$'s by $u$'s.

We'd have $\displaystyle\lim_{x\to 0} \frac{\sin(u)}{x}$ which is not quite complete because we still have $x$'s remaining. If $u = 3s$, then $x = \frac{u}{3}$, and so now we'd have: $\displaystyle\lim_{x\to 0} \frac{\sin(u)}{\frac{u}{3}}$, or $\displaystyle\lim_{x\to 0} 3 \cdot \frac{\sin(u)}{u}$. This is better. Now note that because $u = 3x$ that $u \to 0$ as $x \to 0$ and $x \to 0$ as $u \to 0$, so finally we have

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{u\to 0} 3 \cdot \frac{\sin(u)}{u}$$

Another example is the following. Starting with

$$f'(a) \equiv \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

we'd like to see that this is the same as

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Substituting $x = a + h$, (to make the $f(a+h) - f(a)$ match the $f(x) - f(a)$), then we'd see that $h = x - a$ (which would be needed to substitute for the $h$ in the denominator. That would give us:

$$f'(a) \equiv \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(x) - f(a)}{x - a}$$

and we still have to dismiss that $h \to 0$. But as $h \to 0$, in $x = a + h$, note that $x \to a$, and as $x \to a$ in $h = x - a$, note that $h \to 0$. Thus:

$$f'(a) \equiv \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$