Essential Mathematical Skills

The purpose of this course is to build facility with the collection of skills and concepts which regularly appear while working with Algebra, Trigonometry, and Calculus.

Whenever we are trying to carry out a computation in one of these three, we will typically have a need to reach for some prior idea or tool to progress through the computation. Those frequently-needed ideas and tools are the topic of this course. At their most basic level, it is possible to avoid these ideas and tools through the use of either a calculator or a short-cut, but this is false economy (meaning that while it seems to *save* time and trouble, it is actually *costly*). If these concepts and tools are not used directly and often, then they will remain undeveloped and cumbersome to use when they are needed. However, if they are regularly employed when relevant, then over time they will become more and more natural to use. When you reach for them, you will find them ready at hand. That is, they will have become a part of your understanding and experience.

You have each probably watched a serial television show or read a book series in which certain interactions between the characters are funny or meaningful because you know the characters from the original few shows or chapters. Had you started into the second season or the second book without having experienced the first, many of the interactions wouldn't make any sense, or be interesting or funny. Learning or experiencing Mathematics is much the same. Paying particular attention to these foundational concepts and skills will mean that later episodes/concepts and skills will make more sense.

Here are a few items worthy of note.

1. Sometimes questions seem ambiguous to us and require clarification before we can answer them. That is not a bug, but rather a feature. That is, it is a normal and good part of solving any problem. Usually, if we know what a question is asking, we have a clue to finding its solution. Thus, figuring out what a question is asking gets us part way to a solution. Moreover, *it is in trying to be precise that we begin to understand what we are thinking.* These are just a few of the reasons that Mathematics endeavors to be precise.
2. While sometimes a problem will have more than one answer, even if there is no ambiguity in the question, almost always a problem will have more than one solution. An answer is what the question is literally seeking, whereas a solution is a path of reasoning one's way to an answer. Each of us will likely approach the solution to a problem in a different way because (a) each of us might originally observe a different clue than the others of us, and (b) each of us has a little different background of experiences (among other reasons). Thus, it isn't surprising that we arrive at an answer in a different way, and your teachers appreciate each solution for its own unique qualities. One approach is not lesser than another.
3. While answers are good, they are not as important as the solutions. We will be attentive to both in this class, but our main focus will be on solutions. Thus, a shortcut, or instant answer without an accompanying procedure, is *never* acceptable. An answer to a question will help us with that one question only, *if* we can convince anyone of its validity without an accompanying solution. However, a solution will help us again and again in the solving of other problems. It is better to know just 6 big ideas and be able to use them over and over again to solve other problems, than to know 100 quick shortcuts. What is so bad about shortcuts? (a) Shortcuts typically can only be used in limited circumstances. (b) We have to also remember when each shortcut is useful. (c) Since shortcuts are not accompanied by mathematical principles, they are difficult to remember. (d) They are shortcuts because they avoid referencing or even incidentally using mathematical principles. As a result, we never become familiar with important mathematical principles and thus every new idea that we encounter truly seems to come out of nowhere, rather than following logically from the previous solutions which we've been using.

This booklet is intended to provide a mathematical overview of each of the topics that it addresses, with only a very small set of illustrative exercises. Your teacher in the course will be providing much more extensive exercise sets for you to work, which will add depth to the basic ideas illustrated here. The mind is always drawing conclusions from whatever it sees and experiences; if the only examples that you see are too simple, then the mind may draw conclusions which are too simplistic as well. Finally, this booklet is designed to present these topics in a relatively efficient fashion suitable for someone who is already familiar with many topics. That is, we might refer to equations long before we actually talk about the solving of equations so that when we do discuss the solving of equations we can do so within the environment of having all manner of equations before us at once. That is, while advertised as being "mathematical", this booklet does not recreate all of Algebra and Trigonometry from the ground, up.

Finally, here are a few recommendations for studying Mathematics.

- For many people, it is more effective to work some every day, rather than a lot all at once. That is, two intense 25 minute sessions per day might be more beneficial than a 3 hour session twice per week. Try to find what works best for you.
- Strive to understand the terminology and the notation that is being used, and use it carefully yourself. This is a key to knowing what a question is asking or what a line of your solution is saying and is asking you to do next. Also, if you use notation arbitrarily or carelessly, you can mislead yourself. In particular, use parentheses like you are being paid to do so and use "=" between two expressions only if you are willing to bet that same money on their equality.
- Speak with your teacher when you have questions. That will help your teacher understand your thought process and help you move forward more efficiently.
- Adopt a valid procedure, either your teacher's or a valid procedure of your own, and use it every time. When in doubt, write down what your intuition tells you. If this leads to a correct solution then you can assume that you can trust that intuition. If this leads to an error then you and your teacher can correct that intuition and eventually it will not mislead you any longer.
- Write down on your paper what you are thinking, rather than trying to do too much in your head first. Research shows that we can only hold around 4 different ideas in our working memory at once. Having your thoughts offloaded onto paper will make it easier to solve the problem by giving your mind more freedom to weigh other thoughts and by giving you a visual of what you know. In addition, thoughts are not always as complete as we think they are, and thus even people who understand what they are doing often make mistakes when they are interacting with what are shifting or incomplete thoughts. By writing our thoughts down on paper we can catch the gaps in our thoughts that we would have never found while trying to juggle the various thoughts between the registers in our memory.
- Work basic computations by hand and commit big ideas and formulas to memory. For a number of reasons it is good to do most things by hand unless the work is egregious or super-obvious to you. One reason rooted in the psychology of learning is that unless you wrestle these problems with your mental resources then you'll know them only as separate and artificial ideas. We can only hold about 4 different ideas in our working memory to use for learning new ideas or solving problems, and if we have to use three or more of them to hold the basics, then there is little or no room left for you to hold the additional ideas necessary to solve the problem at hand. On the other hand, through repeated use you can *chunk* many basic ideas into one big idea that only occupies one spot in working memory; this allows you to be much more efficient with both your working memory space and with your thinking in general. [The words 'chunk' and 'chunking' are actual terms for this.]
- When you get stuck, take a break rather than get frustrated. When you have a breakthrough of understanding of something, take a few minutes to write down in words what has just become more clear to you. Initially, each is just a momentary thought pattern which is no more permanent than a wisp of smoke. You don't want to reinforce the frustration or lose that moment of inspiration.

Usually both breakthroughs and difficulties are due to a handful of little things over a long period of time, rather than to whether or not someone was born with an affinity for some skill. Invest in good practices as you are working on these essential skills, and benefit from the resulting accumulation thereafter. (See the picture that opened this section.)

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§**1. Properties of the Real Numbers** These are axioms or absolute properties of the Real Numbers and its operations of "+" and "·".

	Operations	
Property	**Addition**	**Multiplication**
Commutativity:	For any real numbers a, b: $a + b = b + a$	For any real numbers a, b: $a \cdot b = b \cdot a$
Associativity:	For any real numbers a, b, c: $a + (b + c) = (a + b) + c$	For any real numbers a, b, c: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Existence of an Identity:	The real number 0 satisfies $a + 0 = a$ for each real number a.	The real number 1 satisfies $a \cdot 1 = a$ for each real number a.
Existence of Inverses:	For each real number a there is a real number $(-a)$ for which $a + (-a) = 0$ [$-a$ is called the additive inverse of a.]	For each non-zero real number a there is a real number $\frac{1}{a}$ for which $a \cdot \frac{1}{a} = 1$ [$\frac{1}{a}$ is called the multiplicative inverse of a.]
Distributivity of Multiplication over Addition:	For any real numbers a, b, c: $a \cdot (b + c) = a \cdot b + a \cdot c$	
	Subtraction: For real numbers a, b: $a - b \equiv a + (-b)$ (see the additive inverse, above)	**Division:** For real numbers a, b: $\frac{a}{b} \equiv a \cdot \frac{1}{b}$ (see the multiplicative inverse, above)

Exercise Set 1

1. Make a table such as the one above (same entries in the top-most row (the header) and in the left-most column) but rather than writing the property inside the table, give an example of each property at the proper place in the table.

§2. Number-specific issues

Integers

Of course, we all know who/what the Integers are. $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$. It is essentially a listing of the Whole Numbers together with their additive inverses. We've been using them for so long that sometimes we use them a little quickly and make errors. Thus, it wouldn't hurt to review some of the major ideas so that we do not make the same careless errors. Actually, everything we mention is this section is true of all real numbers, but it is easier to illustrate within the Integers.

What is "negative"? vs What is "minus"?

These are used interchangeably by many people, but they are actually different. **Negative** a denotes the additive inverse of a. For instance, -3 is that number for which $-3 + 3 = 0$; that is, that number that when added to $+3$ yields 0. Also, $-(-3)$ is that number for which $-(-3) + (-3) = 0$; that is, that number that when added to -3 yields 0. We can see that $-(-3) = +3$ because both 3 and $-(-3)$, when added to -3 result in 0, and since additive inverses are unique then $3 = -(-3)$.

Subtraction, on the other hand, uses the same symbol but in a somewhat different way. Remember that $a - b \equiv a + (-b)$. Subtracting b IS the addition of the additive inverse of b. Thus, when we see $5 - 3$, the "3" is really positive and we are reading $+5$ minus/take away $+3$. As for the computation: $5 - 3 \equiv 5 + (-3) = 2$. In the case that that we are subtracting a *negative* number: $9 - (-3) \equiv 9 + (-(-3)) = 9 + 3 = 12$.

Thus, "negative" is like a light switch, flipping numbers between positive and negative numbers, while "minus" is the subtraction (or the addition of a negative) that we're used to.

Rational Numbers

The Rational numbers are the collection of all ratios of integers. We typically refer to them as *fractions*. Arithmetic with fractions sometimes trips us up, but there are a few major principles that can help us keep the arithmetic straight.

1. When we multiply two fractions, we multiply straight across.

   For instance: $\dfrac{3}{4} \cdot \dfrac{x}{2} = \dfrac{3 \cdot x}{4 \cdot 2} = \dfrac{3x}{8}$.

2. When we divide two fractions, we multiply by the multiplicative inverse (often called the *reciprocal*) of the denominator.

   For example: $\dfrac{\frac{3x^2}{4}}{\frac{5x^4}{2}} = \dfrac{3x^2}{4} \cdot \dfrac{2}{5x^4} = \dfrac{3x^2 \cdot 2}{4 \cdot 5x^4} = \dfrac{6}{20x^{4-2}} = \dfrac{3}{10x^2}$.

These are both true because of the nature of Multiplication (and because Division is based upon Multiplication). Ask your teacher, if interested, but we'll not go beyond the above examples here.

However, the computation is different with addition and subtraction.

3. When we add (or subtract) two fractions, we must first replace each fraction by equivalent fractions which have the same denominator. At that point we can add (or subtract, respectively) the two numerators, and place that over the denominator which they have in common.

The reason why #3 is different is a matter of units. If you were to add 1 cup + 2 quarts, you would not say 3 cups or 3 quarts or 3 of anything in particular because the items that you are counting between the two cases are not alike. The same is true if you tried to add

$$\frac{1}{16} + \frac{2}{4}$$

Some whole was divided into 16 equal pieces (each of size 1/16) and that specific size is like a "unit". That same whole was divided into 4 equal pieces, (each of size 1/4) and that specific size is like another "unit". That 1 part (1/16th of a gallon) is a much different size than the 2 parts (each 1/4 of a gallon), and so adding the number of parts you have in each case is not possible until you convert each to use the same unit… the same portion of the whole (or gallon). Thus,

$$\frac{1}{16} + \frac{2}{4} = \frac{1}{16} + \frac{2 \cdot 4}{4 \cdot 4} = \frac{1}{16} + \frac{8}{16} = \frac{1+8}{16} = \frac{9}{16}$$

We converted each to the same unit by identifying which factors of one denominator are missing from the factors of the second denominator, and then multiplying that second fraction by "1" in the form of $\frac{\text{missing factors}}{\text{missing factors}}$.

Exercise Set 2

1. Compute $-31 - 17$
2. Compute $-31 - (-11)$
3. Compute $31 - 57$
4. Compute: $\dfrac{7}{12} + \dfrac{7}{15}$
5. Compute $\dfrac{7}{120} - \dfrac{13}{230}$
6. Compute $\dfrac{11}{120} - \dfrac{-13}{230} - \dfrac{7}{150}$

Final Note for §2.

I. There is sometimes confusion with respect to 0 and division. Let's visit division for a moment so that we can clear this up in a definitive way.

$\dfrac{6}{2} = 3$ because (a) $6 = 2 \cdot 3$ and (b) 3 is the only number c for which $6 = 2 \cdot c$.

That is, there exists a number c such that $6 = 2 \cdot c$ (namely $c = 3$), and this number c is unique.

By this same reasoning, $\frac{0}{4} = 0$ because (a) $0 = 4 \cdot 0$ and (b) 0 is the only number c for which $0 = 4 \cdot c$.

Consider $\frac{5}{0}$. Suppose that $\frac{5}{0} = c$ for some real number c. Then $5 = 0 \cdot c \ldots$ but this is impossible. There is no real number c that, when multiplied by 0, results in 5. For this reason we cannot define $\frac{5}{0}$.

Now consider $\frac{0}{0}$. Suppose that $\frac{0}{0} = c$ for some real number c. Then $0 = 0 \cdot c$. Indeed, that is a valid equation. Unfortunately, however, the number c is not unique. $0 = 0 \cdot c$ for every real number. Since the number c is not unique, then we do not define $\frac{0}{0}$.

In summary,

- $\frac{0}{b} = 0$ (as long as $b \neq 0$);

and

- $\frac{b}{0}$ is not defined for any real number b.

II. A relatively common error is the assumption that $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$. We can see that this is incorrect with the example $a = 1$ & $b = 1$. Note $\dfrac{1}{1+1} \neq \dfrac{1}{1} + \dfrac{1}{1}$.

§**3. The Order of Operations:** This is a convention. That is, it is an agreement on a set of tie-breakers for ambiguous mathematical expressions. Since it is just convention, it is nice to have a mnemonic to remember it by. **PEMDAS** ... or **PE(MD)(AS)**

> **P**arentheses: First compute the contents of parentheses
> (or any other 'grouping', such as *the numerator* or *the denominator* of a fractional expression).
> **E**xponents: Compute all Exponents.
> **M**ultiplication & **D**ivision *from left to right*: Compute available Divisions and Multiplications as you encounter them as you pass through the expression from left to right.
> **A**ddition & **S**ubtraction *from left to right*: Compute available Subtractions and Additions as you encounter them as you pass through the expression from left to right.

Exercise Set 3

Carry out the following computations, showing every step of the work.

1. $12 - 3 \cdot (2-4)^2 \div 2$

2. $4^2 + (4-6)^3$

3. $(14 - 2 \cdot (3-7)^2 \div 2)^3$

4. $12 - 3 \cdot (2-4)^3 \div 2$

5. $\dfrac{(14 - 2 \cdot (3-7)^2 \div 2)^3}{4^2 + (4-6)^3}$

6. $\dfrac{12 - 12 \div (-6) \cdot (1-4)^2}{12 - 12 \cdot (1-4)^2}$

7. $\dfrac{14 - 2 \cdot (3-7)^3 \div 2}{(3-5)^3 - 2(4-6)^2}$

8. $\dfrac{123 - 3 \cdot (3-5)^3 \div 2}{123 - 3 \cdot (-2)^2}$

Final Note for §**3.** There are a few things to be careful about with subtraction. While it is based on addition, when we built the table of the properties of the operations "+" and "·", we did not discuss any of the previous properties, but only briefly mentioned that relationship at the end of the table. One reason for that is that subtraction is neither Associative nor Commutative. Notice: $(4-2)-1 \neq 4-(2-1)$ and $5-3 \neq 3-5$. Division is similar. For instance, $6 \div 2 = 6 \cdot \frac{1}{2}$, and so it is based upon an operation (Multiplication) which is Associative and Commutative but Division is neither Associative nor Commutative. Note: $(24 \div 6) \div 2 \neq 24 \div (6 \div 2)$, and $6 \div 2 \neq 2 \div 6$. The fact that neither subtraction nor division are associative means that the *from left to right* is a very important component of the **MD** and the **AS** portions of **PE(MD)(AS)**, even though it is often unwritten.

§**4. Absolute Value**

The absolute value is an excellent example of the advantage of knowing the definition of a term over merely knowing the fuzzy descriptor of what it does. Knowing that the absolute value makes numbers positive might tell you how it is commonly used, but knowing the definition will tell you everything you need to know about solving any question involving the absolute value. It will help you answer "Compute $|1-\pi|$" as $(\pi - 1)$ instead of as $(1+\pi)$.

$$\text{For a real number } a, \quad |a| = \begin{cases} a & \text{if } a \geq 0 \\ -1 \cdot a & \text{if } a < 0. \end{cases}$$

We won't say anything else about the absolute value until we discuss equations and inequalities, but this definition is here for use to refer to, when needed.

§5. Coefficients and Exponents

When trying to do Algebra, it is helpful to know the role of each statement. These statements fall into the following categories.

1. **Properties.** These are statements which are the foundation upon which we build everything. We accept their validity and use them as a guide. We saw these in Section 1.
2. **Theorems.** These are statements whose truth is established from the Properties, and which we use in much the same way as we use the properties. For instance,"For each real number a, $a \cdot 0 = 0$." or that Multiplication distributes across subtraction as well: "For any real numbers a, b, c: $a \cdot (b - c) = a \cdot b - a \cdot c$".
3. **Conventions.** (such as the order of operations) which we agree to follow;
4. **Terminology.** These are the words we use to talk about the components and actions so that we're not saying 'thing' and 'it' over and over and confusing the various its and things. This also allows us to refer to a whole paragraph of meaning with just a single word. [This helps us hold a large chunk of understanding in a single register of memory, rather than using all 4 registers as it might have required otherwise.]
5. **Notation.** This is how we agree to write and express ideas succinctly.

Coefficients and *Exponents* are a part of that notation or book-keeping. Suppose that a is a real number and we wanted to express 17 a's added or multiplied together. That is:

$$a+a+a+a+a+a+a+a+a+a+a+a+a+a+a+a+a$$

or

$$a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$$

Each would be very annoying to have to write...and just think of having to write the sum or product of 120 a's instead of just 17. Thus special ways of writing these have been adopted.

$$17a \equiv a+a+a+a+a+a+a+a+a+a+a+a+a+a+a+a+a$$

The 17 is called the *coefficient* of a and is the number of a's being added.

$$a^{17} \equiv a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$$

The 17 is called the *exponent* of a and is the number of a's being multiplied.

The 17 was just a specific example, but that could have been any natural number. So that we can talk about our general usage without writing one example for every possible combination, will use m and n to represent natural numbers, and a and b to represent any real number. Going forward we will use the following notation:

$n \cdot a = a + a + a + a + \ldots + a$ where there are n copies of a added together; (ie: $17a$ in the example above)

$a^n = a \cdot a \cdot a \cdot a \cdot \ldots \cdot a$ (where there are n copies of a multiplied together; (ie: a^{17} in the example above))

There are 5 common properties of coefficients (used with addition and subtraction) and 5 exactly corresponding properties of exponents (used with multiplication and division). Below, we will write the corresponding properties across from each other. This exact correspondence between coefficients and exponents is not a coincidence, but rather because they each merely count the number of copies of a being added or multiplied, respectively.

Addition, Subtraction, and Coefficients	Multiplication, Division, and Exponents
$ma + na = (m + n)a$ The variable part a must be the same, but then we add the coefficients to get the resulting coefficient.	$a^m \cdot a^n = a^{m+n}$ The base a must be the same, but then we add the exponents to get the resulting exponent.
$ma - na = (m - n)a$ The variable part a must be the same, but then we subtract the coefficients to get the resulting coefficient.	$\frac{a^m}{a^n} = a^{m-n}$ The base a must be the same, but then we subtract the exponents to get the resulting exponent.
$m(na) = (m \cdot n)a$ We multiply the nested coefficients to get the resulting coefficient.	$(a^n)^m = a^{m \cdot n}$. We multiply the nested exponents to get the resulting exponent.
$m(a + b) = ma + mb$ This is the distributive property.	$(ab)^m = a^m \cdot b^m$. Distribution as well, right?
$m(a - b) = ma - mb$ This is the distributive property.	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ Distribution as well, right?
$0 - n \cdot a = -na$	$\frac{1}{a^n} = a^{-n}$
$0 \cdot a = 0$ for each real number a	$a^0 = 1$ for each real number a

Exercise Set 5 Carry out the indicated operations and simplify. [This means to expand all parentheses, combine like factors, and leave all exponents positive.]

1. Multiply: $(4x^2y^3)(6x^4y)$
2. Rewrite using only positive exponents: $\dfrac{1}{x^{-3}}$
3. Rewrite, using only positive exponents: $\dfrac{y^{-4}}{x^{-3}}$
4. Write an equivalent expression without negative exponents: $\dfrac{y^{-4}x^3}{z^{-3}}$
5. Simplify. Leave all exponents in your final answer as positive. $(7x^{-8}y^2)(-9x^3y^{-8})$
6. Simplify. Leave all exponents in your final answer as positive. $(12a^2b^{-4})(-6a^{-4}b^7)$
7. Simplify $\dfrac{7^2 - (-1)^7}{5 \cdot 17 - 9 \cdot 3^2 - 3^2}$
8. Simplify $\dfrac{-2^2 - 5 \cdot (-2)^3}{50 \div 5^2 - (5 - (2 - 19))}$
9. Write an equivalent expression without negative exponents: $\left(\dfrac{2y^{-4}x^3}{w^3z^{-3}}\right)^2$
10. Write an equivalent expression without negative exponents: $(a^2b^{-3}c^2)^2 \cdot (a^{-2}b^5c)$

§5.1 Roots and Radicals

Recall that $\sqrt{a} = c$ if and only if $c > 0$ and $c^2 = a$. In general:

For any even n, $\sqrt[n]{a} = c$ if and only if $c > 0$ and $c^n = a$; for odd values of n, $\sqrt[n]{a} = c$ if and only if $c^n = a$.

Thus, $\sqrt[n]{c^n} = \begin{cases} c & \text{if } n \text{ is odd;} \\ |c| & \text{if } n \text{ is even.} \end{cases}$

Thus, in reality, radicals are exponents. In fact, $\sqrt[n]{a} = a^{1/n}$ and $\sqrt[n]{a^m} = (a^m)^{1/n} = a^{m/n}$

Thus, any property of exponents can be assumed to be a property of roots and radicals.

Conjecture: The notation for radicals and for exponents arose independently, with exponents perhaps coming later.

None-the-less, exponents and radical notation have the relationship above. Many times, particularly in Calculus, the exponential notation is at least more useful if not necessary.

Final Note for §5. While coefficients distribute across sums and differences and exponents distribute across multiplications and divisions (what each sees in their own realm),

Coef: Coefficients do NOT distribute across multiplications and divisions. That is $2(x \cdot y) \neq 2x \cdot 2 \cdot y$ and $2 \cdot \frac{x}{y} \neq \frac{2 \cdot x}{2 \cdot y}$ and likewise

Exp 1: Exponents do NOT distribute across sums and differences. That is $(x+y)^2 \neq x^2 + y^2$ and $(x-y)^2 \neq x^2 - y^2$ (nor is it equal $x^2 + y^2$). Rather $(x+y)^2$ means $(x+y) \cdot (x+y)$, which we'll visit later. This can be seen through the example: $x = 4$ & $y = 3$: $(4+3)^2 \neq 4^2 + 3^2$ and $(4-3)^2 \neq 4^2 - 3^2$ and $(4-3)^2 \neq 4^2 + 3^2$.

Exp 2: As noted above, radicals and roots are essentially exponents, and so $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$. This can be seen through the example $x = 16$ & $y = 9$: $\sqrt{16+9} \neq \sqrt{16} + \sqrt{9}$.

Exp 3: Another offshoot of the fact that exponents do not distribute across sums and differences is: $(x+y)^{-1} \neq x^{-1} + y^{-1}$. Put in more familiar format: $\dfrac{1}{x+y} \neq \dfrac{1}{x} + \dfrac{1}{y}$.
This can be seen through the example: $x = 1$ & $y = 1$: $\dfrac{1}{1+1} \neq \dfrac{1}{1} + \dfrac{1}{1}$

§6 Polynomials

§6.1 Polynomials: Addition & Subtraction

A *Polynomial* is an algebraic expression involving any number of variables added to, subtracted from, or multiplied by real numbers and other variables. Most polynomials that we use will have a single variable such as the following. This is called a polynomial in one variable.

(a) $7x^2 + 3x + \frac{1}{2}$ or (b) $4x + 3x^3 - 5x^4$ or (c) 17

The numbers 7, 3 and $\frac{1}{2}$ in the polynomial $7x^2 + 3x + \frac{1}{2}$ are called its **coefficients**.

The portions separated by + and − signs, such as the $4x$, $3x^3$ and $-5x^4$ are called the **terms** of the polynomial. One might say that each term "**owns**" the sign which precedes it, so it was not an accident that we wrote "$-5x^4$" instead of just "$5x^4$" and if we were really careful we would have written $+4x$, $+3x^3$ and $-5x^4$

The term $+3x^3$ has factors 3 and x^3 bound together by the multiplication between then.

The term without a variable (the term $\frac{1}{2}$ in the polynomial $7x^2 + 3x + \frac{1}{2}$ and the term 0 in the polynomial $4x + 3x^3 - 5x^4 + 0$) is called the **constant term**.

The term whose variable's exponent is largest (the term $7x^2$ in the polynomial $7x^2 + 3x + \frac{1}{2}$ and the term $-5x^4$ in the polynomial $4x + 3x^3 - 5x^4$) is called the **leading term** whether or not it is the first term as the polynomial is read from left to right.

The value of that largest exponent ("2" for the exponent of the leading term $7x^2$ of the polynomial $7x^2 + 3x + \frac{1}{2}$ and "4" for the exponent of the leading term $-5x^4$ in the polynomial $4x + 3x^3 - 5x^4$) is called the **degree** of the polynomial. So the polynomial $4x+3x^3-5x^4$ has degree 4 and the polynomial 17 has degree 0 (because 17 is the same as $17x^0$).

The coefficient of the leading term (the -5 from the leading term $-5x^4$ in the polynomial $4x+3x^3-5x^4$) is called the **leading coefficient**.

Useful convention: The terms of a polynomial can be written in any order:

$$4x+3x^3-5x^4 \quad \text{or} \quad 3x^3-5x^4+4x \quad \text{or} \quad -5x^4+4x+3x^3 \quad \text{or} \quad 4x-5x^4+3x^3 \quad \text{or} \quad 3x^3+4x-5x^4$$

it will be easiest to accomplish the multitude of things we do with polynomials by writing the polynomial in order of decreasing powers of the exponent: $-5x^4+3x^3+4x$ (also referred to as *writing the polynomial in descending order* because the exponents are decreasing from left to right).

Finally, we often refer to polynomials specifically by the number of terms that it has. The polynomial $7x^3$ is called a *monomial* because it has just *one* term, the polynomial $2x+3$ is called a *binomial* because it has *two* terms, and both of the polynomials $7x^2+3x+\frac{1}{2}$ and $4x+3x^3-5x^4$ are called *trinomials* because they have *three* terms. If a polynomial has 4 or more terms, we just call it a *polynomial* (the prefix *poly-* is from the Greek, meaning *many*).

Many times terminology is glossed over because it does not feel like a core idea. While perhaps it isn't a core idea, it is very important to have a common language to use when talking about polynomials or else everything seems more than a little confusing.

Now we will begin to treat polynomials as we do numbers: we add, subtraction, multiply, and find factorizations of them.

Principle for Adding and Subtracting Polynomials:

1. Two polynomial terms are called **like terms** if they have the same variable parts.
2. We add or subtract two polynomials by combining like terms.

If we were to add $5x^2-3x+2$ and $3x^2+7x$ it would look like

$$(5x^2-3x+2)+(3x^2+7x)$$

The polynomial: $5x^2-3x+2$, has 3 terms: $5x^2$ and $-3x$ and 2.

The polynomial: $3x^2+7x$ has 2 terms: $3x^2$ and $5x$.

In computing $(5x^2-3x+2)+(3x^2+7x)$ we combine the like terms: $5x^2$ & $3x^2$, $-3x$ & $7x$, and 2 & 0,

$$\begin{array}{ll} (5x^2-3x+2)+(3x^2+7x) & \\ 5x^2-3x+2+3x^2+7x & \text{expand parentheses} \\ 5x^2+3x^2-3x+7x+2 & \text{identify like terms} \\ (5+3)x^2+(-3+7)x+2 & \text{combine the coefficients} \\ 8x^2+4x+2 & \end{array}$$

Exercise Set 6.1: Carry out the indicated operations and simplify. [This means to expand all parentheses, combine like factors, then combine like terms.]

1. $(x^2-4x+3)+(2x^2+3x-2)$
2. $(x^2-4x+3)-(2x^2+3x-2)$
3. $(x^3-4x+3)+2(4x^3-2x^2+3x-2)$
4. $(x^3-4x+3)-2(4x^3-2x^2+3x-2)$
5. $\left(\frac{1}{3}x^3-\frac{4}{3}x+3\right)+3\left(\frac{1}{6}x^3-\frac{2}{3}x^2+\frac{5}{6}x-\frac{2}{7}\right)$
5. $\left(\frac{1}{3}x^3-\frac{4}{3}x+3\right)-3\left(\frac{1}{6}x^3-\frac{2}{3}x^2+\frac{5}{6}x-\frac{2}{7}\right)$

§6.2 Multiplying Polynomials

A *monomial* is a polynomial with a single term, such as $3x^4$ or $-2x^3$.

Remembering that the x^4 and the x^3 are still just two numbers (we just don't know what they are), then if we were to multiply these two monomials: $(3x^4)\cdot(-2x^3)$ then we could use associativity and commutativity to rearrange the two constants side–by–side and the two variable factors side–by–side:

$$(3\cdot x^4)\cdot(-2\cdot x^3) \quad = \quad ((3)\cdot(-2))\cdot\big((x^4)\cdot(x^3)\big)$$

and then since we know that $3\cdot(-2)=-6$ and $x^4\cdot x^3 = x^{4+3} = x^7$ (because when we multiply two factors with a like base, we add the exponents). Thus

$$(3\cdot x^4)\cdot(-2\cdot x^3) \quad = \quad ((3)\cdot(-2))\cdot\big(x^4\cdot x^3\big) \quad = \quad -6x^{3+4} \quad = \quad -6x^7$$

We should take a moment to clarify what we mean when we say "simplify".

When we say *"Simplify the following expression"* we now have another item to add: Combine like factors.

- Expand Parentheses (almost always done first)
- Resolve cumbersome expressions such as double-operations $2-(-3)$ or $2+(-5)$ (do right after you see them)
- Combine like factors
- Combine like terms (probably done after 'combine like factors' because of the order of operations).

Suppose we were to expand these parentheses: $3(2x+5)$

Remembering that the $2x$ is just another number, we could use the distributive property:

$$3\cdot(2x+5) \quad = \quad 3\cdot(2x)+3\cdot 5$$

We would use the distributive property because those parentheses are 'holding out' the multiplication by 3, and the distributive property indicates how that multiplication by 3 would act upon the $2x+5$ if the parentheses were no longer there. Now we can combine like factors: $3\cdot(2x) = (3\cdot 2)\cdot x$ (which was really just associativity, right?) Hence:

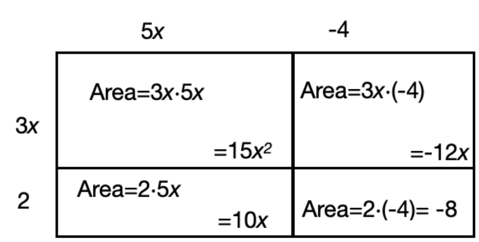
$$3\cdot(2x+5) \quad = \quad 3\cdot(2x)+3\cdot 5 \quad = \quad (3\cdot 2)\cdot x+15 \quad = \quad 6x+15$$

If you had to multiply the two polynomials: $(3x+2)(5x-4)$ it is an extension of the distribution problem above.

$$\begin{aligned}(3x+2)\cdot(5x-4) &= 3x\cdot(5x-4)+2\cdot(5x-4)\\ &= 3x\cdot 5x+3x\cdot(-4)+2\cdot 5x+2\cdot(-4)\\ &= 15x^2-12x+10x-8\\ &= 15x^2-2x-8\end{aligned}$$

However, before we go any further, we are going to try to build a sense of the problem that will help the method of solution seem intuitive and memorable. Thus, the following is NOT a procedure for doing the problem.

Recall, that $(3x+2)(5x-4)$ means $(3x+(2))\cdot(5x+(-4))$



If we were to view the area model for multiplication, then think of a product of those two, which gives the 'area' of the rectangle that these represent. We see the products written inside that rectangle at right. If we add all those products (areas) we get $15x^2+(-12x)+10x+(-8)$. This would be the sum of the 'areas' that make up the large rectangle. The 'area' of the entire rectangle could be found by multiplying the 'length' $(3x+2)$ by the 'width' $(5x-4)$. The area of then entire rectangle, $(3x+2)(5x-4)$, is equal to the sum of the area each of the 4 inner rectangles that make it up. Thus,

$(3x+2)\cdot(5x-4) = 15x^2-2x-8$.

A more typical procedure for computing $(3x+2)\cdot(5x-4)$ might be to distribute the leftmost term of the first factor times each of the terms of the second factor, and then move the next term of the first factor, and distribute it times each term of the second factor, etc.

$$\begin{aligned}(3x+2)\cdot(5x-4) &= 3x\cdot 5x + 3x\cdot(-4) \quad + \quad 2\cdot 5x + 2\cdot(-4)\\ &= 15x^2 + (-12x) + 10x + (-8)\\ &= 15x^2 - 2x - 8\end{aligned}$$

Exercise Set 6.2

Multiply the following polynomial factors and write the expressions in standard form.

1. $2x(3x^2-4x-2)$
2. $-5(3x^2-4x-2)$
3. $(2x-5)(3x^2-4x-2)$
4. $(2x-5y)(2x+5y)$
5. $(2x-5y)^2$
6. $(2x-5y)(3x^2-4xy-2y^2)$
7. $(3x^2-2x-5)(x^2-4x-3)$
8. $(3x-2)(2x^2-3x-1)$
9. $(3x-2y)(2x^2-3xy-y^2)$

10 $(3x-2y)(3x+2y)$

11. $(3x-2y)^2$
12. $(2x^2-x-6)(2x^2-3x-1)$

§7. Factoring

Polynomials and many algebraic expressions are sums and differences of terms of various types. There are many times when we would like to express that polynomial as a product (in the Multiplication Realm) rather than as sums and differences. That is why we factor.

§7.1 Removing the greatest common factor The very first stage of any factoring problem is to try to factor (undistribute) common factors. We might call it 'un-distributing' because it is going in the reverse of the usual direction used with the distributive property. Just as:

$$3\cdot(2x+5) = 3\cdot 2x + 3\cdot 5$$

and then we combine like factors to get

$$3\cdot(2x+5) = 3\cdot 2x + 3\cdot 5 = 6x + 15$$

when presented with

$$(6x+15)$$

we look for a common factor of $6x$ and of 15 (in this case, 3:)

$$6x + 15 = (3\cdot 2x + 3\cdot 5)$$

and then since there is a distribution of 3 to each term, we say that we *factor* out the 3 and write;

$$6x + 15 = (3\cdot 2x + 3\cdot 5) = 3\cdot(2x+5)$$

but what we are really doing is rolling back the distributive property. Typically we won't go through this whole process, but that is what is happening behind the scenes.

We need not be limited by what exists in each term, if we want badly enough to factor something out.

Example 7.1.1: Factor a $2x^{-1}$ out of each term: $\quad 2x - \dfrac{9x^{-1}}{2}$

Solution:

$$2x^1 - \frac{9x^{-1}}{2}$$

$$2x^{-1} \cdot \left(x^2 - \frac{9}{4}\right)$$ Multiply it out, to see that this is, indeed, valid.

Later, after we've discussed factoring differences of squares, we can continue to factor this:

$$2x^{-1} \cdot \left(x^2 - \tfrac{9}{4}\right) = 2x^{-1}\left(x - \tfrac{3}{2}\right)\left(x + \tfrac{3}{2}\right) = \frac{2\left(x-\frac{3}{2}\right)\left(x+\frac{3}{2}\right)}{x}$$

Exercise Set 7.1

1. $x^2y^5 - 4x^3y^2$
2. $4y^5 - 12y^2$
3. $12x^5y^4 - 18x^3y^3$
4. $18x^2y^3 - 12x^3y^4 + 24x^2y^3$
5 $12x^{-2}y^4 - 18x^3y^{-3}$
6. $12(2x-3)^{-2}(x+4)^{5/2} - 18(2x-3)^3(x+4)^{-1/2}$
7. $3x(x+2) - 2(x+2)$
8. $2x^2(2x+5) + (2x+5)$

§7.2 Factoring by grouping

The technique of grouping is used often with algebraic expressions with *4 or more terms*. Basically, we (a) break the expression into two groups, (b) factor out the greatest common factor from each group, and (c) if a common factor emerges we factor it out as well; if a known factorization appears we carry it out as well. Exercises 6–8 in the set immediately above, are examples of step (c). It becomes valuable in these problems to be able to view two agreeing expressions in parentheses as each being a single object or factor without worrying about its complexity inside.

Example 7.2.1: Factor the polynomial by grouping (into to groups of 2 terms each...which is the most common):

$5x^2 + 2x + 5xy + 2y$.

$5x^2 + 2x + 5xy + 2y = (5x^2 + 2x) + (5xy + 2y)$ — we grouped the first two with a parentheses and the second two with a parentheses; we

$= x(5x+2) + y(5x+2)$ — factor out the greatest common factor from each pair but we see a common factor appear the two terms

$= (5x+2)(x+y)$ — we factor out that new common factor. It is factored.

Example 7.2.2: Factor the polynomial by grouping (into to groups of 2 terms each...which is the most common):

$5x^2 + 2x - 5xy - 2y$.

$5x^2 + 2x - 5xy - 2y = (5x^2 + 2x) - (5xy + 2y)$ — we grouped the first two with a parentheses and the second two with a parentheses but the parentheses holds out a negative sign. How do we compensate?

$= x(5x+2) - y(5x+2)$ — We factor out the greatest common factor from each but we see a common factor appear.

$= (5x+2)(x-y)$ — We factor out that new common factor. It is factored.

Exercise Set 7.2: Factor the following by grouping into two pairs of two terms each.

1. $x^2y - 5xy + 2x - 10$
2. $x^2y^2 - xy^2 + 3x - 3$
3. $x^2y^2 + xy^2 - 3x - 3$
4. $xy^2 + 5y^2 - 5bx - 25b$

§7.3 Factoring trinomials by the AC method

Factoring trinomials is what likely comes to mind whenever we hear the word 'factoring', and we probably also think of quadratic polynomials. A quadratic polynomial can be written in the form

$$Ax^2 + Bx + C$$

Many people factor these by trial and error, and for trinomials with leading coefficient '1', this is often easy enough to do. However, in this booklet, we will always use what is called the AC method. Why? Because the AC method will tell you (almost) immediately whether or not the trinomial is factorable, and will reliably lead to the correct factorization, with even a hint or two along the way as to whether or not you have made an error. While it can take more time than a good trial & error guess, it will also save you a lot of time in many other circumstances.

Example 7.3.1: Factor, using the AC method. $2x^2 - 7x - 15$.

Comparing this trinomial to the general quadratic form:

$Ax^2 + Bx + C$
$2x^2 - 7x - 15.$

In this form, we see $A = 2$, $B = -7$, and $C = -15$. The method gets its name by multiplying A and C. $A \cdot C = -30$, and then we look for two factors of AC (that is: -30) which add to B (that is: -7). Since AC is negative in this case, exactly one of the factors must be negative.

$1 \cdot (-30)$ $\quad 1 + (-30) = -29$
$2 \cdot (-15)$ $\quad 2 + (-15) = -13$
$3 \cdot (-10)$ $\quad 3 + (-10) = -7$
$5 \cdot (-6)$ $\quad 5 + (-6) = -1$

The above is a plodding means to find the two factors, but it will eventually lead to success. If you happen to see the correct pair sooner, you can jump to it when you see it rather than go through all these possibilities.

In this case the pair is 3 and -10. This tells us that the middle term can be decomposed into $\quad -7x = 3x - 10x$
We make this replacement and then factor the resulting 4-term polynomial by grouping.

$2x^2 - 7x - 15$

$2x^2 + 3x - 10x - 15$

$(2x^2 + 3x) - (10x + 15)$ $\qquad$ the signs change from $-10x - 15$ to $-(10x + 15)$
because the parens. hold out a '-' sign

$x \cdot (2x + 3) - 5 \cdot (2x + 3)$

$(2x + 3) \cdot (x - 5)$

The following example is way overblown, for the purpose of demonstrating why the AC works.

$$\begin{aligned}
(2x+3)(x-5) &= 2x \cdot x + 2x \cdot (-5) + 3 \cdot 1x + 3 \cdot (-5) \\
&= (2 \cdot 1)x^2 + (2 \cdot (-5)) \cdot x + (3 \cdot 1) \cdot x + 3 \cdot (-5) \\
&= (2 \cdot 1)x^2 + ((2 \cdot (-5)) + ((3) \cdot 1)) \cdot x + ((-3) \cdot 5) \\
&= 2x^2 + ((-10) + 3)x + (3 \cdot (-5)) \\
&= 2x^2 - 7x - 15
\end{aligned}$$

In the middle step, look at all the numerical factors which result in A and C: $2, 1, 3, (-5)$. They each appear as factors in the two coefficients that result in B, but in a different order: $-7 = -10 + 3 = 2 \cdot (-5) + 3 \cdot 1$.

Here is another example.

Example 7.3.2: Factor $2x^2 - 11x - 21$ by the AC method.

$A \cdot C = 2 \cdot (-21) = -42$. We need two factors of -42 which add to -11.

$1 \cdot (-42)$ $\quad 1 + (-42) = -41$
$2 \cdot (-21)$ $\quad 2 + (-21) = -19$
$3 \cdot (-14)$ $\quad 3 + (-14) = -11$

and so that $-11x = 3x - 14x$.

$2x^2 - 11x - 21$

$2x^2 + (3 - 14)x - 21$ we used $(3 - 14)$ because we wanted $-11x$

$2x^2 + 3x - 14x - 21$

$(2x^2 + 3x) - (14x + 21)$ we had to account for the '-' held out by parentheses

$x(2x + 3) - 7(2x + 3)$

$(2x + 3)(x - 7)$

We will do a few additional examples more briefly, In particular, we will just select the pair of factors that we're looking for without illustrating it here.

Example 7.3.3: Factor $x^2 - 4x - 21$ by the AC method.

$A \cdot C = 1 \cdot (-21) = -21$. We need two factors of -21 which add to -4. The factors $3 \cdot (-7) = -21$ should work. So $-4x = 3x - 7x$.

$x^2 - 4x - 21$

$x^2 + (3 - 7)x - 21$ we used $(3 - 14)$ because we wanted $-11x$

$x^2 + 3x - 7x - 21$

$(x^2 + 3x) - (7x + 21)$ we had to account for the '-' held out by parentheses

$x(x + 3) - 7(x + 3)$

$(x + 3)(x - 7)$

Notice that the two factors of -21 also showed up in the final factors of the polynomial:

$(x + 3)(x + (-7)) = (x + 3)(x - 7)$.

In fact, this is always true when the leading coefficient of the trinomial is 1. See the next example:

Example 7.3.4: Factor $x^2 - 12x + 35$ by the AC method.

$A \cdot C = 1 \cdot (35) = 35$. We need two factors of 35 which add to -12. The factors $(-5) \cdot (-7) = -35$ should work.

This time, let's just try: $(x - 5)(x - 7)$ and see if we get back $x^2 - 12x + 35$

$(x - 5)(x - 7) = x^2 - 7x - 5x + 35 = x^2 - 12x + 35$

We didn't even go through the whole AC method. We must be careful, however, because the pair of factors of $A \cdot C$ only directly give us the factors of the trinomial if the trinomial has leading coefficient 1. We'll do one last example.

Example 7.3.5: Factor $4x^2 - 13x - 12$ by the AC method.

$A \cdot C = 4 \cdot (-12) = -48$. We need two factors of -48 which add to -13. The factors $(3) \cdot (-16) = -48$ should work.

$4x^2 - 13x - 12$

$x^2 + (3 - 16)x - 21$ we used $(3 - 14)$ because we wanted $-11x$

$4x^2 + 3x - 16x - 12$

$(4x^2 + 3x) - (16x + 12)$ we had to account for the '-' held out by parentheses

$x(4x + 3) - 4(4x + 3)$

$(4x + 3)(x - 4)$

Exercise Set 7.3: Factor the following trinomials using the AC method.

1. $2x^2 + 11x + 12$
2. $2x^2 - 11x + 12$
3. $2x^2 + 5x - 12$
4. $2x^2 - 5x - 12$
5. $6x^2 + 19x + 10$
6. $6x^2 - 19x + 10$
7. $6x^2 + 11x - 10$
8. $6x^2 - 11x - 10$
9. $12x^2 + 22x - 20$
10. $12x^2y - 22xy - 20y$

Warning!!!

1. If a quadratic is missing the constant term (such as with $2x^2+16x$) then remove the greatest common factor; it will not factor as a product of two binomials.
2. If a quadratic is missing the middle term (such as with $2x^2 - 8$) then see the special factors section below.

§7.4 Special Factors and Products

There are a few products which are particularly useful and recognizable for factoring.

Difference of Squares $(a - b) \cdot (a + b) = a^2 - b^2$.
[Just multiply it out and watch the two middle terms cancel.]

Exercise Set 7.4a: Expand the following products on sight, and then check yourself by multiplying them out directly.

1. $(3x - 4y) \cdot (3x + 4y)$
2. $(2x + 5y) \cdot (2x - 5y)$

Going in reverse can be approximately as automatic. The difference of two squares comes from such a product, so if asked to factor: $4x^2 - 49$, you might recognize it as $(2^2x^2 - 7^2)$ and it is reasonable to conclude that it factors as $(2x - 7)(2x + 7)$.

Exercise Set 7.4b: Factor the following binomials on sight, and then check yourself by multiplying them out.

1. $25x^2 - 4$
2. $25x^2 - 16y^2$

Difference of Cubes $(a - b) \cdot (a^2 + ab + b^2) = a^3 - b^3$.
[Just multiply it out and watch the four middle terms cancel.]

We do not use this to multiply our polynomials, but we do use it for factoring. If we see a difference of two cubes, then this is our means of factoring it. It is not so automatic as factoring the difference of squares (above), but it has to be committed to memory in some way. Notice how the two end terms of the trinomial factor are the squares of the two terms of the binomial factor; the middle term is the -(product of the two terms of the binomial factor).

Example 7.4.1: Factor $8x^3 - y^3$.

We notice that 8 is a perfect cube as well. So

$$8x^3 - y^3 = 2^3x^3 - y^3 = (2x - y) \cdot ((2x)^2 + 2xy + y^2) = (2x - y) \cdot (4x^2 + 2xy + y^2).$$

Exercise Set 7.4c: Factor the following binomials on sight, and then check yourself by multiplying them out.

1. $27x^3 - 8$
2. $64y^3 - 27x^3$

Sums of Cubes $(a + b) \cdot (a^2 - ab + b^2) = a^3 + b^3$.
[Just multiply it out and watch the four middle terms cancel.]

As with the difference of squares do not use this to multiply our polynomials, but we do use it for factoring. If we see a sum of two cubes, then this is our means of factoring it. Notice how the two end terms of the trinomial factor are the squares of the two terms of the binomial factor; the middle term is the -(product of the two terms of the binomial factor). Compare it to the factoring of the difference of cubes.

Example 7.4.2: Factor $8x^3 + y^3$.

We notice that 8 is a perfect cube as well. So

$$8x^3 + y^3 = 2^3x^3 + y^3 = (2x + y) \cdot ((2x)^2 - 2xy + y^2) = (2x + y) \cdot (4x^2 - 2xy + y^2).$$

Exercise Set 7.4d: Factor the following binomials on sight, and then check yourself by multiplying them out.

1. $27x^3 + 8$
2. $64y^3 + 27x^3$

Sums of Squares We cannot factor sums of squares using real number coefficients. Try the always-definitive AC method.

§7.5 Special Products: Finally, there are a few special products that will be helpful to be able to recognize when we are later doing completing the square.

Squares of Binomials: Above we looked at the product of two binomials which differed only by a sign:

$$(a-b)\cdot(a+b) = a^2 + ab - ab - b^2$$

and noted that the two middle terms were, except for the sign, identical; thus they cancelled out. From that came our handy: $(a-b)\cdot(a+b) = a^2 - b^2$.

If, on the other hand, the signs are the same, then those middle terms double-up.

$$(a-b)^2 = (a-b)\cdot(a-b) = a^2-ab-ab-b^2 = a^2-2ab+b^2 \qquad (a+b)^2 = (a+b)\cdot(a+b) = a^2+ab+ab+b^2 = a^2+2ab+b^2$$

This is a useful pattern to remember for (a) if we want to create a square of a binomial, or (b) to be able to immediately recognize and factor a square of a binomial.

§8. Rational Expressions

A *Rational Expression* is a quotient of two polynomials. The discussion is very much like that of rational numbers in §2.

§8.1 Reducing Rational Expressions

Much as with fractions of integers, if we are able to find a common factor in both the numerator and denominator, we can 'cancel' or remove the common factor. There is a slight difference, though. Consider the following problem:

Simplify: $\dfrac{2x^2 + 11x + 12}{2x^2 + 5x - 12}$

$$\frac{2x^2 - 11x + 12}{2x^2 - 5x - 12} = \frac{(2x-3)\cdot(x-4)}{(2x+3)\cdot(x-4)} = \frac{2x-3}{2x+3} \qquad \text{as long as } x \neq 4.$$

The inclusion of "$x \neq 4$" might be somewhat of a picky point, but if $x = 4$, then removing the factor $(x-4)$ by calling $\frac{x-4}{x-4} = 1$ falls apart because is says $\frac{0}{0} = 1$, when $\frac{0}{0}$ is actually not defined.

Warning!!!: A fairly common error is to cancel a term common to both the numerator and denominator $\left(\text{such as } \frac{\cancel{2x^2}-11x+12}{\cancel{2x^2}-5x-12} \Rightarrow \frac{-11x+12}{-5x-12}\right)$. Whenever you are getting ready to use the word *cancel*, always consider what is cancelling what. Here, the student cancelling in this way believes they are cancelling the $2x^2$ through division. However, the $+2x^2$ is *added* to each the numerator and denominator, and thus would only cancelled by a $-2x^2$ in the numerator and another $-2x^2$ in the denominator. Division does not cancel addition (the $+2x^2$), but rather division cancels multiplication. Therefore, in the example above, we first factored both numerator and denominator so that both numerator and denominator were essentially expressions of multiplication. Usually that is why we factor an expression: to create a multiplication out of what used to be sums and differences.

Exercise Set 8.1

1. Simplify: $\dfrac{6x^2 + 19x + 10}{6x^2 + 11x - 10}$

§8.2 Multiplying and Dividing Rational Expressions

As with rational numbers:

1. When we multiply two rational expressions, we multiply straight across (and then reduce as above).

   For instance: $\dfrac{x-2}{x^2+1} \cdot \dfrac{x+1}{2x+3} = \dfrac{(x-2)\cdot(x+1)}{(x^2+1)\cdot(2x+3)}$ we must put parens. around 'the numerator' and 'the denominator', to recongize the implicit grouping of being the numerator and denominator

2. When we divide two rational expressions, we multiply by the multiplicative inverse (often called the *reciprocal*) of the denominator (and then reduce as above).

   For instance: $\dfrac{\frac{x-2}{x^2+1}}{\frac{x+1}{2x+3}} = \dfrac{(x-2)}{(x^2+1)} \cdot \dfrac{(2x+3)}{(x+1)} = \dfrac{(x-2)\cdot(2x+3)}{(x^2+1)\cdot(x+1)}$

§8.3 Adding and Subtracting Rational Expressions

As with the addition and subtraction of rational numbers, we must create equivalent fractions which share the same denominator. In particular (in the case of two summands), we converted each to the same unit by identifying which factors of one denominator are missing from the factors of the second denominator, and then multiplying that second fraction by "1" in the form of $\dfrac{\text{missing factors}}{\text{missing factors}}$.

Example 8.3.1: Add $\frac{2x-1}{x^2-2x-3} + \frac{x+1}{2x^2-6x}$

$$\begin{aligned}
\frac{2x-1}{x^2-2x-3} + \frac{x+1}{2x^2-6x} &= \frac{2x-1}{(x-3)\cdot(x+1)} + \frac{x+1}{2x\cdot(x-3)} \\
&= \frac{2x-1}{(x-3)\cdot(x+1)} \cdot \frac{(2x)}{(2x)} + \frac{x+1}{2x\cdot(x-3)} \cdot \frac{(x+1)}{(x+1)} \\
&= \frac{(2x-1)\cdot(2x)+(x+1)^2}{(x-3)\cdot(x+1)\cdot(2x)} \\
&= \frac{4x^2-2x+x^2+2x+1}{(x-3)\cdot(x+1)\cdot(2x)} \\
&= \frac{5x^2+1}{(x-3)\cdot(x+1)\cdot(2x)}
\end{aligned}$$

we must put parens. around $(2x-1)$ and $(x+1)$ to recognize the grouping 'this was the numerator'

and we leave the denominator factored throughout so that if the final numerator factored, we would be ready to cancel like factors. However $(5x^2+1)$ does not factor.

Exercise Set 8.3: Compute the following Sums and Differences.

1. $\dfrac{2x-1}{x^2-2x-3} + \dfrac{x+1}{2x^2-3x-5}$
2. $\dfrac{2x-1}{x^2-2x-3} - \dfrac{x+1}{x^2-9}$
3. $\dfrac{2x-1}{x^2-2x-3} - \dfrac{x+1}{2x^2-3x-5} - \dfrac{x}{x^2-9}$
4. $\dfrac{2x-1}{x^2-2x-3} - \dfrac{2x-1}{x^2-1}$

§9. The Cartesian Coordinate Plane

The number line has served as our representative of the real number system, and this number line is sufficient to graphically express the solutions to equations and inequalities in one variable (although, as mentioned in the preface, we haven't talked about equations or inequalities yet in this book). If we want to be able to visualize the solutions to equations of two variables, we will need a second number line to represent the real number system for the second variable. Our usual pair of variables is x and y, with x used for the independent variable (whose value we typically choose) and with y used for the dependent variable (whose values are determined based upon the value that has already been chosen for x). The Cartesian Coordinate Plane puts these two number lines perpendicular to each other, intersecting at the point "0" on each line, with the number line representing the variable x being horizontal (and called the x-axis) and the number line representing the variable y being vertical (and called the y-axis), with the negative portion downward. Your instructor will probably have you graph some basic functions by plotting points, and after we introduce piecewise functions, we will graph one of those. Otherwise, we will not go any more deeply into the Cartesian Coordinate System except to say that if we had an equation of 3 variables, we would need to add a 3rd number line, perpendicular to the other two.

§10. Functions

§10.1 Essential Facility with Functions

A relation is a set of order pairs (x, y), or points in the Cartesian plane. The points in the Cartesian plane which correspond to all points of the relation is called the *graph* of the relation. The set of all the first coordinates, x, is called the **Domain** of the relation and the set of all second coordinates, R, is called the **Range** of the relation. A relation is called a *function* if for each value of the variable x in the Domain, there is exactly one value of the variable y in the Range. Thus, the graph of a function f can be viewed as the set of all points $(x, f(x))$ for all values of x in the Domain of the function. In the case that this functional relationship is such that y is writable as an expression in terms of x, then we use the function notation: $y = f(x) = \textit{the expression}$. The graph of the function is therefore the set of all ordered pairs $(x, f(x))$ in the Cartesian plane.

Most of our work with functions in this short class is in the proper use of $f(x) = \textit{the expression}$.

Example 10.1.1 $f(x) = \dfrac{2x+1}{x-5}$

That notation tells us: $y = \frac{2x+1}{x-5}$ is a function with independent variable x and dependent variable y. Input values for the variable x are chosen and substituted into the expression $\left(\frac{2x+1}{x-5}\right)$, whose value is the output (y). Within the function notation, this is handled in a rather seamless way.

For instance,

Compute $f(3)$.

Solution: $f(3) = \frac{2\cdot 3+1}{3-5} = \frac{6+1}{-2} = -\frac{7}{2}$

The very notation $f(3)$ says that $x = 3$ is the input value and that it is to be substituted into each occurrence of x in the expression which is associated with f. We deduce that when $x = 3$ then $y = -\frac{7}{2}$.

Before we get very far, we need to talk about the domain of a function. The domain of the function is (if you refer back to the definition above) the collection of all the first (or x) coordinates of the ordered pairs $(x, f(x))$ of the graph of the function. That is, according to the expression, $f(x)$ must exist as a real number (since it is plotted against the number line known as the y-axis). For the function in the example above, $f(5)$ is not defined because computing $f(5)$ would involve division by zero (see the special note after §5). However, we are able to compute $f(x)$ for any other real number. Thus, the domain of this function f is $(-\infty, 5) \cup (5, \infty)$.

Over the full catalog of functions that we see, the following comprise the only computations which do not result in real numbers.

1. $\frac{a}{0}$ for any real number a.
2. When n is an even natural number, $\sqrt[n]{c}$ for any negative number c.
3. $\log_b(0)$
4. $\log_b(c)$ for any negative number c.

Examples 10.1.2 Given the function $f(x) = \frac{x}{2x+1}$

1. Find the domain of f

   The only of the above which are relevant is #1. Thus,

   $2x + 1 = 0$ means that $x = \frac{-1}{2}$

   The domain is $\left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{-1}{2}, \infty\right)$

2. Compute $f(a)$

   $f(a) = \frac{a}{2a+1}$

3. Compute $f(x+h)$

$$f(x+h) = \frac{x+h}{2(x+h)-1} = \frac{x+h}{2x+2h-1}$$

4. Compute $\frac{f(x+h)-f(x)}{h}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{2(x+h)-1} - \frac{x}{2x-1}}{h}$$

and now simplify.

Exercise Set 10.1: For each of the functions below, find its domain, compute $f(a)$ and $f(x+h)$, and compute and simplify $\frac{f(x+h)-f(x)}{h}$.

1. $f(x) = 3x - 2 - 5x^2$
2. $f(x) = \frac{1}{x-1}$
3. $f(x) = \sqrt{2x+1}$
4. $f(x) = \frac{1}{\sqrt{2x+1}}$

Final Note for §10.1:

I. Note that $f(x+h) \neq f(x) + h$.

Let us look at what this means for a specific function: $f(x) = x^2$.

The notation $f(x+h)$ says to substitute $(x+h)$ *into* the expression x^2. That is:

$$f(x+h) = (x+h)^2 = (x+h)\cdot(x+h) = x^2 + 2xh + h^2$$

The notation $f(x) + h$ says to add h to $f(x)$.

$$f(x) + h = x^2 + h$$

It is clear that $x^2 + 2xh + h^2 \neq x^2 + h$. Similarly, $f(x+h) \neq f(x) + h$.

II. Almost always: $f(x+y) \neq f(x) + f(y)$. The addition sign on the input into the function f entices us to consider that possibility, but functions can have so many different possible actions on inputs that any kind of relationship is very special to the type of function. This can be seen through the example: $f(x) = x + 1$, with $x = 1$ & $y = 2$. Note that $f(1+2) \neq f(1) + f(2)$; $f(1+2) = f(3) = 4$, but $f(1) + f(2) = 2 + 3 = 5$.

§10.2. Piecewise Functions

Example 10.2.1: $f(x) = \begin{cases} x^2+1 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ \frac{1}{x-1} & \text{if } x > 2 \end{cases}$ is called a piecewise function because it uses one of several different expressions, depending upon which portion of its domain the value of x comes from.

The above notation is shorthand for saying that

$f(2) = 3$; and $f(x) = x^2 + 1$ for x in $(-\infty, 2)$; and $f(x) = 4 - x$ for x in $(2, \infty)$.

Hence: $f(6) = \frac{1}{6-1} = \frac{1}{5}$ and $f(0) = 0^2 + 1 = 1$.

§10.3. Composition of Functions, and related ideas

As with numbers, we can add, subtract, multiply, and divide functions to get new ones. However, we can also compose functions; that is, we can apply functions to each other.

Definition: We define: $(f \circ g)(x) \equiv f(g(x))$ and we read that as "f composed with g of x is f of $g(x)$". Similarly, $(g \circ f)(x) \equiv g(f(x))$.

Example 10.3.1: Given $f(x) = 2x + 1$ and $g(x) = x^2 - 1$, compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x) \equiv f(g(x)) = f(x^2 - 1) = 2(x^2 - 1) + 1 = 2x^2 - 2 + 1 = 2x^2 - 1$. That is, $(f \circ g)(x) = 2x^2 - 1$.

$(g \circ f)(x) \equiv g(f(x)) = g(2x+1) = (2x+1)^2 - 1 = (2x+1)\cdot(2x+1) - 1 = 4x^2 + 4x + 1 - 1 = 4x^2 + 4x$.

Exercise Set 10.3 Compute $(f \circ g)(x)$ and $(g \circ f)(x)$ for each of the following pairs of functions.

1. $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$
2. $f(x) = \frac{x}{x-1}$ and $g(x) = \frac{x}{x-1}$
3. $f(x) = 2x + 3$ and $g(x) = \frac{x-3}{2}$
4. $f(x) = \frac{2x}{x+2}$ and $g(x) = \frac{2x}{2-x}$

You might have noticed that for each of these pairs, f and g, both $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. That is not typical, as you know from the earlier example. This occurs precisely when the functions f and g are inverse functions of one another. That is, the action of either of these functions reverses the action of the other. Just checking a few examples: from #1, $f(2) = 9$, and $g(9) = 2$. From #2, $f(3) = 3/2$, and $g(3/2) = 3$. From #3, $f(8) = 19$ and $g(19) = 8$. From #4, $f(2) = 1$ and $g(1) = 2$. We chose specific cases, but the same would have been true if we had started with any x in the domain of f. Also, we began with f in each of these cases. The same would have occurred if we had begun with g and with any element of the domain of g. For instance, looking at #4. The number 2 is not in the domain of g, so we cannot use 2. However, $x = 4$ is in the domain of g and $g(4) = -4$. Computing $f(-4)$ we get $f(-4) = 4$.

Inverse functions will prove useful when we solve equations in §12.

§10.4. The Graphing of Functions

Example 10.4.1: Graph the piecewise function from §10.2. That is, graph $y = f(x) = \begin{cases} x^2+1 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ \frac{1}{x-1} & \text{if } x > 2 \end{cases}$

Recall from the opening of this section that the graph of a function can be viewed as the set of all points $(x, f(x))$ for all values of x in the Domain of the function. It is almost always a good idea to determine the domain of a function before doing anything with it. Taking a quick glance at the expressions used by the functions (any division by zero, any even roots of negative numbers, any logarithms of non-positive numbers), the only possible issue which we detect is that there is a *potential of* division by zero when $x = 1$. However, this is a false alarm because $f(x) = \frac{1}{x-1}$ is only used for x in $(2, \infty)$ and $x = 1$ is not in that interval. Thus, the domain of f is $(-\infty, \infty)$.

Which x do we choose:

- 3 or 4 values of x which are in the interval $(-\infty, 2)$ and thus invoke the use of $f(x) = x^2 + 1$. Suppose $x = -3, -2, -1, 0, 1$. Also (because we're missing many inputs, x: $1 < x < 2$), let's plot $(2, 2^2 + 1)$ as an open circle.

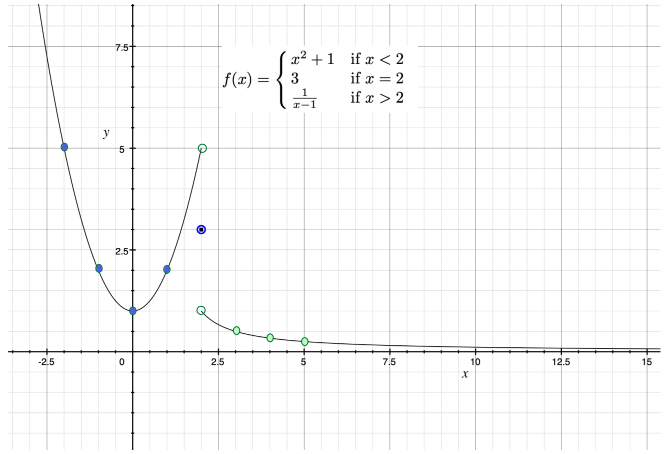
This would look like:

x	$f(x) = x^2 + 1$	y	point
-3	$f(-3) = (-3)^2 + 1$ $= 9 + 1$	10	$(-3, 10)$
-2	$f(-2) = (-2)^2 + 1$ $= 4 + 1$	5	$(-2, 5)$
-1	$f(-1) = (-1)^2 + 1$ $= 1 + 1$	0	$(-1, 2)$
0	$f(0) = (0)^2 + 1$ $= 0 + 1$	1	$(0, 1)$
1	$f(1) = (1)^2 + 1$ $= 1 + 1$	2	$(1, 2)$
	Plot the point below as an open circle		
2	$f(2) = (2)^2 + 1$ $= 4 + 1$	5	$(2, 5)$

- Since $f(2) = 3$, plot the point $(2, 3)$.
- Now we'll choose a few values of x in the interval $(2, \infty)$. Suppose, $x = 3, 4, 5$. Again, because we're missing many inputs $2 < x < 3$, lets plot $(2, \frac{1}{2-1})$ as an open circle.

This would look like:

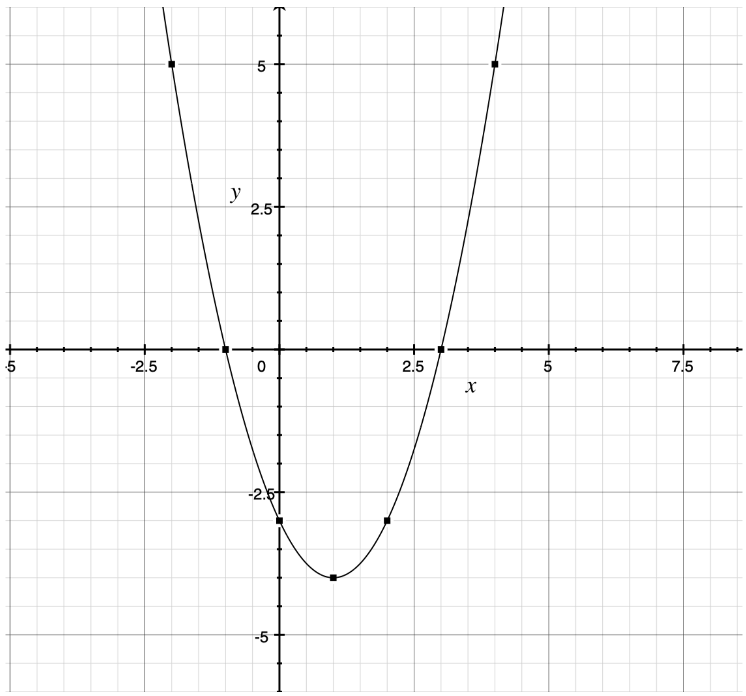
x	$f(x) = \frac{1}{x-1}$	y	point
	Plot the point below as an open circle		
2	$f(2) = \frac{1}{2-1} = \frac{1}{1}$	1	$(2, 1)$
	Plot the remaining points as usual		
3	$f(3) = \frac{1}{3-1} = \frac{1}{2}$	$\frac{1}{2}$	$\left(3, \frac{1}{2}\right)$
4	$f(4) = \frac{1}{4-1} = \frac{1}{3}$	$\frac{1}{3}$	$\left(4, \frac{1}{3}\right)$
5	$f(5) = \frac{1}{5-1} = \frac{1}{4}$	$\frac{1}{5}$	$\left(5, \frac{1}{4}\right)$



Plotting all those points and connect those to the left of $x = 2$ from left to right, and connecting those to the right of $x = 2$ from left to right, we get the graph at right

Example 10.4.2: Graph the function $f(x) = x^2 - 2x - 3$

x	$f(x) = x^2 - 2x - 3$	y	point
-3	$f(-3) = (-3)^2 - 2(-3) - 3$ $= 9 + 6 - 3$	12	$(-3, 12)$
-2	$f(-2) = (-2)^2 - 2(-2) - 3$ $= 4 + 4 - 3$	5	$(-2, 5)$
-1	$f(-1) = (-1)^2 - 2(-1) - 3$ $= 1 + 2 - 3$	0	$(-1, 0)$
0	$f(0) = (0)^2 - 2(0) - 3$ $= 0 + 0 - 3$	-3	$(0, -3)$
1	$f(1) = (1)^2 - 2(1) - 3$ $= 1 - 2 - 3$	-4	$(1, -4)$
2	$f(2) = (2)^2 - 2(2) - 3$ $= 4 - 4 - 3$	-3	$(2, -3)$
3	$f(3) = (3)^2 - 2(3) - 3$ $= 9 - 6 - 3$	0	$(3, 0)$
4	$f(4) = (4)^2 - 2(4) - 3$ $= 16 - 8 - 3$	5	$(4, 5)$



Useful Note about the graph of Quadratic Functions: You probably recognize the graph of the function above to be a parabola, and probably you expected a graph of this shape when you saw the expression for the function: $f(x) = x^2 - 2x - 3$

In the case of this graph, while choosing a few almost-random points, we just happened upon the vertex which is so characteristic of the graph. We can actually find the vertex in advance and that will be very helpful in getting a nice graph: we know the maximum or minimum of the graph, and we know the equation of the vertical line across which the graph has symmetry. This will also tell us which points we should plot in order to get a nice graph more efficiently.

There are two ways to find the vertex of a quadratic function.

By Formula: The x-coordinate of the vertex is $x = \frac{-b}{2a}$ and once we have the x-coordinate, we can plot that point as above. For instance, with the quadratic function above, $a = 1$, $b = -2$, and $c = -3$, so $x = \frac{-b}{2a} = \frac{-(-2)}{2 \cdot 1} = 1$. We can plot the point corresponding to $x = 1$ and (as we did above) get $(1, -4)$. Plot that point on the graph above. Also sketch a dashed-line for the line $x = 1$ (the line of symmetry). After that, all we would have to plot would be $(2,-3)$ (and then from that plot the point $(0,-3)$ one unit on the *other* side of the line of symmetry), and plot $(3,0)$ (and then from that plot the point $(-1,0)$ two units on the *other* side of the line of symmetry), and then the point $(4,5)$ (and then from that plot the point $(-2,5)$ one unit on the *other* side of the line of symmetry). This took a lot less work and we got the same graph and were more sure of its shape from the outset.

By Completing the square: We will talk more about the method of completing the square in §12. The idea is that we have $f(x) = x^2 - 2x - 3$ which we want to write as

$$f(x) = x^2 - 2x - 3 = x^2 - 2x + (1 + (-1) - 3) = (x^2 - 2x + 1) + (-1 - 3) = (x-1)^2 - 4$$

From seeing the function rewritten in this form ($f(x) = (x-1)^2 - 4$), we can see that the vertex is at $(1,-4)$. Since $(x-1)^2$ is never negative, then the function is smallest when $(x-1)^2 = 0$, which is at $x = 1$, and when $x = 1$, the function is -4.

How did we know how to complete $x^2 - 2x$ to the perfect square $x^2 - 2x + 1$ by adding 1 (and then subtracting 1 back again to keep the function at its original value)? There are several ways of seeing this.

The standard way that most people have learned is that, when the leading coefficient is 1, to take half of the middle term and square it. In this case that would be (half of (-2))^2, which is $(-1)^2$ or 1. So we added 1 and then subtracted it back off.

Another way is to think of what d could be in the square: $(x-d)^2$ to give us what we have.

$$\begin{aligned} & x^2 - 2 \cdot x + M \\ (x-d)(x-d) &= x^2 - 2d \cdot x + d^2 \end{aligned}$$

For $-2d = -2$, then $d = 1$. Thus, we need to match $(x-1)(x-1) = x^2 - 2x + 1$; that is why we add one and then subtract it back off.

Example 10.4.3: Find the vertex and line of symmetry of $f(x) = 2x^2 - 5x - 3$ and then plot 3 additional points on one side of the line of symmetry, and sketch its graph.

We will find the vertex in both ways. The x-coordinate of the vertex is $x = \frac{-(-5)}{2\cdot 2} = \frac{5}{4}$. We compute $f\left(\frac{5}{4}\right) = \frac{-49}{8}$. Thus, the vertex of this quadratic function is $\left(\frac{5}{4}, \frac{-49}{8}\right)$.

Finding the vertex the other way, we will need to complete the square. Thus, we will factor out the leading coefficient from the first two terms so that we get a leading coefficient of 1. That leaves the middle coefficient: $\frac{-5}{2}$, and half of that is $\frac{-5}{4}$ and it is that square that we both add and subtract, *but inside the parentheses!*.

$$\begin{aligned} f(x) = 2x^2 - 5x - 3 = 2\left[x^2 - \frac{5}{2}x\right] - 3 &= 2\left[\left(x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2\right) - \left(\frac{5}{4}\right)^2\right] - 3 \\ &= 2\left(x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2\right) - 2\left(\frac{5}{4}\right)^2 - 3 \\ &= 2\left(x - \frac{5}{4}\right)^2 - 2\cdot\frac{25}{16} - 3 \\ &= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} - \frac{24}{8} \\ &= 2\left(x - \frac{5}{4}\right)^2 - \frac{49}{8} \end{aligned}$$

Thus, the vertex is at $\left(\frac{5}{4}, -\frac{49}{8}\right)$. This computation looks a bit forbidding, but it is really a matter of being careful.

The line of symmetry is $x = \frac{5}{4}$, and so plotting 3 points the the right of $\frac{5}{4}$, we'd plot the points corresponding to $x = 2$, $x = 3$, and $x = 4$. These points are $(2,-5)$, $(3,0)$, $(4,9)$. Using symmetry, this would give us the points $\left(\frac{1}{2},-5\right)$, $\left(-\frac{1}{2},0\right)$, and $\left(-\frac{3}{2},9\right)$. Verify. [Note that $x = 2$ is $\frac{3}{4}$ units right of the line of symmetry, and $x = \frac{1}{2}$ is $\frac{3}{4}$

units left of the line of symmetry. That puts $x = 3$ one more unit right of the line of symmetry and $x = -\frac{1}{2}$ one more unit left of the line of symmetry, etc. Now sketch the graph.

Exercise Set 10.4: Plot the graph of each of the following functions.

1. $f(x) = x^3 - x$
2. $f(x) = \begin{cases} 4 - x^2 & \text{if } x < 1 \\ 2x + 4 & \text{if } x \geq 1 \end{cases}$
3. $f(x) = x^2 - 8x + 5$
4. $f(x) = -2x^2 - 8x + 5$

§11. Exponentials and Logarithms

§11.1: What is an Exponential function?

Monomials, such as $2x^3$, are the basic building blocks of polynomials. Every polynomial is made up of sums and differences of such terms. In a monomial the exponent is a constant and the base is a variable.

In exponential function, the base is a constant and the exponent is a variable; for instance: $f(x) = 2^x$.

When $b > 0$ (and $b \neq 1$), then $f(x) = b^x$ is an exponential function.

There are huge difficulties with the domain if b is negative [think of $f(\frac{1}{2})$ for $f(x) = (-2)^x$], and while there are no issues with $f(x) = 1^x$, it is not very interesting.

Exercise Set 11.1: Sketch the graph of the following functions and draw such conclusions as come to mind.

1. $f(x) = 2^x$
2. $f(x) = 3^x$
3. $f(x) = \left(\frac{1}{2}\right)^x$
4. $f(x) = \left(\frac{1}{3}\right)^x$

Conclusions: You might have noticed (A) a horizontal asymptote of $y = 0$ for each, (B) they each pass through $(0, 1)$ and $(1, b)$ (where b is the base), (C) they each have a precipitous rise or fall on one side of the y-axis and the asymptotic behavior on the other side, but perhaps not on the same side.

§11.2: What is a logarithm?

Before we answer this, let's digress a bit. Look at the following table and try to determine the relationship between the function in the left column and the function in the right column.

$f(x) = x^3$	$g(x) = \sqrt[3]{x}$
$f(x) = x^5$	$g(x) = \sqrt[5]{x}$
$f(x) = x^7$	$g(x) = \sqrt[7]{x}$
$f(x) = x^9$	$g(x) = \sqrt[9]{x}$

What happens if you compose them? Yes! In each case, $(f \circ g)(x) = x$, so in each case f and g are inverse functions of one another. Thus, the table could be written instead:

$f(x) = x^3$	$f^{-1}(x) = \sqrt[3]{x}$
$f(x) = x^5$	$f^{-1}(x) = \sqrt[5]{x}$
$f(x) = x^7$	$f^{-1}(x) = \sqrt[7]{x}$
$f(x) = x^9$	$f^{-1}(x) = \sqrt[9]{x}$

The notation of the column of functions on the left is familiar and natural—it is that of the exponent. There is nothing natural about the notation of the functions on the right at all. It is artificial, but crafty.

Each of the functions on the left is a one-to-one function, and thus we know that each has an inverse. The question then becomes, how do we denote that function which is an inverse function to each of these common functions...particularly since there will be one of these pairs for every odd number! If we write the inverse function of $f(x) = x^n$ as $f^{-1}(x) = \sqrt[n]{x}$ (for each odd natural number n), then yes it is artificial...we just made it up out of thin air! However, it is a crafty artifice, because the notation helps us tell all these inverse functions apart and remember who is the inverse function of whom.

Similarly, each exponential function $f(x) = b^x$ is one-to-one and has an inverse. How do we denote the inverse function for $f(x) = 2^x$, and $f(x) = 3^x$, and $f(x) = (1.5)^x$, etc, when there are so very many and our inspiration for unique names will eventually run out. We can take a cue from the power functions and the uniform way of writing the root inverses (see above) to do something similar here.

Exponential Function	**Its Inverse Function**	
$f(x) = 2^x$	$f^{-1}(x) = \log_2(x)$	
$f(x) = 3^x$	$f^{-1}(x) = \log_3(x)$	
$f(x) = (1.5)^x$	$f^{-1}(x) = \log_{1.5}(x)$	
$f(x) = e^x$	$f^{-1}(x) = \log_e(x)$	(shortened to $f^{-1}(x) = \ln(x)$)
$f(x) = 10^x$	$f^{-1}(x) = \log_{10}(x)$	(shortened to $f^{-1}(x) = \log(x)$)

That is about all you have to know about a logarithm function. That is:

$$b^{\log_b(x)} = x \qquad \text{and} \qquad \log_b(b^x) = x$$

There are some properties that go with logarithms, but they are all naturally related to the properties of exponents.

Exponent Property	**Logarithm Property**
$b^m \cdot b^n = b^{m+n}$	$\log_b(xy) = \log_b(x) + \log_b(y)$
$\frac{b^m}{b^n} = b^{m-n}$	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
$(b^m)^r = b^{r \cdot m}$	$\log_b(x^r) = r \cdot \log_b(x)$
$b^1 = b$	$\log_b(b) = 1$
$b^0 = 1$	$\log_b(1) = 0$

Exercise Set 11.2: Compute the following or find the unknown.

1. $\log_2(64) = x$
2. $\log_x(81) = 4$
3. $\log_5(x) = 3$
4. $3^{2\log_3(7)} = x$
5. $\ln(x-1) = 2$
6. $e^{3\ln(x)} = 8$

Express as a single logarithm.

7. $2\log_3(x) - 4\log_3(y) - 3\log_3(z)$
8. $\log_5(x-2) + \log_5(2x-1) - 2\log_5(x-3)$

Express as sums and differences of logarithms of single variables.

9. $\log_7\left(\frac{49x^2y^3}{z^4}\right)$
10. $\ln(ex)$

§12. Equations, Properties of Equality, Solving Equations

Properties of Equality: The following are used constantly in the generation of equivalent equations and in the use of solving equations. It should be noted that not every equation that we can write down, is capable of being solved by algebraic means. What we see below and in most college-level courses are a wide range of equations which can be solved via algebraic means. [The course MAT 442 looks at how to solve these other equations (and this is done to a lesser degree in Calculus as well).]

Equals is Reflexive, Symmetric, and Transitive: For any real numbers a, b, c,

it is true that $a = a$; if $a = b$ then $b = a$; & if $a = b$ *and* $b = c$, then $a = c$.

Substitution Property of Equality:
If $a = b$ then the value of any computation involving a is unchanged when a is replaced by b,

Addition Property of Equality: If a, b, c are real numbers and $a = b$, then $a + c = b + c$.
and since it follows that $a + (-c) = b + (-c)$, then $a - c = b - c$.

Multiplication Property of Equality: If a, b, c are real numbers and $a = b$, then $a \cdot c = b \cdot c$.
and (when $c \neq 0$) it follows that $a \cdot \frac{1}{c} = b \cdot \frac{1}{c}$, and so $\frac{a}{c} = \frac{b}{c}$ as well.

In the process of trying to solve an equation, there are three possible outcomes which could occur.

1. We obtain a specific answer or answers. For example, we end the process with $x = 2$ and $x = -1$. In that case, the equation is called a *conditional* equation. That is, it is true under the condition that $x = 2$ or $x = -1$.
2. We obtain a true statement with no remaining variables. For example, along the way we arrive at something like: $3 = 3$. In this case, the equation can be called an *identity*, and every real number is a solution to that equation.
3. We obtain a false statement with no remaining variables. For example, along the way we arrive at something like: $1 = 3$. In this case, the equation can be called an *contradiction*, and the equation has no solutions.

For us, there are essentially three (overlapping) categories of equations which we can solve: those in which all variable terms are like terms, those in which two or more variable terms are not like terms, and those in which an inverse-function (other than the Addition Property of Equality, or the Multiplication Property of Equality) must be applied. The simplest example of the first category are linear equations.

§12.1 Solving Linear Equations

An equation is a linear equation if it can be rewritten using the above properties of equality to state the equality of two polynomials of degree 1. Initially, the equation might not appear to be linear. [For example, $x(x^3+1) = x^4+2$.] All such equations can be solved via the same basic procedure: (a) expand all parentheses, (b) gather all terms containing the variable to the same side of the equation, and then (c) isolate the variable by finding a way to legally remove all non-variable parts of the equation from the side of equation containing the variable.

Example 12.1.1: Solve for x: $2(3x - 8) = 3x + 8$

$6x - 16 = 3x + 8$	expanding the parentheses
$6x - 16 - 3x = 3x + 8 - 3x$	moving the $+3x$ by adding $-3x$ to both sides
$3x - 16 = 8$	simplify
$3x - 16 + 16 = 8 + 16$	moving the -16 by adding $+16$ to both sides
$3x = 24$	simplify
$\frac{3x}{3} = \frac{24}{3}$	moving the 3 which is multiplied times x by dividing both sides by 3
$x = 8$	simplify

This general pattern emerges: to move some term or factor we do the opposite of whatever is connecting that term or factor to that side of the equation. If it is added, we subtract; if it is subtracted, we add; if it is multiplied, we divide. Secondly, we generally look at a step and decide what has to be done, the next step is an *action* step where we carry out that action, and the following step is a simplification step. At that point we have a brand new equation or expression which is the result of our decision, and we essentially have to start again (as though it were a new problem) and decide what to do next.

§12.2 Equations all of whose variable terms are like terms, but which are not linear

Consider the following problem.

Example 12.2.1: Solve for x: $2(3x^3 - 8) = 3x^3 + 8$

$6x^3 - 16 = 3x^3 + 8$	expanding the parentheses
$6x^3 - 16 - 3x^3 = 3x^3 + 8 - 3x^3$	moving the $+3x^3$ by adding $-3x^3$ to both sides
$3x^3 - 16 = 8$	simplify
$3x^3 - 16 + 16 = 8 + 16$	moving the -16 by adding $+16$ to both sides
$3x^3 = 24$	simplify
$\frac{3x^3}{3} = \frac{24}{3}$	moving the 3 which is multiplied times x^3 by dividing both sides by 3
$x^3 = 8$	simplify

We could continue to isolate x by finding a way to remove the 3rd power. That can be done by a cube-root, we know. Thus, we can continue.

$x^3 = 8$

$\sqrt[3]{x^3} = \sqrt[3]{8}$ — apply the inverse function to both sides

$x = 2$ — simplify

When you think about it, the inverse function of $f(x) = x + c$ is $f^{-1}(x) = x - c$ are inverse functions, so the Addition Property of Equality is the application of a (an inverse) function to both sides. Also, the inverse function of $f(x) = c \cdot x$ is $f^{-1}(x) = \frac{x}{c}$ are inverse functions, so the Multiplication Property of Equality is the application of a (an inverse) function to both sides. Thus, the application of a cube-root to both sides is just an extension of what we've been doing already.

Here is another, similar, example.

Example 12.2.2: Solve for x: $6(5^x - 8) = 3 \cdot 5^x + 3$

$6 \cdot 5^x - 48 = 3 \cdot 5^x + 3$ — expanding the parentheses

$6 \cdot 5^x - 48 - 3 \cdot 5^x = 3 \cdot 5^x + 3 - 3 \cdot 5^x$ — moving the $+3 \cdot 5^x$ by adding $-3 \cdot 5^x$ to both sides

$3 \cdot 5^x - 48 = 3$ — simplify

$3 \cdot 5^x - 48 + 48 = 3 + 48$ — moving the -48 by adding $+48$ to both sides

$3 \cdot 5^x = 51$ — simplify

$\frac{3 \cdot 5^x}{3} = \frac{51}{3}$ — moving the 3 which is multiplied by 5^x by dividing both sides by 3

$5^x = 17$ — simplify

We need an inverse function to $f(x) = 5^x$, and we know that it is $f^{-1}(x) = \log_5(x)$. Thus, we continue.

$5^x = 17$

$\log_5(5^x) = \log_5(17)$ — apply the inverse function to both sides

$x = \log_5(17)$ — simplify

Here is another, similar, example.

Example 12.2.3: Solve for x: $6(\log_3(x) - 8) = 3 \cdot \log_3(x) + 3$

$6 \cdot \log_3(x) - 48 = 3 \cdot \log_3(x) + 3$ — expanding the parentheses

$6 \cdot \log_3(x) - 48 - 3 \cdot \log_3(x) = 3 \cdot \log_3(x) + 3 - 3 \cdot \log_3(x)$

moving the $+3 \cdot \log_3(x)$ by adding $-3 \cdot \log_3(x)$ to both sides

$3 \cdot \log_3(x) - 48 = 3$ — simplify

$3 \cdot \log_3(x) - 48 + 48 = 3 + 48$ — moving the -48 by adding $+48$ to both sides

$3 \cdot \log_3(x) = 51$ — simplify

$\frac{3 \cdot \log_3(x)}{3} = \frac{51}{3}$ — moving the 3 which is multiplied times $\log_3(x)$ by dividing both sides by 3

$\log_3(x) = 17$ — simplify

We need an inverse function to $f(x) = \log_3(x)$, and we know that it is $f^{-1}(x) = 3^x$. Thus, we continue.

$\log_3(x) = 17$

$3^{(\log_3(x))} = 3^{17}$ — apply the inverse function to both sides

$x = 3^{17}$ — simplify

Here is another example.

Example 12.2.4: Solve for x: $2(3x^2 - 8) = 3x^2 + 8$

$6x^2 - 16 = 3x^2 + 8$	expanding the parentheses
$6x^2 - 16 - 3x^2 = 3x^2 + 8 - 3x$	moving the $+3x^2$ by adding $-3x^2$ to both sides
$3x^2 - 16 = 8$	simplify
$3x^2 - 16 + 16 = 8 + 16$	moving the -16 by adding $+16$ to both sides
$3x^2 = 24$	simplify
$\frac{3x^2}{3} = \frac{24}{3}$	moving the 3 which is multiplied times x^2 by dividing both sides by 3
$x^2 = 8$	simplify

We need an inverse function to $f(x) = x^2$. Unfortunately, this function is not one-to-one unless we throw away half of the domain. So we restrict the domain to $[0, \infty)$ to get only the right half of the parabola. This is one-to-one, and it has the inverse $f^{-1}(x) = \sqrt{x}$. So we apply that function to both sides.

$x^2 = 8$	
$\sqrt{x^2} = \sqrt{8}$	apply the inverse function to both sides
$\lvert x\rvert = \sqrt{8}$	(remember our domain restriction)

so $x = +\sqrt{8}$ or $x = -\sqrt{8}$

Here is a final example of this style.

Example 12.2.5: Solve for x: $7(\tan(x) - 1) = 3 \cdot \tan(x) - 3$

$7 \cdot \tan(x) - 7 = 3 \cdot \tan(x) - 3$	expanding the parentheses
$7 \cdot \tan(x) - 7 - 3 \cdot \tan(x) = 3 \cdot \tan(x) - 3 - 3 \cdot \tan(x)$	
	moving the $+3 \cdot \tan(x)$ by adding $-3 \cdot \tan(x)$ to both sides
$4 \cdot \tan(x) - 7 = -3$	simplify
$4 \cdot \tan(x) - 7 + 7 = -3 + 7$	moving the -7 by adding $+7$ to both sides
$4 \cdot \tan(x) = 4$	simplify
$\frac{4 \cdot \tan(x)}{4} = \frac{4}{4}$	moving the 4 which is multiplied times $\tan(x)$ by dividing both sides by 4
$\tan(x) = 1$	simplify

We need an inverse function to $f(x) = \tan(x)$. This function is also not one-to-one, and we must restrict its domain to make it one-to-one. The function $f(x) = \tan(x)$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is one-to-one and its inverse is $f^{-1}(x) = \tan^{-1}(x)$. Thus, we continue with the understanding that applying this inverse function will only give us the single answer in the restricted domain. We will have to restore any remaining answers.

$\tan(x) = 1$	
$\tan^{-1}(\tan(x)) = \tan^{-1}(1)$	apply the inverse function to both sides
$x = \frac{\pi}{4}$	simplify

This is the solution which lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We use the periodicity of the tangent function to obtain the rest of the answers. $x = \frac{\pi}{4} + k\pi$ for any integer k.

§12.3 Equations with multiple variable terms which are alike except for their exponents

Example 12.3.1: Solve the equation: $x^2 - x = 6$.

Try as you might, you are not going to be able to isolate x. We cannot remove the square without taking the square root, but

$$\sqrt{x^2 - x} = \sqrt{6}$$

does not remove the square. If we isolate the squared term and then take the square root:

$x^2 - x + x = 6 + x$

$x^2 = x + 6$

$\sqrt{x^2} = \sqrt{x+6}$

$|x| = \sqrt{x+6}$

we removed the square, but we haven't solved for x because there is still an x on both sides of the equation. So we have to take a different tack.

§12.3.1 The Zero-Product Principle:

Suppose that a and b are real numbers. Then $a \cdot b = 0$ precisely when $a = 0$ or $b = 0$.

Example 12.3.1.1: Solve the equation: $x^2 - x = 6$.

$x^2 - x = 6$	
$x^2 - x - 6 = 6 - 6$	we are trying to get the zero in place
$x^2 - x - 6 = 0$	we have zero, but we need a product equal to zero remember that this is the reason that we factor
$(x-3) \cdot (x+2) = 0$	Now we have set up the Zero-Product Principle
So either $x - 3 = 0$ or $x + 2 = 0$	and we have two linear equations (which we can solve)
$x - 3 + 3 = 0 + 3$ or $x + 2 - 2 = 0 - 2$	
$x = 3$ or $x = -2$.	

Warning: It is an error to factor one side before bringing the other side to be zero. For instance $x(x-1) = 6$ But there are a lot of products that result in 6, so this does not help us. The number 0 is unique in this way.

Example 12.3.1.2: Solve the equation: $\ln^2(x) - \ln(x) = 6$. $[\ln^2(x) \equiv (\ln(x))^2]$

$\ln^2(x) - \ln(x) = 6$	
$\ln^2(x) - \ln(x) - 6 = 6 - 6$	we are trying to get the zero in place
$\ln^2(x) - \ln(x) - 6 = 0$	we have zero, but we need a product equal to zero remember that this is the reason that we factor
$(\ln(x) - 3) \cdot (\ln(x) + 2) = 0$	Now we have set up the Zero-Product Principle
So either $\ln(x) - 3 = 0$ or $\ln(x) + 2 = 0$	and we have two linear equations (which we can solve)
$\ln(x) - 3 + 3 = 0 + 3$ or $\ln(x) + 2 - 2 = 0 - 2$	
$\ln(x) = 3$ or $\ln(x) = -2$.	These equation each have just one variable term; apply inverse to both sides.
$e^{\ln(x)} = e^3$ or $e^{\ln(x)} = e^{-2}$.	$\ln(x)$ and e^x are inverse functions of one another.
$x = e^3$ or $x = e^{-2}$	simplify.

Example 12.3.1.3: Solve the equation: $(2x+1)^2 - (2x+1) = 6$.

Except for the fact that the $(2x+1)$ appears twice, we would be very tempted to expand the parentheses...and this might be ok. However, there is another way that might be fruitful (and we could have done this in the problem above. Because of the similarity of the base on those exponents, we will temporarily replace the base by u. That is $(2x+1) = u$.

$(2x+1)^2 - (2x+1) = 6$	original problem
$u^2 - u = 6$	we make the substitution $(2x+1) = u$; we see that it is like the ones above, so...
$u^2 - u - 6 = 6 - 6$	we try to get the zero in place
$u^2 - u - 6 = 0$	we have zero, but we need a product equal to zero remember that this is the reason that we factor
$(u-3) \cdot (u+2) = 0$	Now we have set up the Zero-Product Principle

$u-3=0$ or $u+2=0$ — we apply ZPP and solve these two linear equations

$u=3$ or $u=-2$

$2x+1=3$ or $2x+1=-2$ — we replace u by what it had replaced originally

$2x+1-1=3-1$
or $2x+1-1=-2-1$ — we solve these two linear equations

$2x=2$ or $2x=-3$

$\frac{2x}{2}=\frac{2}{2}$ or $\frac{2x}{2}=\frac{-3}{2}$ — That is, $x=1$ or $x=-\frac{3}{2}$.

Example 12.3.1.4: Solve the equation: $9x^5=4x^3$.

$9x^5=4x^3$

$9x^5-4x^3=4x^3-4x^3$ — getting all variables on one side of the equals sign

$9x^5-4x^3=0$ — now attempt to factor

$x^3(9x^2-4)=0$ — we can factor further

$x^3(3x-2)\cdot(3x+2)=0$ — now use PZP

So $x^3=0$ or $3x-2=0$ or $3x+2=0$ and we solve each in turn.

$\sqrt[3]{x^3}=\sqrt[3]{0}$ or $3x-2+2=0+2$ or $3x+2-2=0-2$

$x=0$ or $\frac{3x}{3}=\frac{2}{3}$ or $\frac{3x}{3}=\frac{-2}{3}$

$x=0$ or $x=\frac{2}{3}$ or $x=-\frac{2}{3}$

Warning!!! It is tempting to divide both sides by any common factor which they have. When that is a constant, then there is no harm in doing so. That is, the solutions to $2x^2=8$ will be the same as the solutions to $x^2=4$ (think of the Multiplication Property of Equality). However, to divide both sides of the equation by a variable quantity will almost always mean that you will miss a solution. In particular, in the above problem: $9x^5=4x^3$ it would have been tempting to divide both sides by x^3 to eliminate that common factor and get $9x^2=4$. However, this would have lost the solution $x=0$. Moreover, when $x=0$ then dividing both sides by x^3 would have been dividing by 0 and that would render the equation meaningless for $x=0$. That is always a primary risk with dividing by a variable quantity.

Shortcoming of the Zero Product Principle: Once we have *expression* $=0$, we will not always be able to factor the expression. If the expression is of quadratic type, there are two options available: the **Quadratic Formula** and the method of **Completing the Square**. By *quadratic type*, we mean that it has the form of the last two equations: namely it looks like $a\cdot(f(x))^2+b\cdot(f(x))+c=0$.

§**12.3.2 The Quadratic Formula:** By applying the method of completing the square to the quadratic equation $a\cdot x^2+b\cdot x+c=0$, we obtain:

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

The important things to note about using the quadratic formula (to avoid the most common errors)

1. The quadratic expression must have 0 on the other side of the '=' sign before beginning.
2. The signs on a, b, and c include the sign in front of it in the expression $a\cdot x^2+b\cdot x+c=0$
3. The division symbol is under the entire expression: $-b\pm\sqrt{(b^2-4ac)}$
4. The square root symbol is over the entirety of (b^2-4ac) which is why we included it in parentheses.

Example 12.3.2.1: Solve the equation: $x^{2/3} - 3x^{1/3} = 1$.

This equation has unlike terms, so we move over the 1 to get: $x^{2/3} - 3x^{1/3} - 1 = 0$. This is of quadratic type because if we set $u = x^{1/3}$ then $u^2 = x^{2/3}$ and we have the other variable term. With a little practice, we can begin to see that the exponent on one variable term is exactly double the exponent on the other variable term. Substituting in we get:

$u^2 - 3u - 1 = 0$

The only problem is that this will not factor and so we can not use the Principle of Zero Products. However, we can use the Quadratic Formula, with $a = 1$, $b = -3$, and $c = -1$. Substituting in:

$$\begin{aligned} u = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} &= \frac{-(-3) \pm \sqrt{((-3)^2 - 4(1)(-1))}}{2(1)} \\ &= \frac{3 \pm \sqrt{9+4}}{2} \qquad \text{note that } -(-3) = +3, \text{ and } (-3)^2 = 9 \\ &= \frac{3 \pm \sqrt{13}}{2} \\ &= \frac{3}{2} \pm \frac{\sqrt{13}}{2} \end{aligned}$$

Thus,

$u = \frac{3}{2} + \frac{\sqrt{13}}{2}$ or $u = \frac{3}{2} - \frac{\sqrt{13}}{2}$

$x^{1/3} = \frac{3}{2} + \frac{\sqrt{13}}{2}$ or $x^{1/3} = \frac{3}{2} - \frac{\sqrt{13}}{2}$ replace the u by its value as a function of x and now apply a cube to both sides as an inverse to $x^{1/3}$

$\left(x^{1/3}\right)^3 = \left(\frac{3}{2} + \frac{\sqrt{13}}{2}\right)^3$ or $\left(x^{1/3}\right)^3 = \left(\frac{3}{2} - \frac{\sqrt{13}}{2}\right)^3$

$x = \left(\frac{3}{2} + \frac{\sqrt{13}}{2}\right)^3$ or $x = \left(\frac{3}{2} - \frac{\sqrt{13}}{2}\right)^3$

We do not carry out the cube of the right–hand–side because the answer is in the most compact form as it is.

§12.3.3 Completing the Square:

If solving the equation: $x^2 = 9$ we may be tempted to take the square root of both sides:

$x^2 = 9$

$\sqrt{x^2} = \sqrt{9}$

$|x| = 3$

and so $x = 3$ or $x = -3$.

We'd like to replicate this technique by using the method of completing the square.

Example 12.3.3.1: Solve for x: $x^2 - 4x = 2$

We might initially put this equation in the form $x^2 - 4x - 2 = 0$ in order to try to apply the Principle of Zero Products. However, we will find that we cannot factor this quadratic expression. We could use the quadratic formula, but the ability to complete the square pays off again and again in situations where the quadratic formula would not be useful, and so it is good to hone our skill at completing the square. Returning to the original equation, what if we happened to add 4 to both sides of the equation? (Shortly, we will talk about how to come up with the idea to add 4 to both sides.)

$x^2 - 4x = 2$

$x^2 - 4x + 4 = 2 + 4$

$x^2 - 4x + 4 = 6$ and the left-hand-side factors as a perfect square!

$(x-2)(x-2) = 6$

$(x-2)^2 = 6$ and here we are with a situation not unlike that we had with $x^2 = 9$.

$\sqrt{(x-2)^2} = \sqrt{6}$

$|x-2| = \sqrt{6}$

So $x - 2 = +\sqrt{6}$ or $x - 2 = -\sqrt{6}$

Solving each of these: $x = +\sqrt{6} + 2$ or $x = -\sqrt{6} + 2$.

In that example, how did we decide that it would be fruitful to add 4 to both sides?

To mimic the circumstances that we had with $x^2 = 9$ we had to have a perfect square on the variable side of the equation.

$x^2 - 4x + M$

$(x-d)(x-d) = x^2 - 2d \cdot x + d^2$

We need a square: $(x-d)\cdot(x-d)$ that would result in a middle term of $-4x$. Thus, we see that $-2d\cdot x = -4x$ and so $d = 2$ (and $d^2 = 4$). That is why we added 4 to both sides. The standard way that most of us learned how to complete the square was to take $\left(\frac{1}{2} \cdot \text{middle coefficient}\right)^2$ and add that to both sides. Note that $\left(\frac{1}{2} \cdot (-4)\right)^2 = (-2)^2 = 4$.

Crucial to this method is that $a = 1$; that is, the coefficient of x^2 is 1. In summary, these are necessary to set up Completing the Square.

1. The equation must be a quadratic equation (or of quadratic type);
2. The two variable terms are on the same side of the equals sign (and, ideally, the constant term is on the opposite side of the equals sign);
3. The coefficient of x^2 is 1.

Example 12.3.3.2: Solve the equation: $2x^2 - 7 = 3x$

$2x^2 - 7 = 3x$ — we have two unlike variable terms, so we move all to one side;

$2x^2 - 7 - 3x = 3x - 3x$

$2x^2 - 3x - 7 = 0$ — but it can't be factored (ie: PZP cannot be used)

$2x^2 - 3x - 7 + 7 = 0 + 7$ — try Completing the Square...isolate the variable terms

$2x^2 - 3x = 7$ — now we'll divide both sides of the equation by 2 in order to get $1 \cdot x^2$

$\frac{2x^2 - 3x}{2} = \frac{7}{2}$ — ...and now distribute the division

$\frac{2x^2}{2} - \frac{3x}{2} = \frac{7}{2}$

$x^2 - \frac{3}{2} \cdot x = \frac{7}{2}$ — either think of ...

$x^2 - \frac{3}{2} \cdot x + \left(-\frac{3}{4}\right)^2 = \frac{7}{2} + \left(-\frac{3}{4}\right)^2$ — adding $\left(\frac{1}{2} \cdot \left(-\frac{3}{2}\right)\right)^2 = \left(-\frac{3}{4}\right)^2$ to both sides or

$\left(x - \frac{3}{4}\right) \cdot \left(x - \frac{3}{4}\right) = \frac{7}{2} + \left(-\frac{3}{4}\right)^2$ — writing the factors $\left(x - \frac{1}{2} \cdot \frac{3}{2}\right) \cdot \left(x - \frac{1}{2} \cdot \frac{3}{2}\right)$ and seeing that $(-\frac{3}{4})^2$ must be added to the right side to keep the equation balanced

$\left(x - \frac{3}{4}\right)^2 = \frac{7}{2} + \frac{9}{16}$ — either way, we get this result

$\left(x - \frac{3}{4}\right)^2 = \frac{56}{16} + \frac{9}{16}$

$\left(x - \frac{3}{4}\right)^2 = \frac{65}{16}$ — ...and finally we are ready to take the square root of both sides.

$\sqrt{\left(x - \frac{3}{4}\right)^2} = \sqrt{\frac{65}{16}}$

$\left|x - \frac{3}{4}\right| = \sqrt{\frac{65}{16}}$

So $x - \frac{3}{4} = +\sqrt{\frac{65}{16}}$ or $x - \frac{3}{4} = -\sqrt{\frac{65}{16}}$

Hence $x = +\sqrt{\frac{65}{16}} + \frac{3}{4}$ or $x = -\sqrt{\frac{65}{16}} + \frac{3}{4}$

or $x = \frac{\sqrt{65}}{4} + \frac{3}{4}$ or $x = -\frac{\sqrt{65}}{4} + \frac{3}{4}$

You can see that this is a lot of work and so we would almost always choose the Quadratic Formula. However, since Completing the Square is so useful in some other circumstances, we should use it now and then.

§**12.4 Equations involving functions other than polynomials:** (and so some advanced work might need to be done before noting which of the earlier two categories that it falls into)

Equations involving Logarithms typically fall into this category.

Example 12.4.1: Solve for x: $\log_2(x-2) = 2 - \log_2(x+1)$

While they look different, and are different, sums of logarithms can be rewritten as a single object.

$\log_2(x-2) = 2 - \log_2(x+1)$	
$\log_2(x-2) + \log_2(x+1) = 2 - \log_2(x+1) + \log_2(x+1)$	
$\log_2(x-2) + \log_2(x+1) = 2$	we are trying to get at least all the variable terms together.
$\log_2((x-2)(x+1)) = 2$	the sum of two logarithms is the logarithm of the product of their insides (called "arguments")
$\log_2((x-2)(x+1)) = 2$	this now looks like a single variable term so we try to isolate x...apply the inverse (of $\log_2(x)$) to each side
$2^{\log_2((x-2)(x+1))} = 2^2$	the inverse function of $\log_2(x)$ is 2^x
$(x-2)(x+1) = 4$	Simplify! This now looks like a quadratic, but the ZPP does not apply because the product is not equal to zero.
$x^2 - x - 2 = 4$	So we expand the parentheses; it does have two unlike variable terms.
$x^2 - x - 2 - 4 = 4 - 4$	So we move all terms to one side of the equation, leaving 0.
$x^2 - x - 6 = 0$	We factor and plan to use the Principle of Zero Products.
$(x-3)(x+2) = 0$	We factor and plan to use the Principle of Zero Products.
$x - 3 = 0$ or $x + 2 = 0$	PZP
$x = 3$ or $x = -2$	Solve each of those linear equations

But! we have to check both our answers to see if each is in the domain of the original equation. The solution that we found was actually a solution to a quadratic equation that we found along the way. Since $x = -2$ results in $\log_2(-4) = 2 - \log_2(-1)$, we see that $x = -2$ is not in the domain of the original equation. Therefore, only $x = 3$ is a solution.

Note: Looking at the check, you can see how we picked up the additional (non-)solution $x = -2$. By the time that we put the two logarithms together $\log((-4)\cdot(-1))$, the logarithm was looking at the logarithm of a positive number. Thus, this additional value $x = -2$ was found by our solution method, but was not in the domain of the original equation. We must always check our solutions when solving a logarithmic equation.

Another kind of function which demands extra work when it appears in an equation, is a root function.

Example 12.4.2: Solve for x: $2\sqrt{2x+1} - 2 = x$

While these are not like terms, we know that we're not going to be able to solve for x until we get x out from under the square root function, and the only way to legally get rid of the square root is through a square (its inverse*). However, we can't just start squaring things randomly, we have to (a) square whole sides of the equation (because we can only guarantee the continued equality of the two sides if we square each whole side), and (b) we have to square *only* that square root to really have applied the inverse. So we first isolate the square root.

$2\sqrt{2x+1} - 2 = x$	
$2\sqrt{2x+1} - 2 + 2 = x + 2$	remove all from being with the square root
$2\sqrt{2x+1} = x + 2$	simplify
$\left(2\sqrt{2x+1}\right)^2 = (x+2)^2$	apply the inverse function* to both sides
$4(2x+1) = x^2 + 4x + 4$	expand the square to do its job
$8x + 4 = x^2 + 4x + 4$	expand the parentheses.

$8x + 4 - 8x - 4 = x^2 + 4x + 4 - 8x - 4$ We see two variable terms which are not 'like' so we move all to one side of the equals sign

$0 = x^2 - 4x$ simplify

$0 = x(x-4)$ we factor and then use PZP

$x = 0$ or $x - 4 = 0$ set each factor to 0

$x = 0$ or $x = 4$ **but we must check these answers.**

*The square root is an inverse function achieved only by restricting the domain of $f(x) = x^2$, and thus we have reason to be suspicious (in that specific way) of any answers we achieve. In particular, on the way to solving this equation, having squared both sides of the equation, something like $-2 \neq 2$ can exist but once we square both sides: $(-2)^2 = (2)^2$ and the once unequal sides are now equal. That might cause us to pick up an additional solution. This time we actually have to fully test that it is a solution (rather than just checking to see if it is in the domain of the equation).

$2\sqrt{2x+1} - 2 = x$ when $x = 0$ reads: $2\sqrt{1} - 2 = 0$. This is true, so $x = 0$ is a solution.

$2\sqrt{2x+1} - 2 = x$ when $x = 4$ reads: $2\sqrt{9} - 2 = 4$. This is true, so $x = 4$ is a solution.

[It is more common for one of the original 'solutions' to not be a solution.]

Example 12.4.3: Solve for x: $\sqrt{x+2} + \sqrt{7x+2} = 6$

Here we have two terms which are square roots. We'll follow the method above, and isolate a square root at a time.

$\sqrt{x+2} + \sqrt{7x+2} = 6$

$\sqrt{x+2} + \sqrt{7x+2} - \sqrt{7x+2} = 6 - \sqrt{7x+2}$ moving one over

$\sqrt{x+2} = 6 - \sqrt{7x+2}$ simplify

$\left(\sqrt{x+2}\right)^2 = \left(6 - \sqrt{7x+2}\right)^2$ Now applying the inverse* function

$x + 2 = 36 - 12\sqrt{7x+2} + (7x+2)$ next, we simplify

$x + 2 = 38 - 12\sqrt{7x+2} + 7x$ and now get the square root term alone on the right

$x + 2 - 7x - 38 = 38 - 12\sqrt{7x+2} + 7x - 7x - 38$

$-6x - 36 = -12\sqrt{7x+2}$ We try to remove common factors to get smaller numbers

$-6(x+6) = -12\sqrt{7x+2}$ Factor out the (-6) and divide both sides by (-6)

$\frac{-6(x+6)}{-6} = \frac{-12\sqrt{7x+2}}{-6}$

$(x+6) = 2\sqrt{7x+2}$ and now we square both sides

$(x+6)^2 = \left(2\sqrt{7x+2}\right)^2$ and expand

$x^2 + 12x + 36 = 4(7x+2)$ it looks quadratic; we expand the parentheses,

$x^2 + 12x + 36 = 28x + 8$ move all terms to one side, use PZP

$x^2 + 12x + 36 - 28x - 8 = 28x + 8 - 28x - 8$ and simplify

$x^2 - 16x + 28 = 0$ Now factor and use PZP

$(x-2)(x-14) = 0$

$x - 2 = 0$ or $x - 14 = 0$ applying PZP

$x - 2 + 2 = 0 + 2$ or $x - 14 + 14 = 0 + 14$

$x = 2$ or $x = 14$

*The square root was an inverse function achieved only by restricting the domain of $f(x) = x^2$, and thus we have reason to be suspicious (in that specific way) in any answers we achieve. Therefore we need to check each answer. If $x = 2$ $\sqrt{2+2} + \sqrt{7(2)+2} = 6$, and so $x = 2$ is a solution. If $x = 14$ $\sqrt{14+2} + \sqrt{7(14)+2} \neq 6$, and so $x = 14$ is not a solution.

Example 12.4.4: Solve for x: $\dfrac{x-2}{x+2} + \dfrac{2x}{x-3} = \dfrac{20}{x^2 - x - 6}$

There are several ways to approach this problem, and all will involve finding a denominator common to all of the terms. So we begin by factoring the denominators.

$\dfrac{x-2}{x+2} + \dfrac{2x}{x-3} = \dfrac{20}{x^2-x-6}$ factor the denominators

$\dfrac{x-2}{x+2} + \dfrac{2x}{x-3} = \dfrac{20}{(x+2)(x-3)}$ multiply both sides by the common den.: $(x+2)(x-3)$

$(x+2)(x-3)\cdot\left(\dfrac{x-2}{x+2} + \dfrac{2x}{x-3}\right) = (x+2)(x-3)\cdot\dfrac{20}{(x+2)(x-3)}$

$(x+2)(x-3)\cdot\dfrac{x-2}{x+2} + (x+2)(x-3)\cdot\dfrac{2x}{x-3} = 20$ distribute; simplify

$(x-3)(x-2) + (x+2)(2x) = 20$ simplify; but as we simplify, remember that $x=-2$ and $x=3$ are not in the domain of the original equation

$x^2-5x+6+2x^2+4x = 20$ expand parentheses

$3x^2-x+6 = 20$ this is going to be quadratic so we should bring all to one side

$3x^2-x+6-20 = 20-20$ simplify next, and factor

$3x^2-x-14 = 0$ factor and use PZP

$(3x-7)(x+2) = 0$ now apply PZP

$3x-7=0 \quad \text{or} \quad x+2=0$

$3x-7+7=0+7 \quad \text{or} \quad x+2-2=0-2$

$3x=7 \quad \text{or} \quad x=-2$

$\dfrac{3x}{3} = \dfrac{7}{3} \quad \text{or } x=-2$

$x=\frac{7}{3} \quad \text{or} \quad x=-2$

Recall, however, and note above, that $x=-2$ is not in the domain of the original equation. Therefore the only solution is: $x = 7/3$.

Example 12.4.5: Solve for x: $\sin^2(x) - \sin(x) = \cos^2(x)$

This is an example that many equations are just individual cases. Perhaps if we were to convert every function to sines, then we'd be able to solve the resulting equations. Why convert to sines and not to cosines? Because the only cosine is squared and $\cos^2(x) = 1-\sin^2(x)$, whereas it is not as easy to convert the $\sin(x)$ to a cosine.

$\sin^2(x)-\sin(x) = \cos^2(x)$ Substitute $\cos^2(x) = 1-\sin^2(x)$

$\sin^2(x)-\sin(x) = 1-\sin^2(x)$ seeing all the squares, we move everything to one side (thinking PZP)

$\sin^2(x)-\sin(x)+\sin^2(x) = 1-\sin^2(x)+\sin^2(x)$

$2\sin^2(x)-\sin(x) = 1$

$2\sin^2(x)-\sin(x)-1 = 1-1$ simplify

$2\sin^2(x)-\sin(x)-1 = 0$ simplify

$(2\sin(x)+1)(\sin(x)-1) = 0$ factor and use PZP

$2\sin(x)+1=0 \quad \text{or} \quad \sin(x)-1=0$

$2\sin(x)=-1 \quad \text{or } \sin(x)=1$

$\sin(x)=-1/2 \quad \text{or } \sin(x)=1$

We need an inverse function to $f(x)=\sin(x)$. This function is not one-to-one, and we must restrict its domain to make it one-to-one. The function $f(x)=\sin(x)$ on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is one-to-one and its inverse is $f^{-1}(x)=\sin^{-1}(x)$. Thus, we continue

$x=\sin^{-1}(-1/2) \quad \text{or } x=\sin^{-1}(1)$

$x=-\pi/6 \quad \text{or } x=\pi/2$

When we talk about trigonometry, we will also find another solution to $\sin(x) = -\frac{1}{2}$; namely: $x = \frac{7\pi}{6}$.
Using the periodicity of the sine function, the solutions are:

$x = -\frac{\pi}{6} + 2\pi \cdot k$ or $x = \frac{7\pi}{6} + 2\pi \cdot k$ or $x = \pi/2 + 2\pi \cdot k$ for any integer k.

Absolute Value Equations

The solution to these is essentially the application of the definition of the absolute value.

Example 12.4.6 Solve the equation $|3x - 4| = 10$

As we know from the definition of Absolute Value, $|3x-4| = \begin{cases} 3x - 4 & \text{if } 3x - 4 \geq 0 \\ -1 \cdot (3x - 4) & \text{if } 3x - 4 < 0. \end{cases}$

Thus, $|3x - 4| = 10$ can be reformulated as:

$3x - 4 = 10$ or $-1(3x - 4) = 10$.

You can also think of it in these terms: If $|3x - 4| = 10$, then either $3x - 4 = 10$ or $3x - 4 = -10$.

Either way, you get two simple linear equations to solve, with solutions: $x = 14/3$ and $x = -2$.

Exercise Set 12: Solve the following equations for the variable x.

1. $5x - (7x - 4) = 5 - (3x + 2)$
2. $\dfrac{1}{3x+18} - \dfrac{1}{2x+12} = \dfrac{1}{x^2+6x}$
3. $x^2 - 6x = 16$
4. $x^4 - 6x^2 = 16$
5. $x - 6\sqrt{x} = 16$
6. $\ln^2(x) - 6\ln(x) = 16$
7. $x^{-2} - 6x^{-1} = 16$
8. $\dfrac{1}{x^2} - \dfrac{6}{x} = 16$
9. $\sqrt{1-3x} - 1 = x$
10. $\sqrt{x-4} + \sqrt{x+1} = 5$
11. $e^{2x} = 4e^x + 5$
12. $\dfrac{17}{e^x + 4} = 2$
13. $21 - 4e^{0.1x} = 5$
14. $\log_5(x - 4) = 1 - \log_5(x)$
15. $\log_2(3 - x) = 5 - \log_2(1 - 2x)$
16. $\ln(x) = 4 - \ln(x - 2)$
17. $\tan(x) = -\sqrt{3} \quad 0 \leq x < 2\pi$
18. $2\cos^2(x) + 5\cos(x) = 6 \quad 0 \leq x < 2\pi$
19. $3\cos^2(x) + 2\sin(x) = 3 \quad 0 \leq x < 2\pi$
20. $5\log_3(x + 1) = 4 - 2\log_3(x + 1)$
21. $|2x - 9| = -2$
22. $|2x - 9| = 11$

Final Remarks for §12: As you see, there is a huge variety of equations, but there are some regular features of solving them.

1. Most steps involve applying some function or operation to each side of the equation. (Even adding a quantity to both sides of the equation is the action of a function, but we include 'operation' in the previous sentence because most people naturally think of 'adding 2 to both sides' or 'dividing both sides by 3' rather than 'applying $f(x) = x + 2$ to both sides' or 'applying $g(x) = \frac{x}{3}$ to both sides'.)
2. This function or operation is applied to **entire sides** of the equation, not to each term on both sides. In the case of multiplying both sides by 3, the multiplication *will distribute* to all individual terms on each side of the equation, but that is the byproduct of that specific case, not a general principle. Squaring both sides of the equation (for instance, applying $f(x) = x^2$ to both sides of the equation), notably **does not** apply to individual terms on each side, but decidedly to *each side as a whole*.
3. What is done to one side of the equation must be done to the other side of the equation as well.
4. *Usually* we expand any parentheses that appear in an equation that we are trying to solve, but not always.
5. Typically, an inverse function is applied to extract the variable x from the inside of some function, so that it is free to be written as $x =$ *some solution*.

6. If there is more than one such function term, and they cannot be combined in some way or removed separately, then we will probably want to move all terms to one side of the equation (with 0 on the other side), and then either factor the equation and use the Principle of Zero Products, or use the Quadratic Formula.

§13 Inequalities

§13.1 Ordering of the real numbers

You are familiar with the real number line. If a and b are real numbers, then we write $a < b$ if a is to the left of b on the real number line.

Trichotomy Property: Given real numbers x and y, exactly one of the following is true:

i. $a < b$
ii. $a = b$
iii. $b < a$

§13.1 Solving Linear Inequalities

The solving of linear inequalities is almost identical to that of linear equations, but there are two huge differences.

1. With inequalities, a solution is an interval, rather than a single real number.
2. When multiplying or dividing both sides of an inequality by a negative number, we must reverse the order of the inequality.

You can watch # 2 in practice by looking at

$1 < 2 < 3 < 4 < \ldots$

and then noticing that

$\ldots - 4 < -3 < -2 < -1 \quad$ (that is $-1 > -2 > -3 > -4 > \ldots$).

Thus, if you multiply each by (-1) you would need to reverse the order of the inequality.

Example 13.1.1: Find the solution set of $2(x - 1) < 5x + 4$

$2(x-1) < 5x + 4$

$2x - 2 < 5x + 4$

$2x - 2 - 5x < 5x + 4 - 5x$

$-3x - 2 < 4$

$-3x - 2 + 2 < 4 + 2$

$-3x < 6$

$\frac{-3x}{-3} > \frac{6}{-3}$ notice the reversal of the inequality at this same instant

$x > -2$

In interval notation, this is $(-2, \infty)$.

If the original inequality had been: $2(x - 1) \leq 5x + 4$, then the solution would have been: $[-2, \infty)$. That is, the bracket '[' indicates 'or equal to', and the parentheses '(' indicates a strict inequality.

Warning!!!: While the above might include an abundance of steps, there is a sequence of steps which is important to observe without dropping any portion thereof.

The last 3 steps were (and should be):

$-3x < 6$

$\frac{-3x}{-3} > \frac{6}{-3}$

$x > -2$

Many people, believing that they are saving time, follow up:

$-3x - 2 + 2 < 4 + 2$

$-3x < 6$

by immediately dividing this very last step by (-3), adjusting the step that was already there, and waiting to reverse the order of the inequality until the next step (after the horse is already out of the barn). In this case it looks like:

$-3x - 2 + 2 < 4 + 2$	
$\frac{-3x}{-3} < \frac{6}{-3}$	doing two steps in one, dividing both sides by (-3) but unable to reverse the inequality at the same time because the inequality had already been written as '<' upon simplifying the previous step
$x > \ -2$	and now reversing the inequality to '>' when there no longer appears to be a reason to do so

This is bad form, and therefore invites mistakes and misinterpretations. When you are tempted to skip a step, realize that it usually takes just seconds to write out the full step, and that it risks the accuracy of the whole problem. In contrast, you have spent hours studying so that you can get the problems correct. It doesn't serve you well to jeopardize the reward of hours of study just for the saving of mere seconds.

§13.2 Solving Absolute Value Inequalities:

Example 13.2.1: Find the solution set of $|2x - 3| \leq 7$

When you think of numbers whose absolute value is less than 7, that would be those real numbers between -7 and 7, including the origin. Thus, a convenient method of solution is:

$|2x - 3| \leq 7$

$-7 \leq 2x - 3 \leq 7$ and isolating the x as usual, but maintaining the relationship with all 3 expressions.

$-7 + 3 \leq 2x - 3 + 3 \leq 7 + 3$

$-4 \leq 2x \leq 10$

$\frac{-4}{2} \leq \frac{2x}{2} \leq \frac{10}{2}$

$-2 \leq x \leq 5$ or $[-2, 5]$

Example 13.2.2: Find the solution set of $|5 - 3x| > 7$

When you think of numbers whose absolute value is greater than 7, that would be those real numbers greater than 7 and those real numbers less than -7. Thus, a method of solution is:

$|5 - 3x| > 7$

$5 - 3x > 7$ or $5 - 3x < -7$ and solving each of these as the linear inequalities which they are.

$5 - 3x - 5 > 7 - 5$ or $5 - 3x - 5 < -7 - 5$

$-3x > 2$ or $-3x < -12$

$\frac{-3x}{-3} < \frac{2}{-3}$ or $\frac{-3x}{-3} > \frac{-12}{-3}$

$x < \frac{-2}{3}$ or $x > 4$

In interval notation, this would be: $(-\infty, -2/3) \cup (4, \infty)$.

§13.3 Nonlinear Inequalities

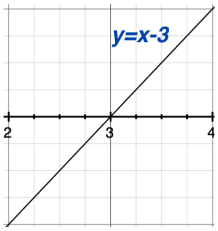
Solving nonlinear inequalities are typically much simpler than solving nonlinear equations, because our solution methods are so limited.

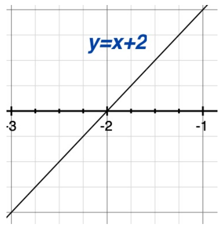
1. Bring all terms to one side of the inequality, with 0 on the other side;
2. Factor completely'
3. Use the fact that the product of two negative numbers or of two positive numbers is a positive number, and the product of a positive number and a negative number is a negative number.

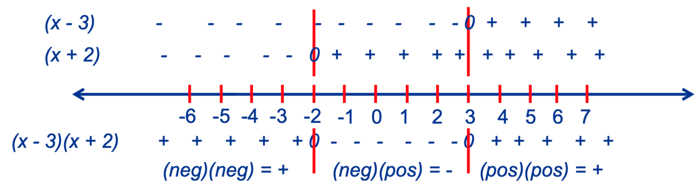
Example 13.3.1: Find the solution set to $x^2 - x > 6$.

$x^2 - x > 6$	bring all terms to one side of the inequality
$x^2 - x - 6 > 6 - 6$	
$x^2 - x - 6 > 0$	and now factor
$(x-3)\cdot(x+2) > 0$	and now we analyze signs

Since the individual factors are lines ($y = x - 3$ and $y = x + 2$), they are easy to graph (having positive slope with identifiable zeros) and their graphs provide an easy guide to sketching the summary chart (below, right) which maps when each factor is negative, zero, or positive.





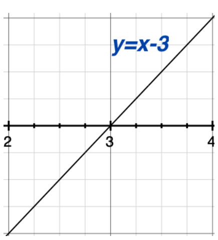


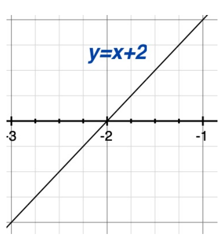
The bottom line shows when the product of the two is negative, zero, or positive. Since we are wanting to find when $(x-3)\cdot(x+2) > 0$ then the solution set to this inequality is $(-\infty, -2) \cup (3, \infty)$

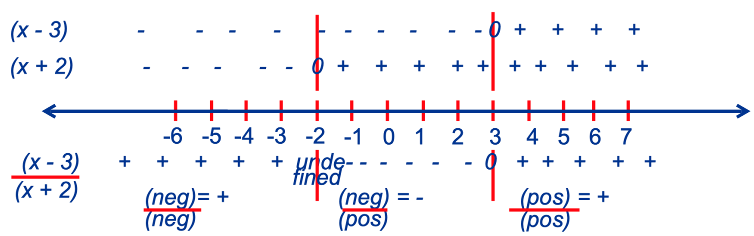
Example 13.3.2: Find the solution set to $\frac{2x-1}{x+2} \leq 1$.

$\frac{2x-1}{x+2} \leq 1.$	bring all terms to one side of the inequality
$\frac{2x-1}{x+2} - 1 \leq 1 - 1$	
$\frac{2x-1}{x+2} - 1 \leq 0$	
$\frac{2x-1}{x+2} - \frac{x+2}{x+2} \leq 0$	and now we put into a single rational expression
$\frac{2x-1-(x+2)}{x+2} \leq 0$	
$\frac{2x-1-x-2}{x+2} \leq 0$	
$\frac{x-3}{x+2} \leq 0$	

Since the numerator and denominator are lines ($y = x - 3$ and $y = x + 2$), they are easy to graph (having positive slope with identifiable zeros) and their graphs provide an easy guide to sketching the summary chart (below, right) which maps when each of the numerator and denominator are negative, zero, or positive..







The bottom line shows when the quotient of the two is negative, zero, positive or undefined. Since we are wanting to find when $\frac{x-3}{x+2} \leq 0$ then the solution set to this inequality is $(-2, 3]$

Exercise Set 13: Find the solution set to each of the following inequalities and express your answer in interval notation.

1. $\frac{5x}{3} \leq \frac{4+3x}{2}$
2. $7 - 2(1-x) > 5 + 3(x-2)$
3. $1 < 5 - 3x < 7$
4. $|2x - 3| < 7$
5. $|4x - 3| > 17$
6. $x^2 + 2x \geq 3$
7. $\dfrac{x}{x+2} \leq 3$
8. $x(x-2) \leq 8$

§14. Angles: Degrees and Radians, Special Angles

To get all on the same page, we begin with some early definitions. A *line* is a primitive object, but we think of a line as a one-dimensional figure with no width which extends infinitely in both directions. A *ray* can be thought of as a half–line, beginning at some point and extending infinitely in one direction. In Trigonometry, we think of a ray in the Cartesian plane as beginning at the origin $(0,0)$ and extending infinitely along a line in some direction. We can call the origin the *initial point* of a ray.

1. An *angle* is standard position is the union of two rays with initial points at the origin. One of those rays is called the *initial side* and the other is called the *terminal side*. The origin (the common initial point of both rays) is called the *vertex* of the angle.
2. Angles are placed with the initial side on the positive x-axis. If measured from the positive $x-$axis, going counter–clockwise (CCW) towards the terminal side, the angle is considered *positive* and if measured from the positive x-axis, going clockwise (CW) towards the terminal side, the angle is considered *negative*.
3. Historically (probably from the Babylonians, who used base 60) an angle has been measured in degrees, with $360°$ being a complete circle; this unit of measurement is arbitrary and that comes with drawbacks. Thus, Mathematics uses a different unit of measurement: the *radian*. One radian is the measure of a central angle of a circle which subtends (encloses) an arc the length of one radius of the circle. Put another way, if a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in radians is the length of that arc that is subtended (enclosed) by that angle. With this unit of measurement, the length of the arc s of a circle of radius r is given by $s = r \cdot \theta$, where θ is the measurement of the central angle which subtends that arc. Thus, since the circumference of a circle is $C = 2\pi \cdot r$, then $360° = \pi$ radians.
4. The conversion between degrees and radians: $180° = \pi$ radians and so

   To convert degrees to radians, multiply degrees by $\dfrac{\pi \text{ radians}}{180°}$ To convert radians to degrees, multiply radians by $\dfrac{180°}{\pi \text{ radians}}$

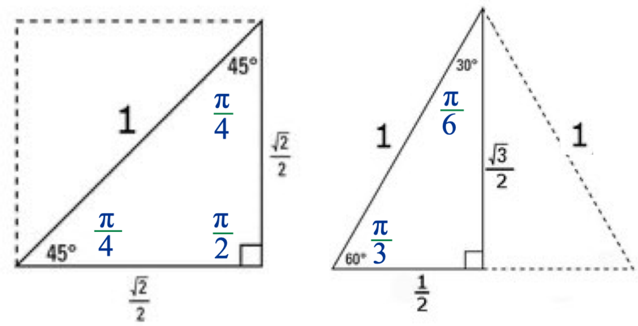
   Radians are uncomfortable at first, but if we start using them as much as possible as our default way of referring to angles, we will become accustomed to them. For reasons that we will soon discover, the following angles are called *special angles* and they are the primary angles which we will be using. Check a few to see that use of the conversion formulas above will transform one column into the other column.

Special Angle (deg)	Special Angle (rad)	Special Angle (deg)	Special Angle (rad)
$0°$	0 radians	$180°$	$\pi = \frac{3\pi}{3} = \frac{4\pi}{4} = \frac{6\pi}{6}$ rad
$30°$	$\frac{\pi}{6}$ radians	$210°$	$\frac{7\pi}{6}$ radians
$45°$	$\frac{\pi}{4}$ radians	$225°$	$\frac{5\pi}{4}$ radians
$60°$	$\frac{\pi}{3}$ radians	$240°$	$\frac{4\pi}{3}$ radians
The remaining are a multiple of one of the above.			
$90°$	$\frac{\pi}{2}$ radians	$270°$	$\frac{3\pi}{2}$ radians
$120°$	$\frac{2\pi}{3}$ radians	$300°$	$\frac{5\pi}{4}$ radians
$135°$	$\frac{3\pi}{4}$ radians	$315°$	$\frac{7\pi}{4}$ radians
$150°$	$\frac{5\pi}{6}$ radians	$330°$	$\frac{11\pi}{6}$ radians
$180°$	π radians	$360°$	$2\pi = \frac{6\pi}{3} = \frac{8\pi}{4} = \frac{12\pi}{6}$ rad

Looking over this chart occasionally and thinking of multiples of $30°$ to be multiples of $\frac{\pi}{6}$ and multiples of $45°$ to be multiples of $\frac{\pi}{4}$, etc, will help you remember them.

The first thing that might come to one's mind at this point is "**Why are these called 'Special Angles'?**"

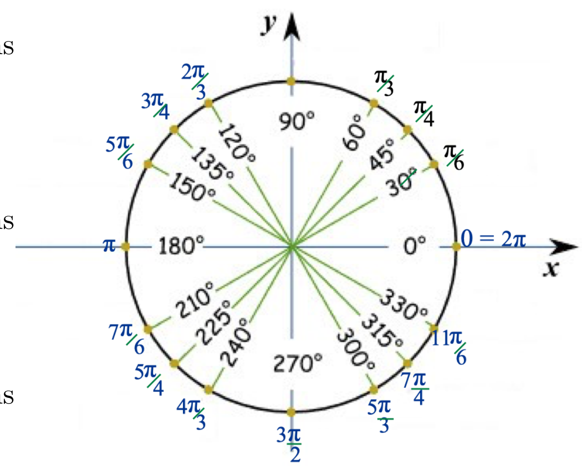
According to Wikipedia, the term "trigonometry" was derived from Greek *τριγωνον* (*trigōnon*), "triangle" and *μέτρον* (metron), "measure". The angles 0°, 30°, 45°, 60°, and 90° ($0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ radians, respectively) are the only angles in the first quadrant for which the values of the six trigonometric can be computed exactly, without the use of a calculator.



Triangles (the 30–60–90, and the 45–45–90 right triangles) can be used to determine these values for $30° = \frac{\pi}{6}$ radian, the $60° = \frac{\pi}{3}$ radian, and the $45° = \frac{\pi}{4}$ radian special angles.

This diagram provides all the elements necessary for a proof of the stated lengths of the sides. We will make use of these in the next several sections.

Exercise Set 14



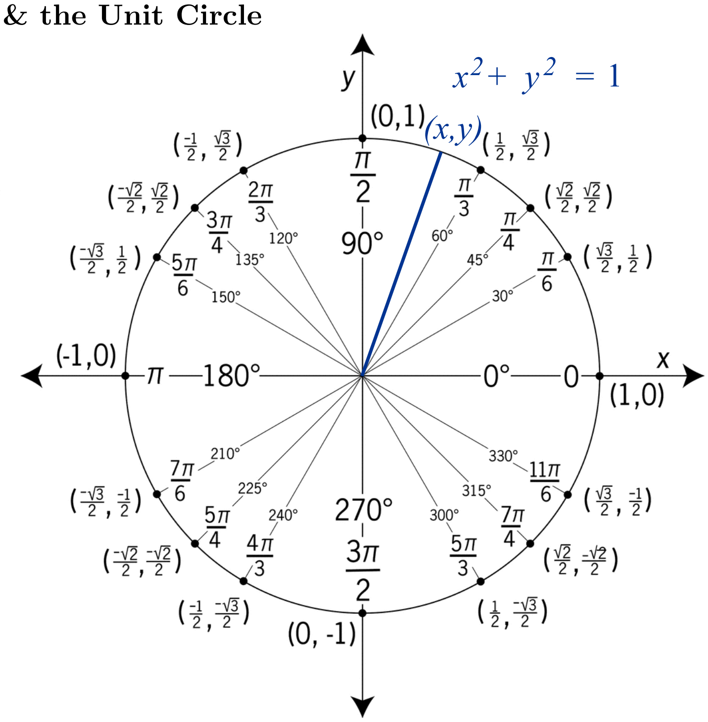
Day 1. Look at the above table and the picture at right and select those angles which are a multiple of 45° or $\frac{\pi}{4}$ radians
  a. Convert one of them from degrees to radians
  b. Convert another of them from radians to degrees
  c. Try to memorize their radian and degree value.

Day 2. Look at the above table and the picture at right and select those angles which are a multiple of 60° or $\frac{\pi}{3}$ radians
  a. Convert one of them from degrees to radians
  b. Convert another of them from radians to degrees
  c. Try to memorize their radian and degree value.

Day 3. Look at the above table and the picture at right and select those angles which are a multiple of 30° or $\frac{\pi}{6}$ radians
  a. Convert one of them from degrees to radians
  b. Convert another of them from radians to degrees
  c. Try to memorize their radian and degree value.

§15 Trigonometric Functions Right Triangles, & the Unit Circle

We will begin with the unit circle.



Let θ be an angle in standard position, and let (x, y) be a point on the terminal side of the angle and for which $x^2 + y^2 = 1$. That is, (x, y) is a point on the unit circle. We define:

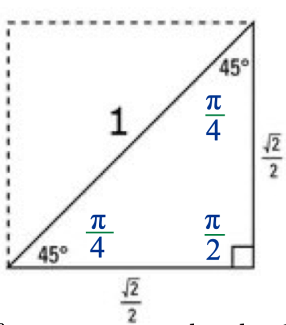
$\sin(\theta) = \frac{y}{1}$	$\csc(\theta) = \frac{1}{y}$
$\cos(\theta) = \frac{x}{1}$	$\sec(\theta) = \frac{1}{x}$
$\tan(\theta) = \frac{y}{x}$	$\cot(\theta) = \frac{x}{y}$

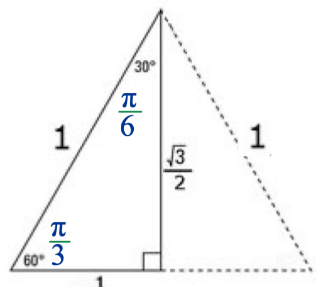
Note: If the radius of the circle is r, then

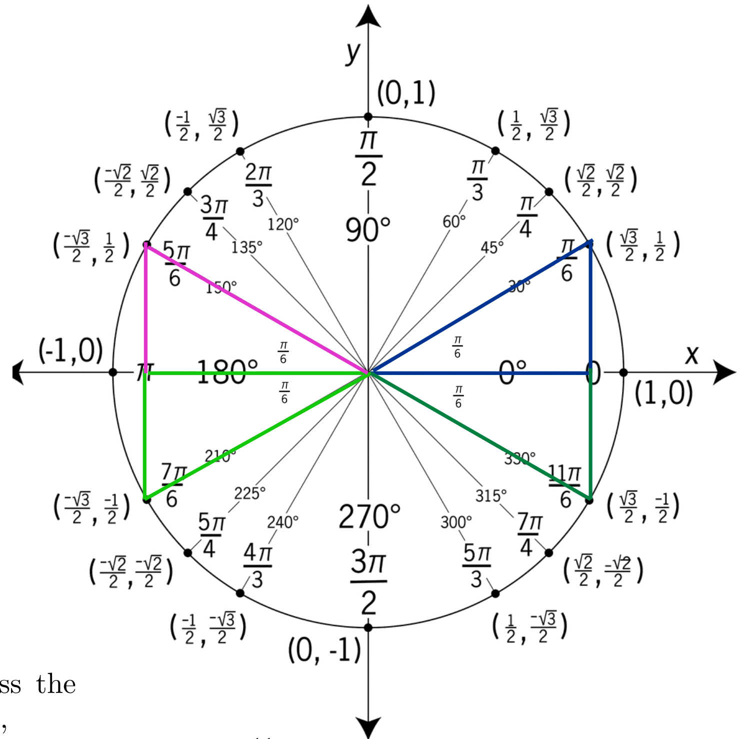
$\sin(\theta) = \frac{y}{r}$	$\csc(\theta) = \frac{r}{y}$
$\cos(\theta) = \frac{x}{r}$	$\sec(\theta) = \frac{r}{x}$
$\tan(\theta) = \frac{y}{x}$	$\cot(\theta) = \frac{x}{y}$

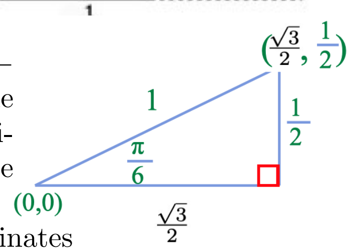
and it does not matter which definition you use, because for fixed θ the pictured triangle(s) for different values of r are all similar. Thus, the ratios of corresponding sides of those triangles are equal.

First, let us verify the points on the edge of the unit circle. Just because someone has written something, does not make it true.

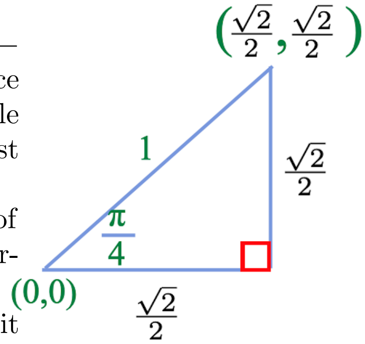


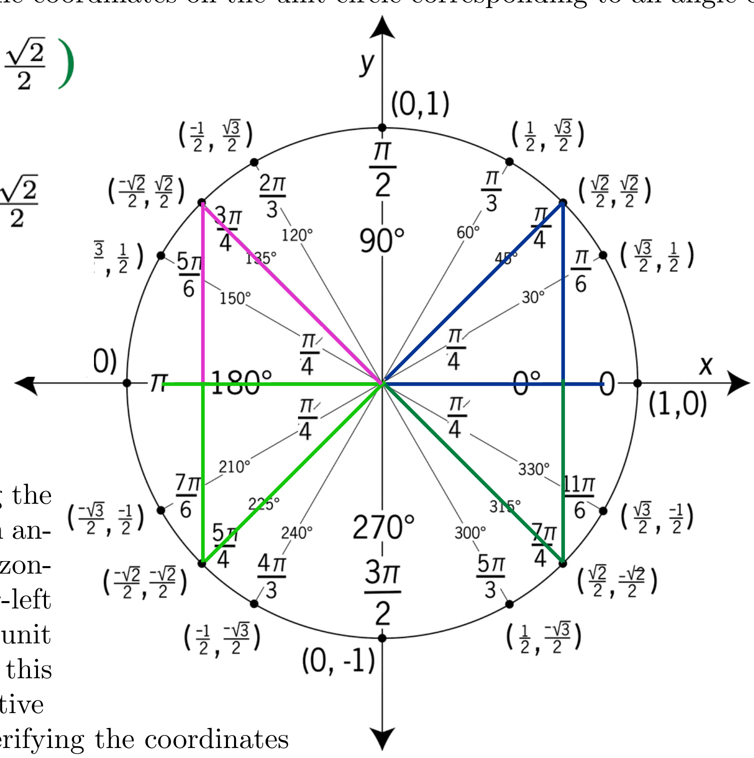






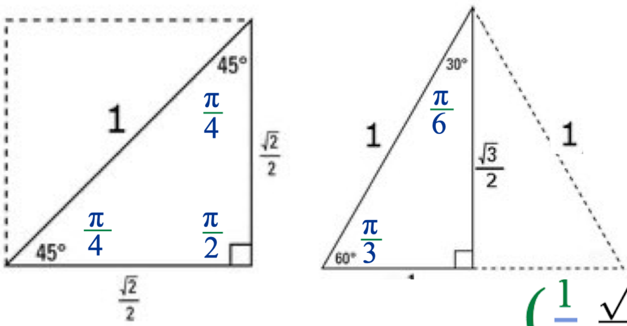
If we were to take the $30-60-90$ triangle out of the equilateral triangle and orient it as right and then place it over the bolded triangle in the unit circle, the coordinates of the upper-right vertex of the triangle verify the coordinates corresponding to an angle of $\frac{\pi}{6}$ on the unit circle. This same triangle can be flipped upside down across the positive x-axis to cover the lower-right bolded triangle, verifying the coordinates on the unit circle corresponding to an angle of $\frac{11\pi}{6}$. This same triangle can be flipped horizontally across the negative y-axis to cover the lower-left bolded triangle verifying the coordinates on the unit circle corresponding to an angle of $\frac{7\pi}{6}$. Finally, this triangle can be flipped vertically across the negative x-axis to cover the upper-right bolded triangle, verifying the coordinates on the unit circle corresponding to an angle of $\frac{5\pi}{6}$.

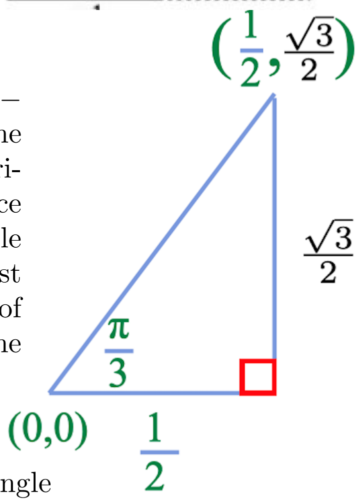


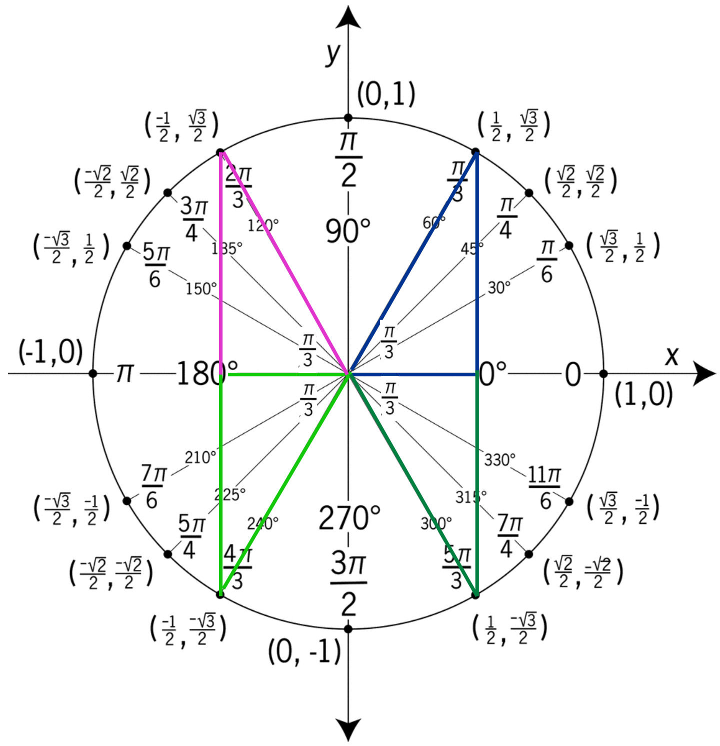


If we were to take the $45-45-90$ triangle and place it over the bolded triangle in the unit circle in the first quadrant, the coordinates of the upper-right vertex of the triangle verify-the coordinates corresponding to an angle of $\frac{\pi}{4}$ on the unit circle. This same triangle can be flipped upside down across the positive x-axis to cover the lower-right bolded triangle, verifying the coordinates on the unit circle corresponding to an angle of $\frac{7\pi}{4}$. This same triangle can be flipped horizontally across the negative y-axis to cover the lower-left bolded triangle verifying the coordinates on the unit circle corresponding to an angle of $\frac{5\pi}{4}$. Finally, this triangle can be flipped vertically across the negative x-axis to cover the upper–left bolded triangle, verifying the coordinates on the unit circle corresponding to an angle of $\frac{3\pi}{4}$.

The coordinates of $(1,0)$ and $(-1,0)$ where the unit circle meets the x-axis and the coordinates $(0,1)$ and $(0,-1)$ where the unit circle meets the y-axis are clearly correct.

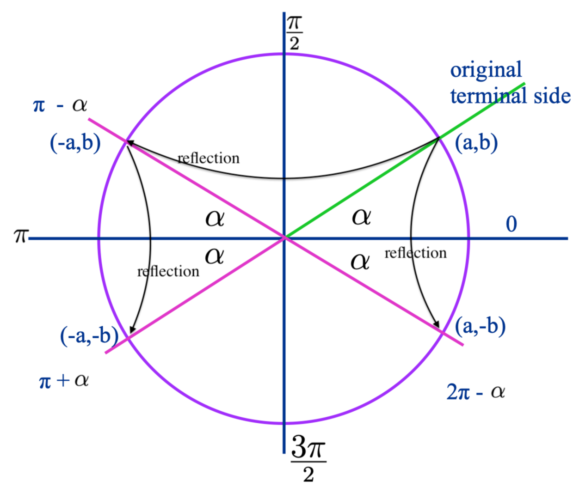






If we were to take the $30-60-90$ triangle out of the equilateral triangle and orient it as right and then place it over the bolded triangle in the unit circle in the first quadrant, the coordinates of the upper-right vertex of the triangle verify the coordinates corresponding to an angle of $\frac{\pi}{3}$ on the unit circle. This same triangle can be flipped upside down across the positive x-axis to cover the lower-right bolded triangle, verifying the coordinates on the unit circle corresponding to an angle of $\frac{5\pi}{3}$. This same triangle can be flipped horizontally across the negative y-axis to cover the lower–left bolded triangle verifying the coordinates on the unit circle corresponding to an angle of $\frac{4\pi}{3}$. Finally, this triangle can be flipped vertically across the negative x-axis to cover the upper–right bolded triangle, verifying the coordinates on the unit circle corresponding to an angle of $\frac{2\pi}{3}$.

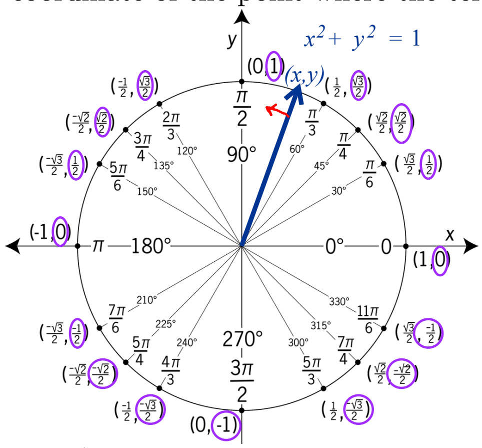
Important Note: The above work verified the coordinate pairs of the edge of the unit circle for these special angles, but in the process a pattern occurred which will be useful to us in the future. In particular, consider the general results as we reflect a terminal side across the coordinate axes. In the picture at right, the original angle was the angle α with terminal side in the first quadrant. If we reflect this terminal side and its corresponding ordered pair across the x-axis, we get a ray which makes an angle α with the negative x axis. Convince yourself that this is the terminal side of the angle $\pi-\alpha$. The new corresponding point is exactly the same height as the original ordered pair and so the y-coordinate is the same, but its location is the same distance left of the y-axis as the original point was right of the y-axis. So the x-coordinate will be the negative of the original y-coordinate. The corresponding point to $\pi-\alpha$ is $(-a, b)$.



If we reflect the terminal side of $\pi-\alpha$ and corresponding point $(-a, b)$ across the x-axis, the new ray lies in the 3rd Quadrant and makes an angle of α with the negative x-axis. Convince yourself that this ray is the terminal side of the angle $\pi+\alpha$. The corresponding point is the same distance left of the y-axis as is $(-a, b)$ but it is as far below the x-axis as $(-a, b)$ was above the x-axis. Hence the corresponding point must be $(-a, -b)$.

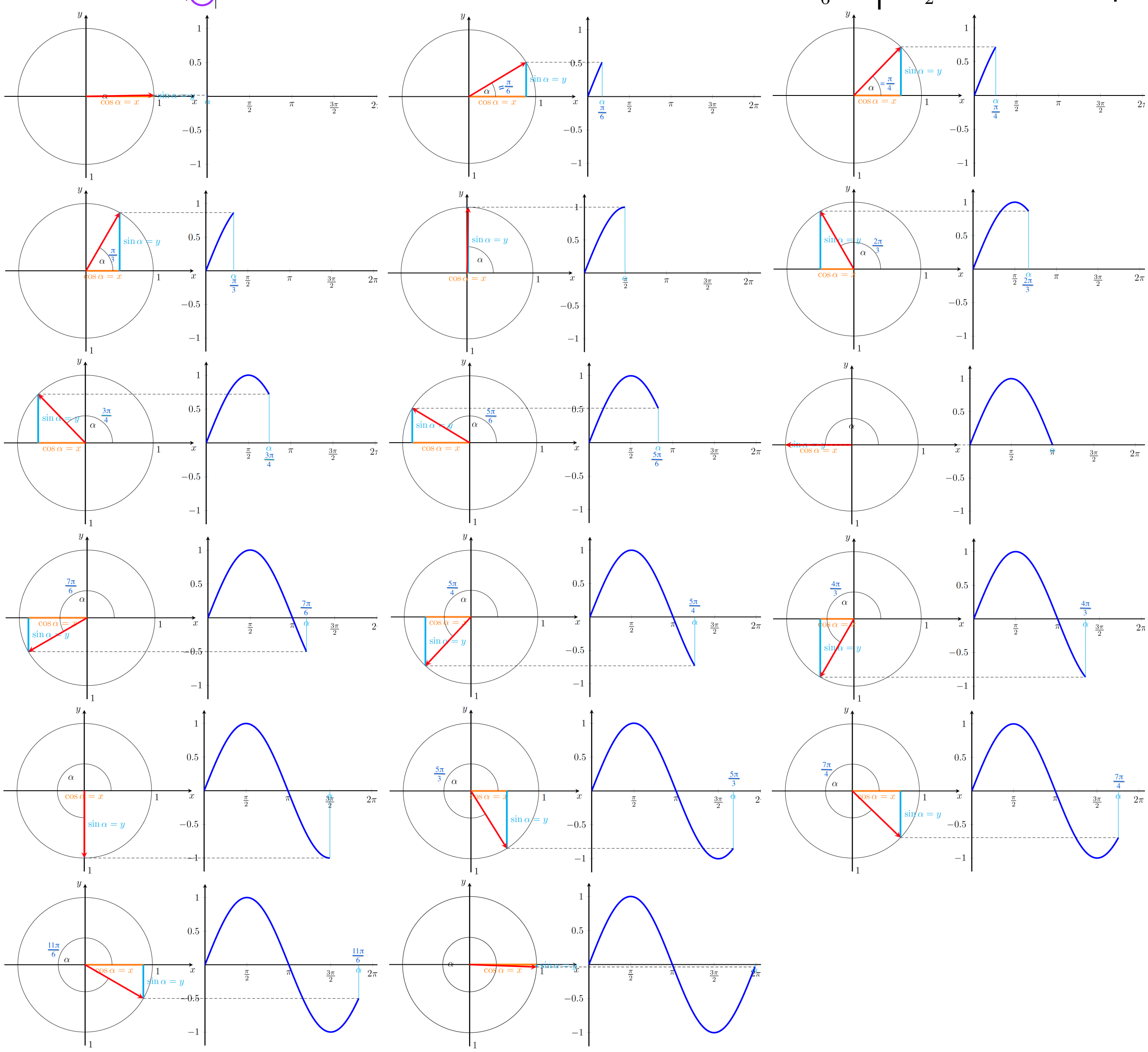
Finally, if we reflect the original angle α and its corresponding point (a, b) across the x axis, it makes an angle of α with the positive x-axis. Convince yourself that this ray is the terminal side of the angle $2\pi-\alpha$. The corresponding point is the same distance right of the y-axis as is (a, b) but it is as far below the x-axis as (a, b) was above the x-axis. Hence the corresponding point must be $(a, -b)$.

One way that we get more familiar with a function is through looking at its graph. Let's start with the Sine function, $\sin(x)$. If you use the unit circle to define the six trigonometric functions (below, center), and consult the unit circle (below, left). Note that we will always consider the argument (or input) of trigonometric functions to be in radians. Using the unit circle to obtain the values of the sine function, the sine function is always the y-coordinate of the point where the terminal side of the angle θ intersects the unit circle.

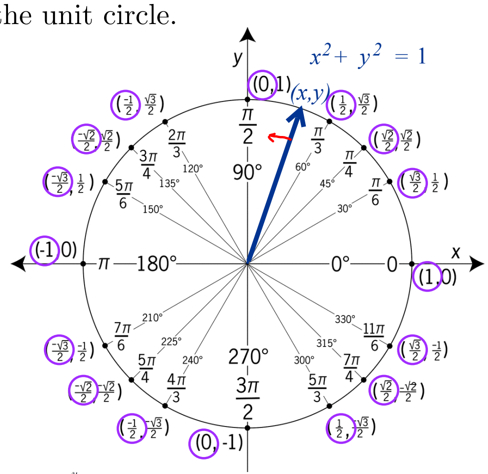


$$\sin(\theta) = \frac{y}{1} \qquad \csc(\theta) = \frac{1}{y}$$
$$\cos(\theta) = \frac{x}{1} \qquad \sec(\theta) = \frac{1}{x}$$
$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{x}{y}$$

Angle θ	$\sin(\theta)$	Angle θ	$\sin(\theta)$
0	0	$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{3\pi}{2}$	-1
$\frac{\pi}{2}$	1	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{11\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	2π	0



Now we'll graph the Cosine function, $\cos(x)$. If you use the unit circle to define the six trigonometric functions (below, center), and consult the unit circle (below, left). Using the unit circle to obtain the values of the cosine function, the cosine function is always the x-coordinate of the point where the terminal side of the angle θ intersects the unit circle.



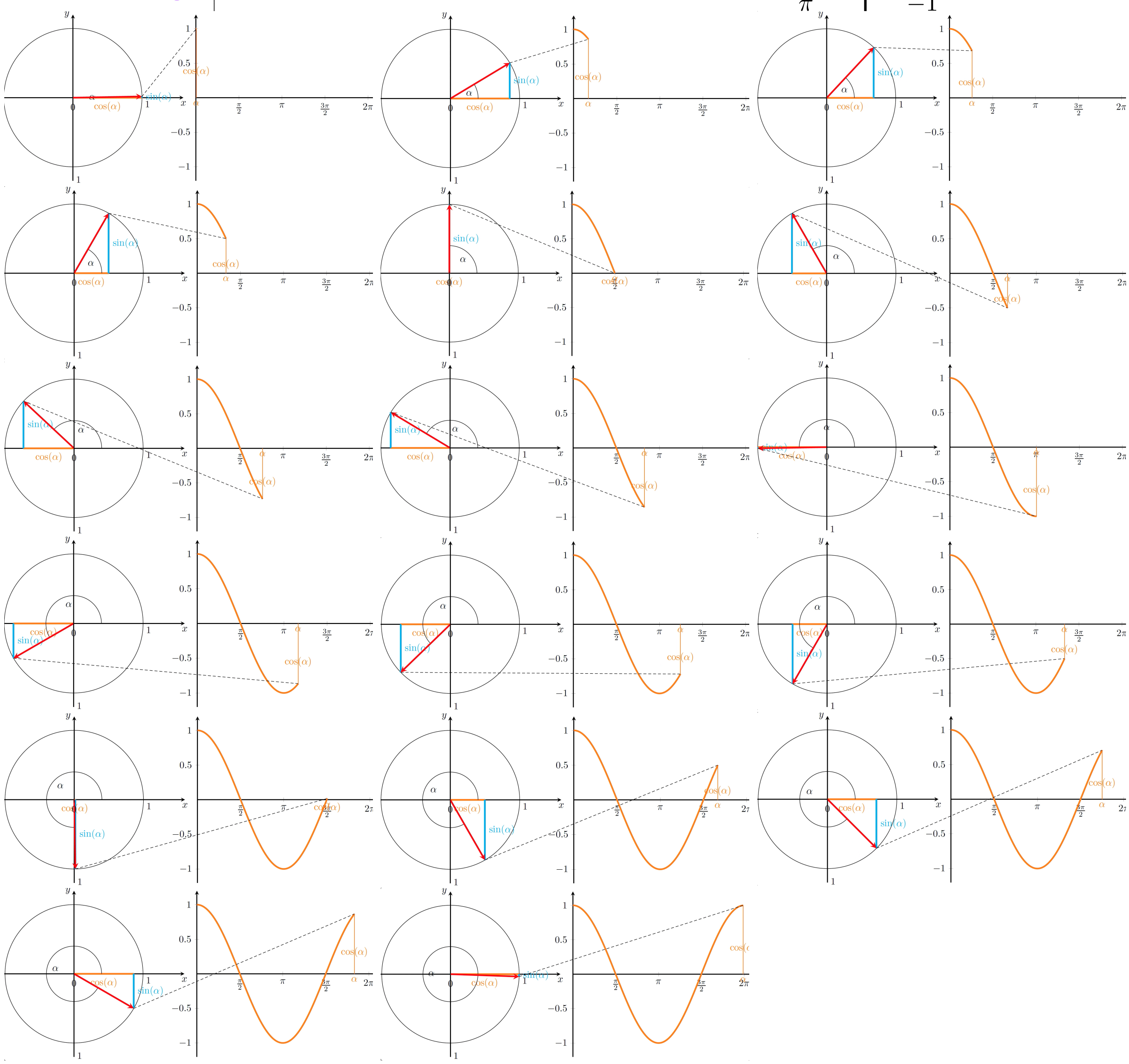
$\sin(\theta) = \frac{y}{1}$ $\qquad \csc(\theta) = \frac{1}{y}$

$\cos(\theta) = \frac{x}{1}$ $\qquad \sec(\theta) = \frac{1}{x}$

$\tan(\theta) = \frac{y}{x}$ $\qquad \cot(\theta) = \frac{x}{y}$

Angle θ	$\cos(\theta)$
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$
π	-1

Angle θ	$\cos(\theta)$
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$
$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{2}$	0
$\frac{5\pi}{3}$	$\frac{1}{2}$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$
2π	1



What relationships might we have noted from the verification of the values on the Unit Circle and the graphing of the Sine and Cosine?

1. As mentioned on the previous page, for any given angle α, the points on the Unit Circle corresponding to α, $\pi - \alpha$, $\pi + \alpha$, and $2\pi - \alpha$ have coordinates which only differ from one another in their sign (+ or $-$).

This provides a very big clue to the computation of these functions and their inverse functions, and so we will define a term to help make this more readily usable.

Definition: Let θ be an angle with its initial side on the positive x axis. The *reference angle* of θ is *the* positive acute angle which the terminal side of θ makes with the x-axis.

Aside: In Mathematics, the choice of the word *the* is not an accident. Draw an angle and measure the various angles with the terminal side makes with the x-axis. Only one is an acute angle. For instance, the angle $\frac{5\pi}{6}$ is an angle whose terminal side is in the 2nd Quadrant and it makes an angle of $\frac{5\pi}{6}$ with the positive x axis, but it also makes an angle of $\frac{\pi}{6}$ with the negative x axis. [For reference angles, we aren't concerned with the direction (CCW vs CW) but we are computing the absolute value of the angle between the two, however traced.] That is all we might think about initially, but going an additional revolution CCW from $\frac{5\pi}{6}$, it makes an angle of $\frac{17\pi}{6}$ (which is the sum of $\frac{5\pi}{6}$ and 2π), with the positive x-axis, and also $\frac{29\pi}{6}$ and $\frac{41\pi}{6}$ with the positive x-axis, etc, each time adding a full revolution. This terminal side also makes an angle of $\frac{11\pi}{6}$ with the negative x-axis (measuring CW back to the negative x-axis), and $\frac{23\pi}{6}$, and ... More about this way of thinking later.

To summarize, the following angles all share the reference angle $\frac{\pi}{6}$.

Quadrant	**1st Quadrant**	**2nd Quadrant**	**3rd Quadrant**	**4th Quadrant**
Angle	$\frac{\pi}{6}$	$\pi - \frac{\pi}{6}$	$\pi + \frac{\pi}{6}$	$2\pi - \frac{\pi}{6}$
Terminal Side	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$
Ordered Pair	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

The following angles all share the reference angle $\frac{\pi}{4}$,

Quadrant	**1st Quadrant**	**2nd Quadrant**	**3rd Quadrant**	**4th Quadrant**
Angle	$\frac{\pi}{4}$	$\pi - \frac{\pi}{4}$	$\pi + \frac{\pi}{4}$	$2\pi - \frac{\pi}{4}$
Terminal Side	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
Ordered Pair	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

The following angles all share the reference angle $\frac{\pi}{3}$,

Quadrant	**1st Quadrant**	**2nd Quadrant**	**3rd Quadrant**	**4th Quadrant**
Angle	$\frac{\pi}{3}$	$\pi - \frac{\pi}{3}$	$\pi + \frac{\pi}{3}$	$2\pi - \frac{\pi}{3}$
Terminal Side	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
Ordered Pair	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Moreover, in each case the coordinates are identical, **except** for the sign.

In summary, these angles all have the same reference angle.

(a) $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$ all have reference angle $\frac{\pi}{6}$.
hence $\sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) = -\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{11\pi}{6}\right)$ (the last two negative because y is negative in QIII, QIV)
and $\cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{11\pi}{6}\right) = -\cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{7\pi}{6}\right)$ (the last two negative because x is negative in QII, QIII)
and $\tan\left(\frac{\pi}{6}\right) = \tan\left(\frac{7\pi}{6}\right) = -\tan\left(\frac{5\pi}{6}\right) = -\tan\left(\frac{11\pi}{6}\right)$ (the last two negative because $\frac{y}{x}$ is negative in QII, QIV)

(b) $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$ all have reference angle $\frac{\pi}{4}$. . . . see the next Exercise set.

(c) $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and $\frac{5\pi}{3}$ all have reference angle $\frac{\pi}{3}$. . . . see the next Exercise set.

(d) 0 and π both have reference angle 0. . . . see the next Exercise set.

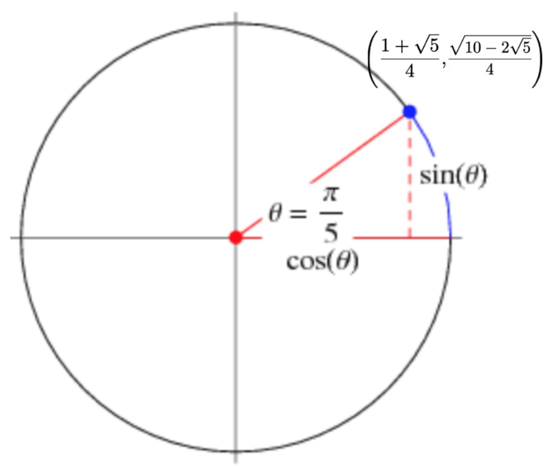
(e) $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ both have reference angle $\frac{\pi}{2}$. . . . see the next Exercise set.

Remark: You can see that we have spent a lot of time developing the coordinate pairs on the unit circle and we've been computing the values of the trigonometric functions of these special angles using that unit circle. We have a calculator at our disposal, why not use the calculator go obtain each of those values? As mentioned in the preface, unless you engage with these problems with your mental resources then you'll know the ideas only as separate and artificial ideas. This is especially true for something like Trigonometry where many of the ideas are particularly new to you. If one has to use three or more memory registers to hold the basic ideas, then there is little or no room left to help you progress. In addition, through repeated use you can *chunk* many basic ideas into one big idea that only occupies one spot in working memory; this allows you to be much more efficient with both your working memory space and with your thinking in general. Thus, if a special angle is involved (some multiple of $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$ or π) or if one of the values $\frac{\pm 1}{2}$, $\frac{\pm\sqrt{3}}{2}$, $\frac{\pm\sqrt{2}}{2}$, $\pm\sqrt{3}$, $\frac{\pm 1}{\sqrt{3}}$, or ± 1 which comes from a function value of a special angle, then we will work those by hand.

Exercise Set 15.1

1. We have some notion that $\sin\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{4}\right)$, $\sin\left(\frac{5\pi}{4}\right)$, $\sin\left(\frac{7\pi}{4}\right)$ will be similar. If you knew that $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$, immediately, without looking at the unit circle, see if you can find the value of the other three.
2. Similarly, if you knew that $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, immediately, without looking at the unit circle, see if you can find the value of the other three.
3. Finally, if you knew that $\tan\left(\frac{3\pi}{4}\right) = -1$, immediately, without looking at the unit circle, see if you can find the value of $\tan\left(\frac{\pi}{4}\right)$, $\tan\left(\frac{5\pi}{4}\right)$, $\tan\left(\frac{7\pi}{4}\right)$.
4. Compute $\sec\left(\frac{\pi}{3}\right)$ using the unit circle. From that, without looking at the unit circle, see if you can find the value of $\sec\left(\frac{2\pi}{3}\right)$, $\sec\left(\frac{4\pi}{3}\right)$, $\sec\left(\frac{5\pi}{3}\right)$.
5. Compute $\sin\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{3\pi}{2}\right)$; compute $\cos\left(\frac{\pi}{2}\right)$ and $\cos\left(\frac{3\pi}{2}\right)$.
6. Compute $\sin(\pi)$ and $\sin(0)$; compute $\cos(\pi)$ and $\cos(0)$; compute $\tan(\pi)$ and $\tan(0)$.

Example 15.1: Given the unit circle at right, compute the value of each of the six trigonometric functions at $\theta = \frac{\pi}{5}$.



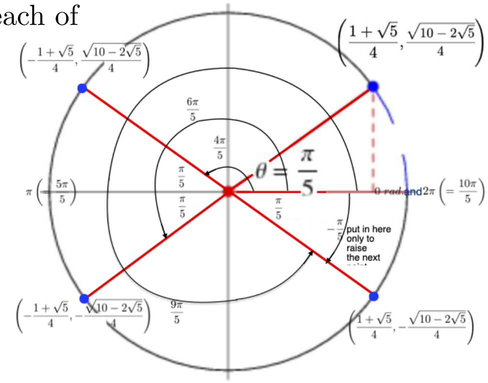
1. $\sin\left(\frac{\pi}{5}\right) =$
2. $\cos\left(\frac{\pi}{5}\right) =$
3. $\tan\left(\frac{\pi}{5}\right) =$
4. $\cot\left(\frac{\pi}{5}\right) =$
5. $\sec\left(\frac{\pi}{5}\right) =$
6. $\csc\left(\frac{\pi}{5}\right) =$

Solutions:

1. $\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{10-2\sqrt{5}}}{4}$ (the y-coordinate of the point on the edge of the unit circle where the terminal side of the angle intersects it)
2. $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$ (the x-coordinate of the point on the edge of the unit circle where the terminal side of the angle intersects it)
3. $\tan\left(\frac{\pi}{5}\right) = \frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}$ (the ratio of the y-coordinate to the x-coordinate of the point on the edge of the unit circle where the terminal side of the angle intersects it)
4. $\cot\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$ (the ratio of the x-coordinate to the y-coordinate of the point on the edge of the unit circle where the terminal side of the angle intersects it)
5. $\sec\left(\frac{\pi}{5}\right) = \frac{4}{1+\sqrt{5}}$ (the reciprocal of the x-coordinate of the point on the edge of the unit circle where the terminal side of the angle intersects it)
6. $\csc\left(\frac{\pi}{5}\right) = \frac{4}{\sqrt{10-2\sqrt{5}}}$ (the reciprocal of the y-coordinate of the point on the edge of the unit circle where the terminal side of the angle intersects it)

Example 15.2: We now know that $\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{10-2\sqrt{5}}}{4}$. On the unit circle above, draw the rest of the angles in from 0 to 2π whose reference angle is $\frac{\pi}{5}$ and label the angle and the point on the edge of the unit circle which corresponds to it.

Example 15.3: Use this unit circle to compute the value of each of the 6 trigonometric functions at $\theta = \frac{4\pi}{5}$ at $\theta = \frac{9\pi}{5}$.



Solutions:

1a. $\sin\left(\frac{4\pi}{5}\right) = \frac{\sqrt{10-2\sqrt{5}}}{4}$ 1b. $\sin\left(\frac{9\pi}{5}\right) = -\frac{\sqrt{10-2\sqrt{5}}}{4}$

2a. $\cos\left(\frac{4\pi}{5}\right) = -\frac{1+\sqrt{5}}{4}$ 2b. $\cos\left(\frac{9\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$

3a. $\tan\left(\frac{4\pi}{5}\right) = -\frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}$ 3b. $\tan\left(\frac{9\pi}{5}\right) = -\frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}$

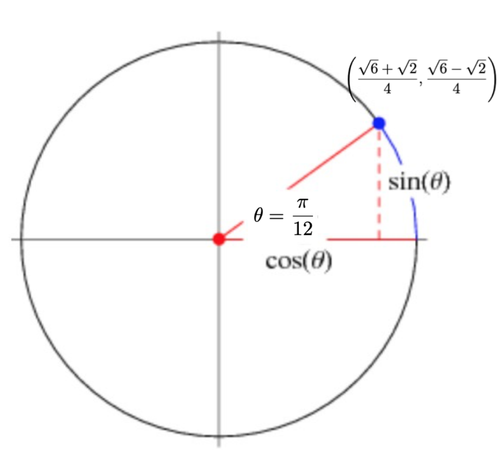
4a. $\cot\left(\frac{4\pi}{5}\right) = -\frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$ 4b. $\cot\left(\frac{9\pi}{5}\right) = -\frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$

5a. $\sec\left(\frac{4\pi}{5}\right) = -\frac{4}{1+\sqrt{5}}$ 5b. $\sec\left(\frac{9\pi}{5}\right) = \frac{4}{1+\sqrt{5}}$

6a. $\csc\left(\frac{4\pi}{5}\right) = \frac{4}{\sqrt{10-2\sqrt{5}}}$ 6b. $\csc\left(\frac{9\pi}{5}\right) = -\frac{4}{\sqrt{10-2\sqrt{5}}}$

Exercise Set 15.2

Using the unit circle at right, compute the value of each of the following.



1a. $\sin\left(\frac{\pi}{12}\right) =$

1b. $\cos\left(\frac{\pi}{12}\right) =$

1c. $\tan\left(\frac{\pi}{12}\right) =$

1d. $\cot\left(\frac{\pi}{12}\right) =$

1e. $\sec\left(\frac{\pi}{12}\right) =$

1f. $\csc\left(\frac{\pi}{12}\right) =$

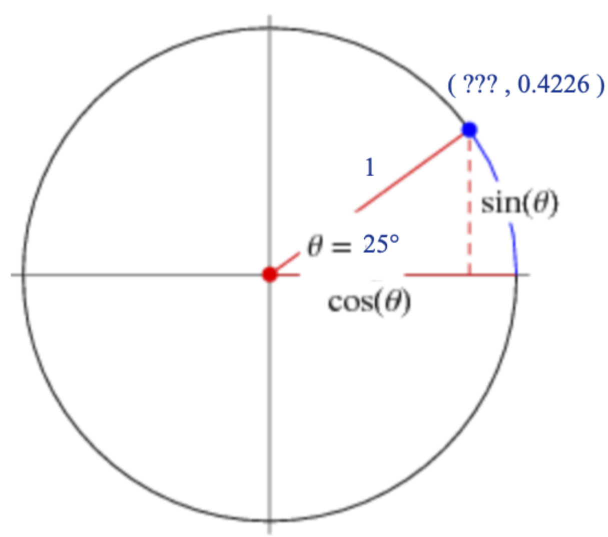
2. On the unit circle above, draw the rest of the angles in from 0 to 2π whose reference angle is $\frac{\pi}{12}$ and label the angle and the point on the edge of the unit circle which corresponds to it.

3. Use the resulting unit circle to compute the value of each of the six trigonometric functions at the angle whose terminal side is in Quadrant II and whose reference angle is $\frac{\pi}{12}$.

4. Use the resulting unit circle to compute the value of each of the six trigonometric functions at the angle whose terminal side is in Quadrant III and whose reference angle is $\frac{\pi}{12}$.

5. Use the resulting unit circle to compute the value of each of the six trigonometric functions at the angle whose terminal side is in Quadrant IV and whose reference angle is $\frac{\pi}{12}$

Example 15.4: Suppose that $\sin(25°) \simeq 0.4226$. Compute $\cos(\theta)$.



Solution: At right is a picture of what we know. Given a unit circle, the terminal side of a 25° angle intersects the unit circle at a point with y-coordinate $y = 0.4226$. We then know two sides of a right triangle, and can use the Pythagorean Theorem to find that the other leg is: $x = 0.9063$ Thus, the complete ordered pair is $(0.9063, 0.4226)$.

Thus, $\cos(25°) = 0.9063$

Of course, from this ordered pair you can compute the value of $\tan(25°)$, $\cot(25°)$, $\sec(25°)$, $\csc(25°)$.

Example 15.5: Find each of the remaining angles, $0° \leq \theta < 360°$ which have the same reference angle, and compute the value of each of the 6 trigonometric functions at each of these 4 angles. Sketch these angles, with accompanying points on the unit circle above.

Solution: The other angles are: Second Quadrant: $180° - 25° = 155°$; Third Quadrant: $180° + 25° = 205°$; Fourth Quadrant: $360° - 25° = 335°$.

Rather than include a picture with this solution, let's think our way to determining the point on the unit circle corresponding to the terminal side of each angle.

In the second quadrant, the variable x is negative and the variable y is positive. Thus, when the terminal side of the angle $\theta = 25°$ and its corresponding point $(0.9063, 0.4226)$ are reflected across the y-axis, we will get an angle with terminal side $\theta = 155°$ and corresponding point $(-0.9063, 0.4226)$. This allows us to compute the values of the six trigonometric functions for $\theta = 155°$.

In the third quadrant, the variable x is negative and the variable y is negative. Thus, when the terminal side of the angle $\theta = 25°$ and its corresponding point $(0.9063, 0.4226)$ are reflected across the y-axis, (as above) and then immediately across the x-axis, we will get an angle with terminal side $\theta = 205°$ and corresponding point $(-0.9063, -0.4226)$. This allows us to compute the values of the six trigonometric functions for $\theta = 205°$.

In the fourth quadrant, the variable x is positive and the variable y is negative. Thus, when the terminal side of the angle $\theta = 25°$ and its corresponding point $(0.9063, 0.4226)$ are reflected across the x-axis, we will get an angle with terminal side $\theta = 335°$ and corresponding point $(0.9063, -0.4226)$. This allows us to compute the values of the six trigonometric functions for $\theta = 335°$.

Example 15.6: Suppose that $0 \leq \alpha < 2\pi$, and $\sin(\alpha) = -\frac{\sqrt{2}}{2}$. If $\tan(\alpha)$ is positive, what is α and what is $\tan(\alpha)$. [First, try to solve this without looking at the unit circle; if you can't quite remember, then look at the unit circle and answer the question. We'll provide the solution after the next example.]

Example 15.7: Suppose that $0 \leq \alpha < 2\pi$, and $\cos(\alpha) = -\frac{\sqrt{3}}{2}$. If $\sin(\alpha)$ is negative, what is α and what is $\sin(\alpha)$. [First, try to solve this without looking at the unit circle; if you can't quite remember, then look at the unit circle and answer the question. We'll provide the solution after the solution to Example 15.7.]

Solution to Example 15.6: We are given that $\sin(\alpha) = -\frac{\sqrt{2}}{2}$, and this might remind us that the reference angle is $\frac{\pi}{4}$. In addition, the sine is negative, which means that the terminal side of α is in Quadrant III or Quadrant IV (where the variable y upon which sine is based, is negative). That is not enough information to specify a single angle, but once we are given that $\tan(\alpha)$ is positive, then we know that the terminal side of α must be in Quadrant I or Quadrant III, where x, y both have the same sign. Only Quadrant III meets both of these requirements. So α is in Quadrant III with reference angle $\frac{\pi}{4}$ and this makes $\alpha = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

Solution to Example 15.7: We are given that $\cos(\alpha) = -\frac{\sqrt{3}}{2}$, and this might remind us that the reference angle is $\frac{\pi}{6}$. In addition, the cosine is negative, which means that the terminal side of α is in Quadrant II or Quadrant III (where the variable x upon which cosine is based, is negative). That is not enough information to specify a single angle, but once we are given that $\sin(\alpha)$ is negative, then we know that the terminal side of α must be in Quadrant III or Quadrant IV, where the variable y is negative. Only Quadrant III meets both of these requirements. So α is in Quadrant III with reference angle $\frac{\pi}{3}$ and this makes $\alpha = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Exercise Set 15.3

1. Given that $\cos\left(\frac{5\pi}{18}\right) \simeq 0.6428$ sketch the unit circle and the terminal side of the angle $\alpha = \frac{5\pi}{18}$, and find the value of the $\sin\left(\frac{5\pi}{18}\right)$

2. From the information found in problem 1, label the point on the unit circle corresponding to the terminal side of $\alpha = \frac{5\pi}{18}$. Moreover, find all angles between 0 and 2π whose reference angle is $\alpha = \frac{5\pi}{18}$ and add them to your unit circle.

3. Label the unit circle from problem 2, and label the points on the unit circle corresponding to each of the terminal sides drawn in problem 2. There should be one in each quadrant.

4. Compute the value of each of the six trigonometric functions at $\alpha = \frac{23\pi}{18}$. ... at $\alpha = \frac{31\pi}{18}$.

5. Find the reference angle of the angle $\alpha = \frac{10\pi}{7}$. Draw a unit circle, the terminal side of this angle, and label the terminal side.

6. If we know that $\sin\left(\frac{10\pi}{7}\right) \simeq -0.9750$, find the value of $\cos\left(\frac{10\pi}{7}\right)$. Update the unit circle from problem 5 with the ordered pair which corresponds to the angle $\alpha = \frac{10\pi}{7}$

7. Compute the value of each of the 6 trigonometric functions at $\alpha = \frac{10\pi}{7}$.

8. Sketch on the unit circle of problem 6, the angle whose terminal side is in the 4th Quadrant and whose reference angle is the same as that of $\alpha = \frac{10\pi}{7}$. What is this angle? Label the point where this terminal side intersects the unit circle.

9. Compute the value of each of the 6 trigonometric functions at $\alpha = \frac{11\pi}{7}$.

10. Suppose that α is an angle with terminal side in the 3rd Quadrant, and for which $\cos(\alpha) = -0.8090$. Find $\sin(\alpha)$.

11. Find the value of each of the remaining 4 trigonometric functions at the angle in problem 10.

12. Suppose that β is an angle whose terminal side is in the 2nd quadrant, and which has the same reference angle as the angle α from problems 10 and 11. Find the value of $\sin(\beta)$, $\cos(\beta)$, $\tan(\beta)$, $\cot(\beta)$, $\sec(\beta)$, $\csc(\beta)$.

13. Suppose that $\tan(\theta) = +1$. Find the first two positive angles, θ, which satisfy $\tan(\theta) = 1$.

14. Suppose that $\cos(\alpha) = -\frac{1}{2}$. Find the first two positive angles, α, which satisfy $\cos(\alpha) = -\frac{1}{2}$.

15. Looking back at problem 14, find the first two positive angles, γ, for which $\cos(\gamma) = \frac{1}{2}$.

16. Looking back at problem 15, with one of those angles γ, $0 \le \gamma < 2\pi$ and $\cos(\gamma) = -\frac{1}{2}$. Suppose $\tan(\gamma)$ is negative. Which quadrant are we in, and what is γ.

17. Draw the terminal side of the angle $\theta = \frac{7\pi}{3}$ on the unit circle. What is a reasonable value for $\cos\left(\frac{7\pi}{3}\right)$?

18. Draw the terminal side of the angle $\theta = \frac{13\pi}{3}$ on the unit circle. What is a reasonable value for $\cos\left(\frac{13\pi}{3}\right)$?

19. What is the reference angle for each of the angles in problems 14, 15, 16, 17, 18?

§16. Trigonometric Functions of Real Numbers, Periodicity

Let t be a real number, and let (x, y) be a point on the unit circle which is t units of arc from the point $(1, 0)$. Since it is a unit circle, the terminal side of an angle of t radians also passes through (x, y). We define:

$\sin(\theta) = \frac{y}{1}$ $\qquad$ $\csc(\theta) = \frac{1}{y}$

$\cos(\theta) = \frac{x}{1}$ $\qquad$ $\sec(\theta) = \frac{1}{x}$

$\tan(\theta) = \frac{y}{x}$ $\qquad$ $\cot(\theta) = \frac{x}{y}$

This is essentially identical to the definition given at the opening of §15, except back at that time we were probably thinking of the angle θ as being between 0 and 2π radians. This time we began with a real number t. You might have noticed in the exercises above $t = \frac{7\pi}{3}$ wrapped around past 2π and $t = \frac{13\pi}{3}$ did so a second time. However, since it shared the same terminal side with the angle $\frac{\pi}{3}$ you probably correctly assumed that $\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \cos\left(\frac{13\pi}{3}\right)$. Two angles which share the same terminal side are said to be *co-terminal*. In addition, in Example 15.3 for the angle $\theta = \frac{\pi}{5}$, we drew in the angle $\theta = -\frac{\pi}{5}$, which is co-terminal with the angle $\frac{9\pi}{5}$. Thus, $\cos\left(-\frac{\pi}{5}\right) = \cos\left(\frac{9\pi}{5}\right)$. You can probably guess, then, that every 2π units, the graphs of these functions will repeat, and that is essentially true, but we'll have to slow down and take a closer look. We have spent a fair amount of effort getting everything on the table: the angles, the unit circle, the underlying geometry that makes reference angles important, and the definitions of the six functions, and how to compute them at values for which we know the corresponding point on the unit circle. Now, we need to step back and take better look at what we have.

Domain:

1. The domain of $f(x) = \sin(x)$ is $(-\infty, \infty)$
2. The domain of $f(x) = \cos(x)$ is $(-\infty, \infty)$

However, the rest have some difficulties, generally surrounding division by zero. Every time the angle reaches a coordinate axis, one of x or y is zero, and that means division by zero for two of the remaining 4 functions. For instance, when the value of θ is 0 radians or π radians, then the y-coordinate of the corresponding point on the unit circle is 0, and so neither the cotangent nor the cosecant are defined at 0 or π radians (or at any integer multiple of π). When the value of θ is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, then x-coordinate of the corresponding point on the unit circle is 0 and so neither the tangent or the secant are defined at $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ radians (or at any odd multiple of $\frac{\pi}{2}$). In summary,

3. The domain of $f(x) = \tan(x)$ is $\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\ldots$
4. The domain of $f(x) = \cot(x)$ is $\ldots(-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup (2\pi, 3\pi)\ldots$
5. The domain of $f(x) = \sec(x)$ is $\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\ldots$
6. The domain of $f(x) = \csc(x)$ is $\ldots(-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup (2\pi, 3\pi)\ldots$

These are functions: This seems an opportune moment to remind ourselves that these six trigonometric functions are *functions* acting on real numbers. In other words:

$$\sin(t) \neq \sin \cdot t$$

The parentheses are to hold the real number input of the function, it does not indicate multiplication by juxtaposition. As an example: $\sin\left(\frac{\pi}{6}\right)$ is $\frac{1}{2}$, and that $\sin \cdot \frac{\pi}{6}$ has no meaning at all. The same is true for $\cos(t)$, $\tan(t)$, $\cot(t)$, $\sec(t)$, and $\csc(t)$. The value inside the parentheses is called the *argument* of the function (rather than the angle) from here onward.

Fundamental Identities

$\csc(t) = \dfrac{1}{\sin(t)}$ $\qquad \sec(t) = \dfrac{1}{\cos(t)}$ $\qquad \cot(t) = \dfrac{1}{\tan(t)}$

$\tan(t) = \dfrac{\sin(t)}{\cos(t)}$ $\qquad \cot(t) = \dfrac{\cos(t)}{\sin(t)}$

$\cos^2(t) + \sin^2(t) = 1$ $\qquad 1 + \tan^2(t) = \sec^2(t)$ $\qquad \cot^2(t) + 1 = \csc^2(t)$

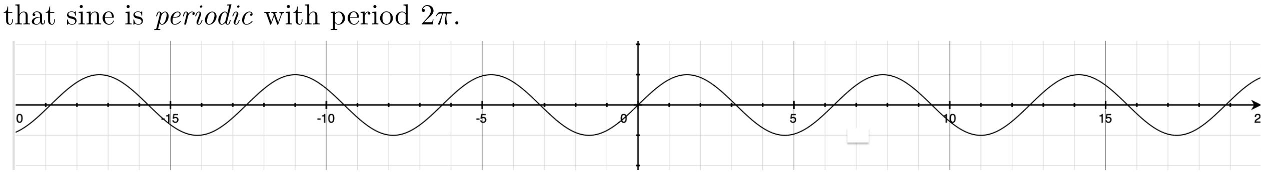
The first five of these follow from the very way that they are defined. The next three are called the *Pythagorean Identities* because they follow from the Pythagorean Theorem. For instance, the first of these, $\cos^2(t)+\sin^2(t) = 1$, follows directly from the fact that the point $(\cos(t), \sin(t))$ is on the unit circle. The other two follow from this one; the identity $1 + \tan^2(t) = \sec^2(t)$ results from dividing both sides of $\cos^2(t) + \sin^2(t) = 1$ by $\cos^2(t)$; the identity $\cot^2(t) + 1 = \csc^2(t)$ results from dividing both sides of $\cos^2(t) + \sin^2(t) = 1$ by $\sin^2(t)$.

Another immediate result of the definition of the sine and the cosine:

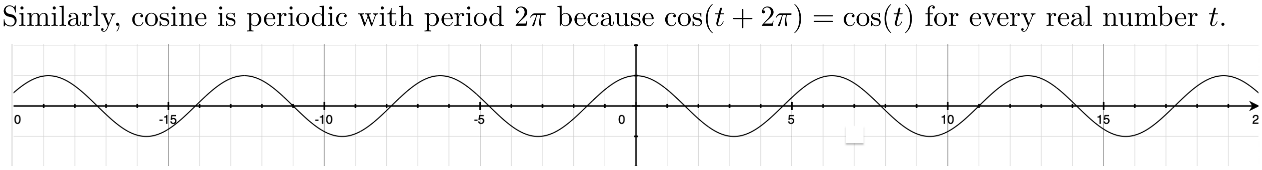
1. $-1 \leq \sin(t) \leq 1$ for all real numbers t.
2. $-1 \leq \cos(t) \leq 1$ for all real numbers t.

Graphs; Periodicity

We have already graphed the sine and cosine for $0 \leq \theta < 2\pi$. As we noted already, for $\theta \geq 2\pi$, the values of sine begin to repeat. Because of the nature of the definition, $\sin(t + 2\pi) = \sin(t)$ for every real number t. Thus, we say that sine is *periodic* with period 2π.

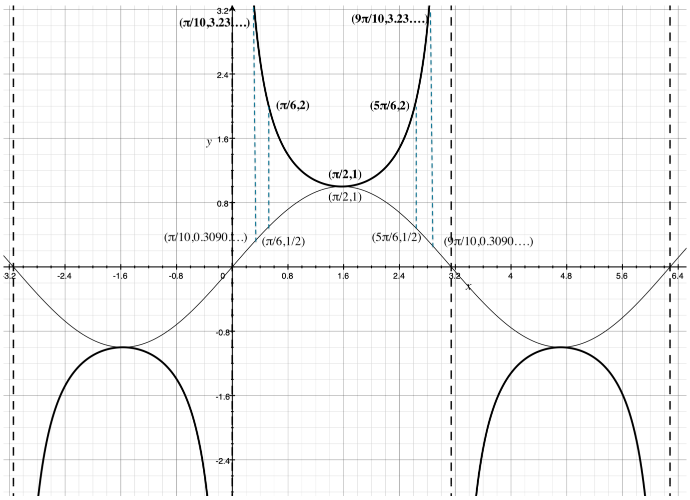


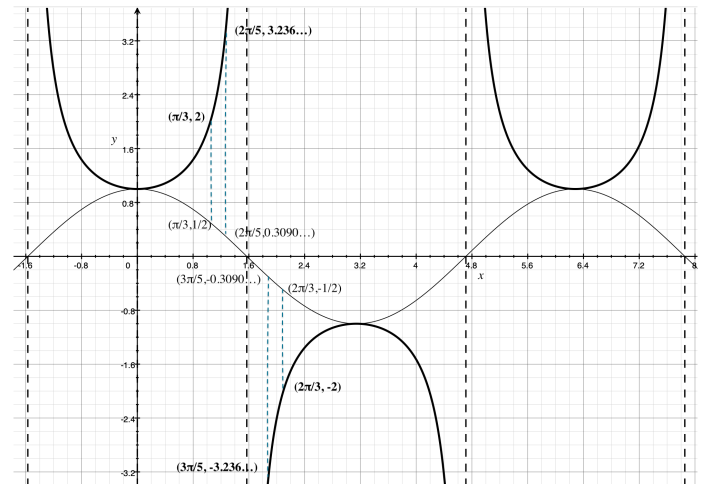
Similarly, cosine is periodic with period 2π because $\cos(t + 2\pi) = \cos(t)$ for every real number t.



The easiest two graphs to obtain after this, are those for the cosecant and secant, since they are just the reciprocals of the sine and cosine, respectively. For instance, where the sine is 1, the cosecant will be 1/1; where the sine is 1/2, the cosecant will be 1/(1/2) or 2. Where the sine is 0, the cosecant will be undefined and the graph will have a vertical asymptote. Similarly for the pair: secant and the cosine.

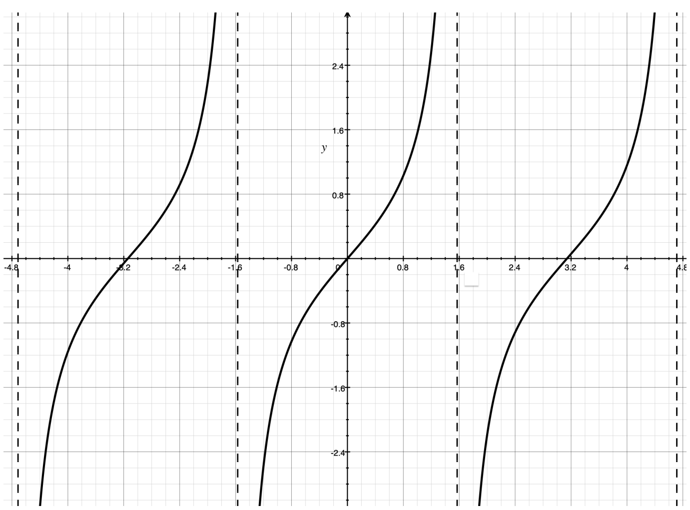
The graph of the cosecant is on the left-side and the graph of the secant is on the right side. The graph of the cosecant is bolded, while the lighter graph of the sine is there as reference, so that we can see the reciprocal relationship. Likewise, the graph of the secant is bolded, while the lighter graph of the cosine is there as reference, so that we can see the reciprocal relationship.

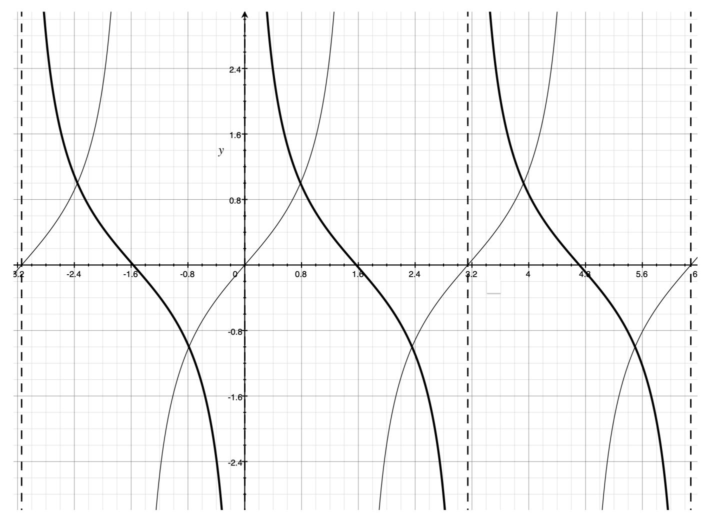




That they are periodic, with period 2π, shows up in the graph as well.

The graph of the tangent and cotangent are below. The graph of the tangent is on the left side, and the graph of the cotangent is the bolded graph in the right-hand graph. The lighter graph of the tangent is also there as a reference so that we can see its reciprocal relationship.





However, we can see that $\tan(t+\pi) = \tan(t)$ for every real number t, and $\cot(t+\pi) = \cot(t)$ for every real number t, and so both the tangent and the cotangent have period π.

Exercise Set 16.1

1. On the graph of the sine and cosine on the previous page, go through and label the x-axis at all zeros and where each maximum and minimum occurs, by writing its radian equivalent below the number line.
2. On the graph of the cosecant and secant and tangent and cotangent, go through and label the x-axis at each vertical asymptote, by writing its radian equivalent below the number line.
3. Use the unit circle and the definition of tangent to label the points on the graph of the tangent which correspond to the special angles.

Co-functions: You might have wondered about the naming convention leading to a **co**sine when there was already a sine, and a **co**secant when there was already a secant, and a **co**tangent when there was already a tangent. Analytically speaking, that is because of the following relationships

$\sin\left(\frac{\pi}{2}-t\right)=\cos(t)$ $\quad\tan\left(\frac{\pi}{2}-t\right)=\cot(t)$ $\quad\sec\left(\frac{\pi}{2}-t\right)=\csc(t)$

$\cos\left(\frac{\pi}{2}-t\right)=\sin(t)$ $\quad\cot\left(\frac{\pi}{2}-t\right)=\tan(t)$ $\quad\csc\left(\frac{\pi}{2}-t\right)=\sec(t)$

and the angles t and $(\frac{\pi}{2}-t)$ are **co**mplementary angles. If one is familiar with graphs, then this relationship can be seen in the graphs above if one notes that $f\left(\frac{\pi}{2}-t\right)=f\left(-\left(t-\frac{\pi}{2}\right)\right)$ and that the graph of $y=f\left(-\left(t-\frac{\pi}{2}\right)\right)$ can be found from the graph of $y=f(t)$ by reflecting the graph of f across the y-axis shifting it $\frac{\pi}{2}$ units right.

Exercise Set 16.2

1. For the graph of $y=\sin(t)$ above, obtain the graph of $y=\sin(-t)$ by sketching on the graph of $y=\sin(t)$ above, its reflection across the y-axis. Take that graph and shift it $\frac{\pi}{2}$ units right, to see that you get the graph of $y=\cos(x)$ (which is on the graph just below it).
2. For the graph of $y=\cos(t)$ above, obtain the graph of $y=\cos(-t)$ by sketching on the graph of $y=\cos(t)$ above, its reflection across the y-axis. Take that graph and shift it $\frac{\pi}{2}$ units right, to see that you get the graph of $y=\sin(x)$ (which is on the graph just above it).
3. For the graph of $y=\csc(t)$ above, obtain the graph of $y=\csc(-t)$ by sketching on the graph of $y=\csc(t)$ above, its reflection across the y-axis. Take that graph and shift it $\frac{\pi}{2}$ units right, to see that you get the graph of $y=\sec(x)$ (which is on the graph just to its right).
4. For the graph of $y=\sec(t)$ above, obtain the graph of $y=\sec(-t)$ by sketching on the graph of $y=\sec(t)$ above, its reflection across the y-axis. Take that graph and shift it $\frac{\pi}{2}$ units right, to see that you get the graph of $y=\csc(x)$ (which is on the graph just to its left).
5. For the graph of $y=\tan(t)$ above, obtain the graph of $y=\tan(-t)$ by sketching on the graph of $y=\tan(t)$ above, its reflection across the y-axis. Take that graph and shift it $\frac{\pi}{2}$ units right, to see that you get the graph of $y=\cot(x)$ (which is on the graph just to its right).
6. For the graph of $y=\cot(t)$ above, obtain the graph of $y=\cot(-t)$ by sketching on the graph of $y=\cot(t)$ above, its reflection across the y-axis. Take that graph and shift it $\frac{\pi}{2}$ units right, to see that you get the graph of $y=\tan(x)$ (which is on the graph just to its left).

§17. Trigonometric Functions, Graphs, Amplitude, Phase Shift, Periodicity

In our discussion of cofunctions above, we showed that the two cofunctions are related by a reflection and a phase shift. We have done this before, but we haven't called it a phase shift. Recall the following, given the graph of $y=f(t)$.

- Given a positive number b, the graph of $y=f(t-b)$ is the graph of $y=f(t)$ shifted right by b units, and the graph of $y=f(t+b)$ is a shift of the graph of $y=f(t)$ by b units left. With trigonometric functions, that is called a *phase shift*.
- The graph of $y=f(-t)$ a reflection of the graph of $y=f(t)$ across the y-axis and the graph of $y=-f(t)$ is a reflection of the graph of $y=f(t)$ across the x-axis.
- The graph of $y=A\cdot f(t)$ causes vertical stretching of the graph of $y=f(t)$ by a factor of A if $|A|>1$ and vertical compression of the graph of $y=f(t)$ if $|A|<1$. If f is a sine or a cosine function, the number $|A|$ is called the *Amplitude* of the sine (respectively, cosine) function.
- In contrast, the graph of $y=f(k\cdot t)$ causes horizontal compression of the graph of $y=f(t)$ by a factor of k if $|k|>1$ and horizontal stretching of the graph of $y=f(t)$ if $|k|<1$.

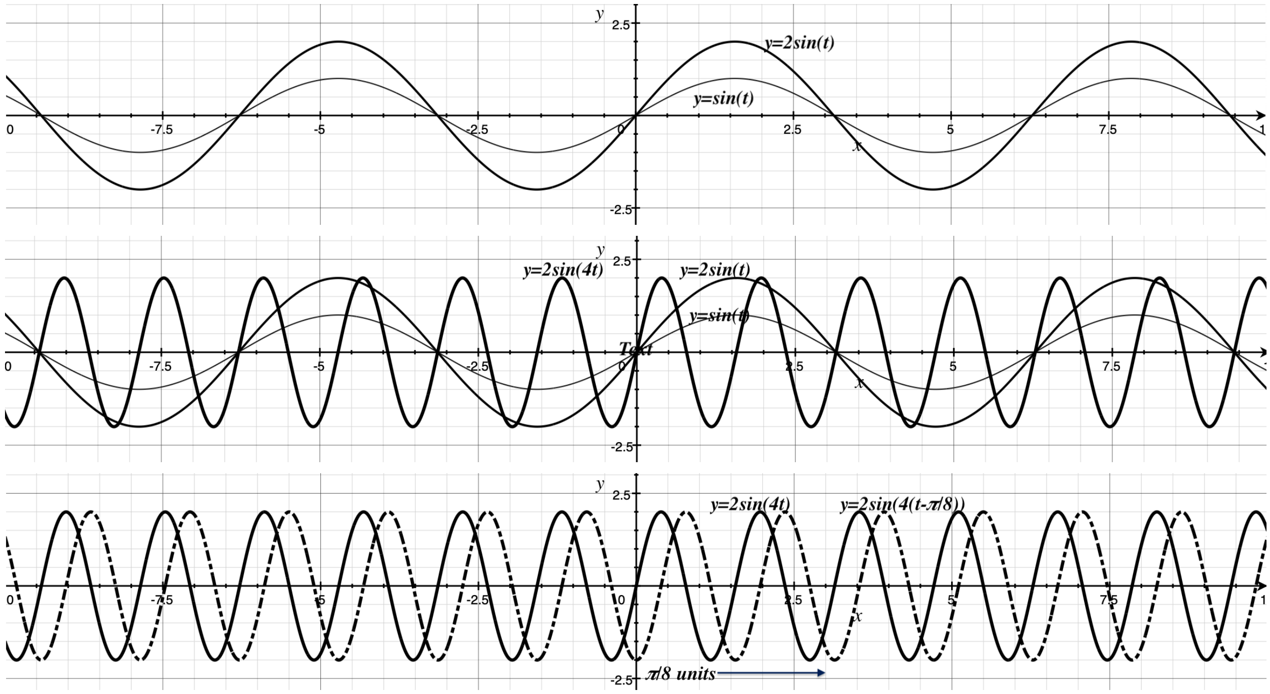
Example 17.1: Beginning with the graph of $y=\sin(t)$,

- The graph of $y=2\cdot\sin(t)$ will have amplitude $A=2$, meaning that it will essentially look the same as the graph of $y=\sin(t)$ except its maxima and minima will be at $y=2$ and $y=-2$, respectively. Its period is unchanged.
- The graph of $y=2\cdot\sin(4t)$ will have amplitude $A=2$, like the above graph, and its period will be affected by a factor of 4. What will be the period? The graph of $y=\sin(t)$ goes through a full period over the interval

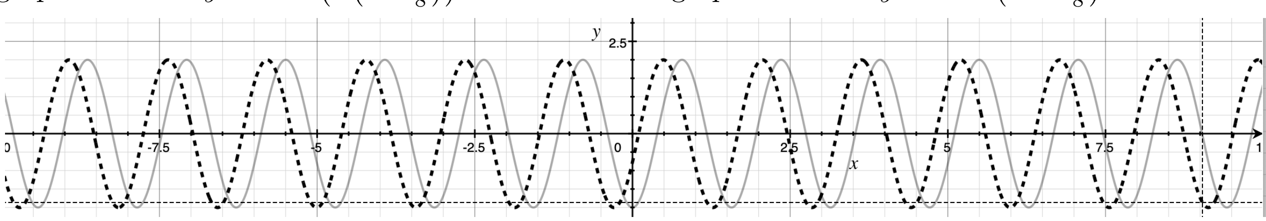
$0 \le t \le 2\pi$. However, this function has the argument $4t$, and so we solve the inequality $0 \le 4t \le 2\pi$ to get: $0 \le t \le \frac{\pi}{2}$. Thus, $y = 2 \cdot \sin(4t)$ has period $\frac{\pi}{2}$.

- The graph of $y = 2 \cdot \sin\left(4\left(t - \frac{\pi}{8}\right)\right)$ is the graph of the function $y = 2 \cdot \sin(4t)$ shifted by $\frac{\pi}{8}$ units to the right. The phase shift is $\frac{\pi}{8}$.

See the graphs below.



Important Note: There is a difference between $f(kt - b)$ and $f(k(t - b))$. Notice in the example above with: $y = 2\sin\left(4\left(t - \frac{\pi}{8}\right)\right)$. If we multiply through by 4, we get: $y = 2\sin\left(4t - \frac{4\pi}{8}\right)$; that is: $y = 2\sin\left(4t - \frac{\pi}{2}\right)$. The graph of $y = 2\sin\left(4t - \frac{\pi}{2}\right)$ and $y = 2\sin\left(4t - \frac{\pi}{8}\right)$ are certainly not the same. See the graph below. The bolded graph is that of $y = 2\sin\left(4\left(t - \frac{\pi}{8}\right)\right)$ and the dashed graph is that of $y = 2\sin\left(4t - \frac{\pi}{8}\right)$.



Thus, before deciding upon the phase shift, make sure that you have factored out the multiplier k from within the argument.

Exercise Set 17

1. Sketch the graph of $y = -3\sin\left(2t - \frac{\pi}{2}\right)$ in stages by first sketching the graph of $y = \sin(t)$ over several periods, then the graph of $y = -3\sin(t)$ over the same interval, and then the graph of $y = -3\sin(2t)$, and finally the graph of $y = -3\sin\left(2t - \frac{\pi}{2}\right)$ (remember to factor out the 2 inside the argument, as mentioned in the note above. Sketch each of these without plotting any points, but just from knowing what the graph of $y = \sin(t)$ looks like.
2. Sketch the graph of $y = -3\cos\left(2t - \frac{\pi}{2}\right)$ in stages by first sketching the graph of $y = \cos(t)$ over several periods, then the graph of $y = -3\cos(t)$ over the same interval, and then the graph of $y = -3\cos(2t)$, and finally the graph of $y = -3\cos\left(2t - \frac{\pi}{2}\right)$ (remember to factor out the 2 inside the argument, as mentioned in the note above. Sketch each of these without plotting any points, but just from knowing what the graph of $y = \cos(t)$ looks like.

§18. Trigonometric Functions, Identities

Rather than spend a lot of time deriving every identity, we are going to provide the main set of identities up front and then use them to show how to prove other identities.

Trigonometric Identities

Reciprocal Identities

$\csc(t) = \dfrac{1}{\sin(t)}$ $\qquad \sec(t) = \dfrac{1}{\cos(t)}$ $\qquad \cot(t) = \dfrac{1}{\tan(t)}$

$\tan(t) = \dfrac{\sin(t)}{\cos(t)}$ $\qquad \cot(t) = \dfrac{\cos(t)}{\sin(t)}$

Pythagorean Identities

$\cos^2(t) + \sin^2(t) = 1$ $\qquad 1 + \tan^2(t) = \sec^2(t)$ $\qquad \cot^2(t) + 1 = \csc^2(t)$

Even–odd Identities

$\sin(-x) = -\sin(x)$ $\qquad \cos(-x) = \cos(x)$ $\qquad \tan(-x) = -\tan(x)$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - t\right) = \cos(t)$ $\qquad \tan\left(\frac{\pi}{2} - t\right) = \cot(t)$ $\qquad \sec\left(\frac{\pi}{2} - t\right) = \csc(t)$

$\cos\left(\frac{\pi}{2} - t\right) = \sin(t)$ $\qquad \cot\left(\frac{\pi}{2} - t\right) = \tan(t)$ $\qquad \csc\left(\frac{\pi}{2} - t\right) = \sec(t)$

Addition and Subtraction Formulas

$\sin(s+t) = \sin(s)\cos(t) + \cos(s)\sin(t)$ $\qquad \sin(s-t) = \sin(s)\cos(t) - \cos(s)\sin(t)$

$\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$ $\qquad \cos(s-t) = \cos(s)\cos(t) + \sin(s)\sin(t)$

$\tan(s+t) = \frac{\tan(s)+\tan(t)}{1-\tan(s)\tan(t)}$ $\qquad \tan(s-t) = \frac{\tan(s)-\tan(t)}{1+\tan(s)\tan(t)}$

Double-Angle Formulas

$\sin(2t) = 2\sin(t)\cos(t)$ $\qquad \cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t) = 2\cos^2(t) - 1$ $\qquad \tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)}$

Half-Angle Formulas (but first we need this set)

$\sin^2(t) = \frac{1-\cos(2t)}{2}$ $\qquad \cos^2(t) = \frac{1+\cos(2t)}{2}$ $\qquad \tan^2(t) = \frac{1-\cos(2t)}{1+\cos(2t)}$

$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\dfrac{1-\cos(u)}{2}}$ $\qquad \cos\left(\frac{u}{2}\right) = \pm\sqrt{\dfrac{1+cos(u)}{2}}$ $\qquad \tan\left(\frac{u}{2}\right) = \frac{1-\cos(u)}{\sin(u)} = \frac{\sin(u)}{1+\cos(u)}$

Product-To-Sum Formulas

$\sin(u)\cos(v) = \frac{1}{2}\cdot[\sin(u+v) + \sin(u-v)]$ $\qquad \cos(u)\sin(v) = \frac{1}{2}\cdot[\sin(u+v) - \sin(u-v)]$

$\cos(u)\cos(v) = \frac{1}{2}\cdot[\cos(u+v) + \cos(u-v)]$ $\qquad \sin(u)\sin(v) = \frac{1}{2}\cdot[\cos(u+v) - \cos(u-v)]$

Sum-To-Product Formulas

$\sin(s) + \sin(t) = 2\cdot\sin\left(\frac{s+t}{2}\right)\cos\left(\frac{s-t}{2}\right)$ $\qquad \sin(s) - \sin(t) = 2\cdot\cos\left(\frac{s+t}{2}\right)\sin\left(\frac{s-t}{2}\right)$

$\cos(s) + \cos(t) = 2\cdot\cos\left(\frac{s+t}{2}\right)\cos\left(\frac{s-t}{2}\right)$ $\qquad \cos(s) - \cos(t) = -2\cdot\sin\left(\frac{s+t}{2}\right)\sin\left(\frac{s-t}{2}\right)$

It should be noted that the Pythagorean Identities each have two automatic extensions:

1. $\cos^2(t) + \sin^2(t) = 1$ implies that $\cos^2(t) = 1 - \sin^2(t)$ and $\sin^2(t) = 1 - \cos^2(t)$
2. $1 + \tan^2(t) = \sec^2(t)$ implies that $1 = \sec^2(t) - \tan^2(t)$ and $\tan^2(t) = \sec^2(t) - 1$
3. $\cot^2(t) + 1 = \csc^2(t)$ implies that $\cot^2(t) = \csc^2(t) - 1$ and $1 = \csc^2(t) - \cot^2(t)$

In addition, the reciprocal identities imply:

$\sin(t)\cdot\csc(t) = 1$ $\qquad \cos(t)\cdot\sec(t) = 1$ $\qquad \tan(t)\cdot\cot(t) = 1$

Example 18.1: Prove the identity: $\tan(\theta) + \cot(\theta) = \sec(\theta)\csc(\theta)$

In proving identities we work only with one side of the equals sign, transforming continually into a new expression using algebra and known trigonometric identities, until we obtain the other side of the equals sign. There are a few techniques that are used over and over again, and after seeing a few examples, you will probably have seen most of them. It is usually best start with the most complicated side (although either would be fine). One common technique is to convert everything to sines and cosines so that parts can be more readily combined.

$$\begin{aligned}
\tan(\theta) + \cot(\theta) &= \sec(\theta)\csc(\theta) \\
\frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)} &= \sec(\theta)\csc(\theta) \\
\frac{\sin(\theta)\cdot\sin(\theta)}{\cos(\theta)\sin(\theta)} + \frac{\cos(\theta)\cos(\theta)}{\cos(\theta)} &= \sec(\theta)\csc(\theta) \\
\frac{\sin^2(\theta) + \cos^2(\theta)}{\cos(\theta)\sin(\theta)} &= \sec(\theta)\csc(\theta) \\
\frac{1}{\cos(\theta)\sin(\theta)} &= \sec(\theta)\csc(\theta) \\
\frac{1}{\cos(\theta)} \cdot \frac{1}{\sin(\theta)} &= \sec(\theta)\csc(\theta) \\
\sec(\theta)\cdot\csc(\theta) &= \sec(\theta)\csc(\theta)
\end{aligned}$$

Another common technique is to multiply a fraction by 1 in the form of a conjugate over itself. The *conjugate* of $a-b$ is $a+b$ and that is useful because $(a-b)(a+b) = a^2 - b^2$. This is really nice with something like $(1-\cos(t))$, because if we multiply by its conjugate, we get

$$(1-\cos(t))\cdot(1+\cos(t)) = 1 - \cos^2(t) = \sin^2(t)$$

and this is especially useful because it turns a binomial into a monomial. As you know, $\frac{1}{a-b} \neq \frac{1}{a} - \frac{1}{b}$ and so $\frac{1}{a-b}$ is irreducible. However, with this technique:

$$\begin{aligned}
\frac{1}{1-\cos(t)} = \frac{1\cdot(1+\cos(t))}{(1-\cos(t))\cdot(1+\cos(t))} = \frac{1+\cos(t)}{1-\cos^2(t)} &= \frac{1+\cos(t)}{\sin^2(t)} \\
&= \frac{1}{\sin^2(t)} + \frac{\cos(t)}{\sin^2(t)} \\
&= \csc^2(t) + \frac{\cos(t)}{\sin(t)} \cdot \frac{1}{\sin(t)} \\
&= \csc^2(t) + \cot(t)\csc(t)
\end{aligned}$$

If we were asked to prove this identity (which we have just proven) I would call the left hand side more complicated because it at least appears to be an irreducible fraction.

Example 18.2: Prove the identity: $\dfrac{\cos(t)}{1-\sin(t)} = \sec(t)+\tan(t)$

$$\begin{aligned}
\frac{\cos(t)}{1-\sin(t)} &= \sec(t)+\tan(t)\\
\frac{\cos(t)\cdot(1+\sin(t))}{(1-\sin(t))\cdot(1+\sin(t))} &= \sec(t)+\tan(t)\\
\frac{\cos(t)(1+\sin(t))}{1-\sin^2(t)} &= \sec(t)+\tan(t)\\
\frac{\cos(t)+\cos(t)\sin(t))}{\cos^2(t)} &= \sec(t)+\tan(t)\\
\frac{\cos(t)}{\cos^2(t)}+\frac{\cos(t)\cdot\sin(t)}{\cos^2(t)} &= \sec(t)+\tan(t)\\
\frac{1}{\cos(t)}+\frac{\sin(t)}{\cos(t)} &= \sec(t)+\tan(t)\\
\sec(t)+\tan(t) &= \sec(t)+\tan(t)
\end{aligned}$$

Exercise Set 18 Prove the following trigonometric identities.

1. $\dfrac{\cos(u)\cdot\sec(u)}{\tan(u)} = \cot(u)$
2. $\dfrac{1+\tan(x)}{1-\tan(x)} = \dfrac{\cos(x)+\sin(x)}{\cos(x)-\sin(x)}$
3. $\sec(t)-\tan(t) = \dfrac{1}{\sec(t)+\tan(t)}$
4. $\sin(x-\pi) = -\sin(x)$
5. $(\sin(x)+\cos(x))^2 = 1+\sin(2x)$
6. $\dfrac{\sin(3x)+\cos(3x)}{\cos(x)-\sin(x)} = 1+4\sin(x)\cos(x)$

§19. Inverse Trigonometric Functions

In this section we will define the inverse functions for the trigonometric functions. Of course, none of the six trigonometric functions are one-to-one, and so none of them really has an inverse function, as they are. This was the same with $f(x)=x^2$: it was not one-to-one so we restricted its domain: $f(x)=x^2$ on $[0,\infty)$ and since that function is one-to-one, then *it* has an inverse function. The positive side is that we can now reverse a square, but with that power comes the responsibility to restore those possible solutions lost by the domain restriction. That is, when we solve $x^2=9$, we must remember $x=-3$ as well as the result $x=+3$ of taking a square root. We will have to restrict each of the six trigonometric functions.

1. Since $f(x)=\sin(x)$ goes through each element of its range (which is $[-1,1]$) exactly once over the subset $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then we restrict the sine function to: $f(x)=\sin(x)$ on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
2. Since $f(x)=\cos(x)$ goes through each element of its range (which is $[-1,1]$) exactly once over the subset $[0,\pi]$, then we restrict the cosine function to: $f(x)=\cos(x)$ on $[0,\pi]$.
3. Since $f(x)=\tan(x)$ goes through each element of its range (which is $(-\infty,\infty)$) exactly once over the subset $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then we restrict the tangent function to: $f(x)=\tan(x)$ on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Of course, each of the trigonometric functions passes through its full range, over and over again, so we could have chosen any of the other intervals where this happened. However, the above restrictions are standard. The above 3 are also the three trigonometric functions which are used most often. The restrictions for some of the remaining three functions below are sometimes not the same between textbooks.

4. Since $f(x)=\cot(x)$ goes through each element of its range (which is $(-\infty,\infty)$) exactly once over the subset $(0,\pi)$, then we restrict the cotangent function to: $f(x)=\cot(x)$ on $(0,\pi)$.
5. Since $f(x)=\sec(x)$ goes through each element of its range (which is $(-\infty,-1]\cup[1,\infty)$) exactly once over the subset $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$, then we restrict the secant function to: $f(x)=\sec(x)$ on $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$.

6. Since $f(x) = \csc(x)$ goes through each element of its range (which is $(-\infty, -1] \cup [1, \infty)$) exactly once over the subset $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, then we restrict the cosecant function to: $f(x) = \csc(x)$ on $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

Under these restrictions, these six restricted functions are one-to-one on their restricted domain, and therefore each has an inverse function. Thus, putting this all together in one place:

1. Since $f(x) = \sin(x)$ goes through each element of its range (which is $[-1, 1]$) exactly once over the subset $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then we restrict the sine function to: $f(x) = \sin(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
   Thus, $y = \sin^{-1}(x)$ satisfies:
   $$\begin{aligned} \sin(\sin^{-1}(x)) = x &\quad \text{for } -1 \le x \le 1 \\ \sin^{-1}(\sin(x)) = x &\quad \text{for } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2} \end{aligned}$$
   However, if solving an equation and we do not know that $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then we have to account for the possibilities. For instance, if we are to solve $\sin(x) = \frac{1}{2}$, and we rely upon the inverse sine function, we will find:
   $\sin^{-1}(\sin(x)) = \sin^{-1}\left(\frac{1}{2}\right)$
   $x = \frac{\pi}{6}$
   but we know that the sine also attains this value at $x = \frac{5\pi}{6}$, and also for every 2π units because the sine function has period 2π. Thus the solution to the equation
   $\sin(x) = \frac{1}{2}$
   is $x = \frac{\pi}{6} + k \cdot 2\pi$ and $x = \frac{5\pi}{6} + k \cdot 2\pi$ for every integer k.
   The other direction is automatic, given that the ranges are respected, with no additional solutions attached. That is, the solution to
   $\sin^{-1}(x) = \frac{\pi}{3}$ is
   $\sin(\sin^{-1}(x)) = \sin\left(\frac{\pi}{3}\right)$
   $x = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ with no other solutions.
   However, because of the domain restriction, not every such equation has a solution. In particular,
   - $\sin^{-1}(x) = \pi$ has no solution because π is not in the restricted domain of $\sin(x)$ from which $\sin^{-1}(x)$ came.
   just as
   - $\sqrt{x} = -4$ has no solution because -4 is not in the restricted domain of $y = x^2$ from which $\sqrt{x}$ came.
2. Since $f(x) = \cos(x)$ goes through each element of its range (which is $[-1, 1]$) exactly once over the subset $[0, \pi]$, then we restrict the cosine function to: $f(x) = \cos(x)$ on $[0, \pi]$.
   Thus, $y = \cos^{-1}(x)$ satisfies:
   $$\begin{aligned} \cos(\cos^{-1}(x)) = x &\quad \text{for } -1 \le x \le 1 \\ \cos^{-1}(\cos(x)) = x &\quad \text{for } 0 \le x \le \pi \end{aligned}$$
3. Since $f(x) = \tan(x)$ goes through each element of its range (which is $(-\infty, \infty)$) exactly once over the subset $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then we restrict the tangent function to: $f(x) = \tan(x)$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
   Thus, $y = \tan^{-1}(x)$ satisfies:
   $$\begin{aligned} \tan(\tan^{-1}(x)) = x &\quad \text{for } -\infty < x < \infty \\ \tan^{-1}(\tan(x)) = x &\quad \text{for } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2} \end{aligned}$$

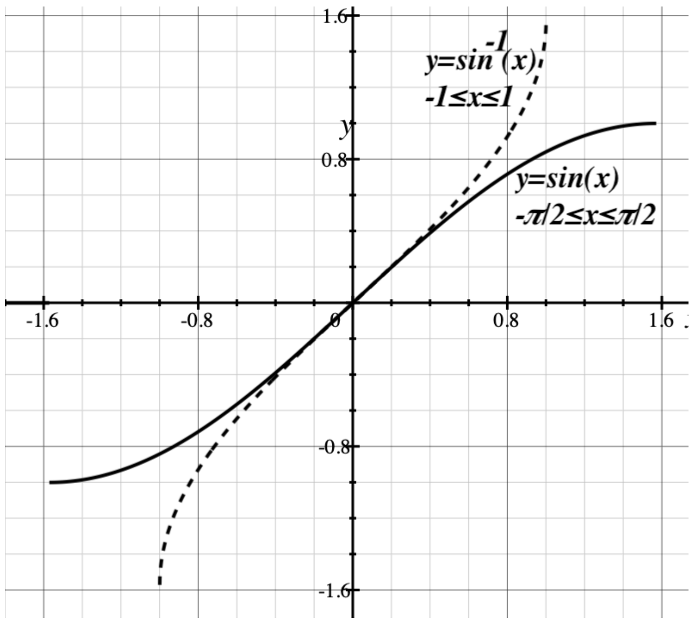
Each of the statements made about the use of the inverse sine, can be made about the inverse cosine and inverse tangent above. Regarding the remaining three inverse trigonometric functions, we can make similar statements about their use as we did for the three above. However, they are rarely used; if we spend time understanding the three functions above and their inverse functions, then we will understand these.

4. Since $f(x) = \cot(x)$ goes through each element of its range (which is $(-\infty, \infty)$) exactly once over the subset $(0, \pi)$, then we restrict the cotangent function to: $f(x) = \cot(x)$ on $(0, \pi)$.
5. Since $f(x) = \sec(x)$ goes through each element of its range (which is $(-\infty, -1] \cup [1, \infty)$) exactly once over the subset $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, then we restrict the secant function to: $f(x) = \cot(x)$ on $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.
6. Since $f(x) = \csc(x)$ goes through each element of its range (which is $(-\infty, -1] \cup [1, \infty)$) exactly once over the subset $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, then we restrict the secant function to: $f(x) = \cot(x)$ on $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

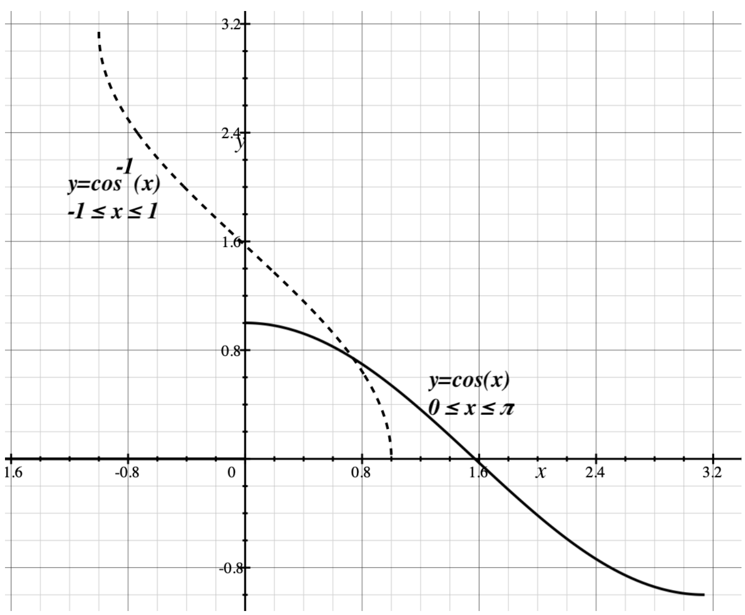
Important Note: The notation $\sin^{-1}(x)$ that we often use for the inverse sine function leads to some possible ambiguity. Recall that $x^{-1} = \frac{1}{x}$ when -1 is thought of as an exponent. In the notation for the inverse sine function, the -1 is not an exponent, so $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$!!! From context, we know that this means that $y = \sin^{-1}(x)$ is the inverse sine functions. Some books use the notation $y = \arcsin(x)$ instead of $y = \sin^{-1}(x)$. Some books distinguish between these two different notations as well!

Now we will look at some graphs.

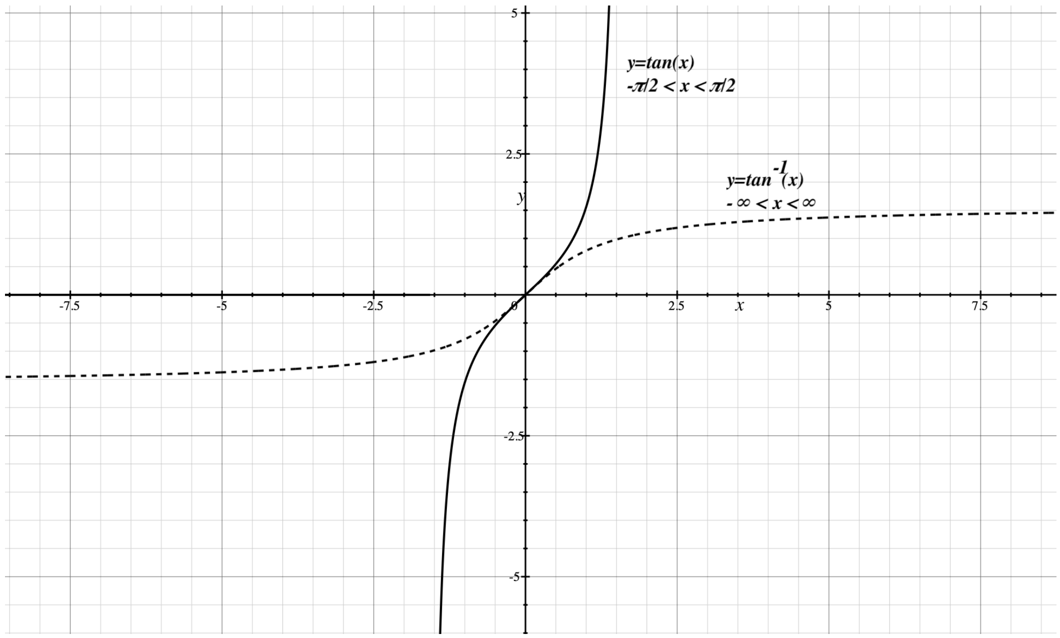
First is the graph of the sine function with restricted domain (bolded) and the inverse sine function (dashed).



The cosine function with restricted domain (bolded) and the inverse cosine function (dashed).

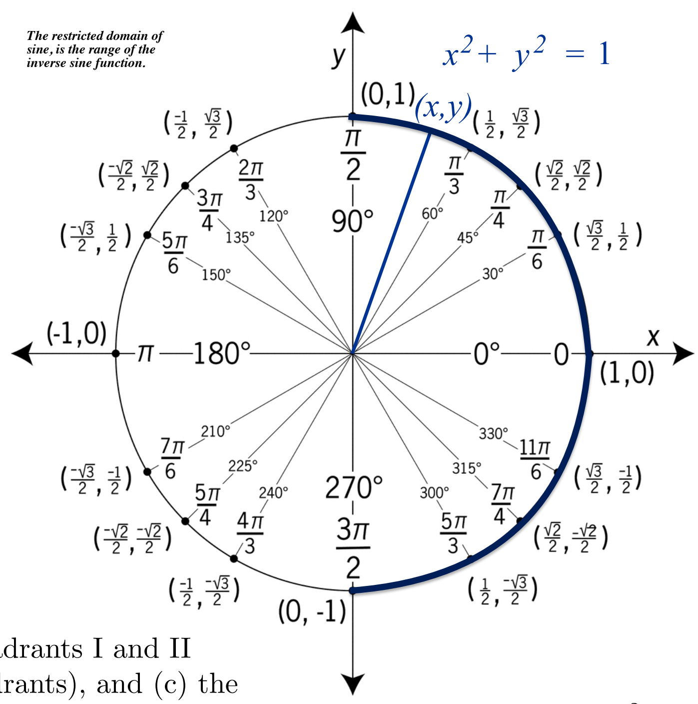


The tangent function with restricted domain (bolded) and the inverse tangent function (dashed).



Examples 19.1: Compute the following.

1. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$



We are looking for an angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ for which $\sin(\theta) = \frac{\sqrt{3}}{2}$. We have bolded the restricted domain of the sine, which is the range of the inverse sine. Since $\sin(\theta) = y$, we scan the unit circle for a y-coordinate which is $\frac{\sqrt{3}}{2}$. From that we see $\theta = \frac{\pi}{3}$. So

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

2. Find all θ, with $0 \leq \theta < 2\pi$ for which $\sin(\theta) = \frac{\sqrt{3}}{2}$.

Of course, since $\sin(\theta) = y$, we can scan the entire unit circle looking for y-coordinates which are $\frac{\sqrt{3}}{2}$. We see $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$.

Alternatively, suppose that we first only thought of the angles: $\theta = \frac{\pi}{3}$, but didn't scan the unit circle for the second answer. We could still reason our way to the additional answer. Our clues here are that (a) the sine is positive ($\sin(\theta) = \frac{\sqrt{3}}{2} > 0$), (b) the sine is positive in Quadrants I and II (because the y-coordinates are positive in those two quadrants), and (c) the reference angle of our first answer is $\frac{\pi}{3}$. There is only one other answer, it is in Quadrant II, and it is $\theta = \frac{2\pi}{3}$.

Why would we choose the second method? Consider the next problem.

3. Find all θ, with $0 \leq \theta < 2\pi$ for which $\sin(\theta) = 0.93$.

The unit circle is of no help here. We take the inverse sine of both sides: $\sin^{-1}(\sin(\theta)) = \sin^{-1}(0.93)$ and obtain one of the angles: $\theta \simeq 1.1944$ radians. Now we have to reason our way to any additional answers. Our clues here are: (a) the sine is positive (0.93 is positive) in Quadrants I and II (because the y-coordinates are positive in those two quadrants) and (b) the reference angle is approximately 1.1944. The answer that we have has its terminal side in Quadrant I, so there is only one other answer and it is in Quadrant II with reference angle of about 1.1944. So the other answer is $\theta = \pi - 1.1944 \simeq 1.9472$ radians.

4. Find all θ, with $0 \leq \theta < 2\pi$ for which $\sin(\theta) = -0.93$.

Again, the unit circle is of no help. We take the inverse sine of both sides: $\sin^{-1}(\sin(\theta)) = \sin^{-1}(-0.93)$ and obtain one of the angles: $\theta \simeq -1.1944$ radians. Now we have to reason our way to our answers. Our clues here are: (a) the sine is negative (-0.93 is negative) in Quadrants III and IV (because the y-coordinates are negative in those two quadrants) and (b) the reference angle is approximately 1.1944. The answer that we have ($\theta = -1.1944$ radians) has its terminal side in Quadrant IV but the angle -1.1944 radians is not in the interval $[0, 2\pi)$. So we are looking for two answers: the answer in Quadrant III is $\pi + 1.1944 = 4.3360$ radians and the one in Quadrant IV is $2\pi - 1.1944 = 5.0888$ radians.

5. Find all real numbers t for which $\sin(t) = \frac{\sqrt{3}}{2}$.

Above, we found the two angles in $[0, 2\pi)$ whose sine is $\frac{\sqrt{3}}{2}$: $t = \frac{\pi}{3}$ and $t = \frac{2\pi}{3}$. Because the sine is periodic with period 2π, the remaining answers are:

$$t = \tfrac{\pi}{3} + k \cdot 2\pi \quad \text{for each integer } k$$
$$t = \tfrac{2\pi}{3} + k \cdot 2\pi \quad \text{for each integer } k$$

6. Find all values of θ, with $0 \leq \theta < 2\pi$ for which $\tan(\theta) = -1$.

Since $\tan(\theta) = \frac{y}{x}$ we scan the unit circle for where that ratio is -1. We find that at $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

If we remembered, for instance, that $\tan\left(\frac{3\pi}{4}\right) = -1$, and wanted to find any other angles, we would reason based on knowing (a) the tangent is negative ($\tan(\theta) = -1$), (b) the tangent is negative in Quadrant II and Quadrant IV where the x and y coordinates have opposite sign, and (c) the reference angle of $\frac{\pi}{4}$. The answer that we already have ($\theta = \frac{3\pi}{4}$) is in Quadrant II, so we are looking for the angle in Quadrant IV with reference angle $\frac{\pi}{4}$. It is: $2\pi - \frac{\pi}{4}$ which is $\frac{8\pi}{4} - \frac{\pi}{4}$, or $\frac{7\pi}{4}$. Thus, the answers are: $\theta = \frac{3\pi}{4}$, $\theta = \frac{7\pi}{4}$.

7. Compute: $\sin(\sin^{-1}(2))$

This does not exist. Since the range of the sine function is $[-1, 1]$, then the domain of the inverse sine function is $[-1, 1]$.

8. Compute: $\sin(\sin^{-1}(0.12121))$

$\sin(\sin^{-1}(0.12121)) = 0.12121$. The reversal in this direction is automatic.

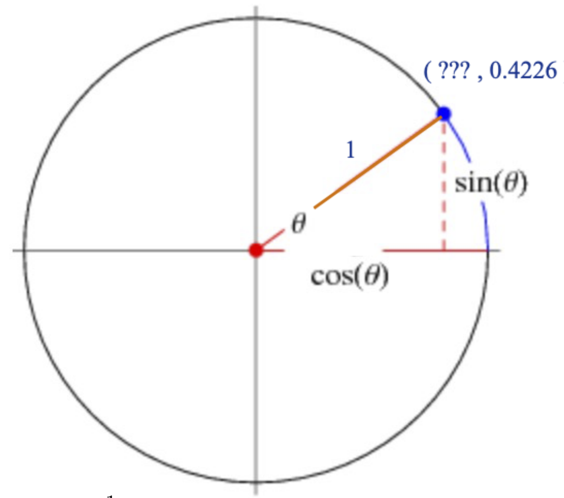
9. Compute: $\sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$

Since the angle $-\frac{\pi}{3}$ satisfies $-\frac{\pi}{2} \leq -\frac{\pi}{3} \leq \frac{\pi}{2}$, then $\sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

10. Compute: $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

The angle $\frac{4\pi}{3}$ is not in the restricted domain of the sine. Therefore, the answer will not be $\frac{4\pi}{3}$. In computing $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$, the answer *will be in the restricted domain* of sine: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, we are looking for an angle θ in the restricted domain for which $\sin(\theta) = \sin\left(\frac{4\pi}{3}\right)$. That means that the two angles have to have the same reference angle. The angle $\frac{4\pi}{3}$ is in Quadrant III and has reference angle $\frac{\pi}{3}$. Since it is in the third quadrant, we know that $\sin\left(\frac{4\pi}{3}\right)$ is a negative number. Thus, θ will also have to have its terminal side in a Quadrant where the sine is negative: in Quadrant IV. We are looking for an angle with terminal side in Quadrant IV and whose reference angle is $\frac{\pi}{3}$. Our first thought might be $\frac{5\pi}{3}$ but that is not in the restricted domain either. But if we measure clockwise from the positive x axis to the terminal side is $-\frac{\pi}{3}$. Hence, $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{\pi}{3}$.

11. Compute: $\cos\left(\sin^{-1}(0.4226)\right)$



This might cause us to draw the following.

Using the Pythagorean Theorem, we would find the x-coordinate to be $x = 0.9063$. Thus,
$\cos\left(\sin^{-1}(0.4226)\right) = 0.9063$

Exercise Set 19

1. Find the exact value of

1. $\tan^{-1}(\sqrt{3})$	2. $\cos^{-1}(-\frac{1}{2})$	3. $\sin^{-1}(0)$
4. $\cot^{-1}\left(\frac{\sqrt{3}}{3}\right)$	5. $\csc^{-1}(-2)$	6. $\sin^{-1}(-1)$
7. $\tan^{-1}(\tan(\frac{\pi}{4}))$	8. $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$	9. $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$
10. $\tan(\tan^{-1}(-1))$	11. $\sin(\sin^{-1}(\pi))$	12. $\tan(\cos^{-1}(-0.23))$

§20. Trigonometric Equations

Recalling our discussion of equations, there are something like 2 different main categories of trigonometric equations which we could see: (A) ones in which there is a single variable trigonometric term, or which through identities or algebra could be reduced to a single variable trigonometric term, or (B) those with multiple variable trigonometric terms which can not be combined into a single trigonometric terms no matter what method we employ. The first kind we treat as we would a linear style equation and we isolate the variable by employing a lot of inverse operations, and the second kind where we move everything to one side of the equals sign and factor or use a quadratic method.

We'll do a series of examples.

Example 20.1: Find all values of α, $0 \le \alpha < 2\pi$ for which $2\cos(\alpha) + 1 = 0$. (Category A)

We will isolate α but this begins by isolating the cosine,

$$2\cos(\alpha) + 1 = 0$$

$$2\cos(\alpha) + 1 - 1 = 0 - 1$$

$$\frac{2\cos(\alpha)}{2} = -\frac{1}{2}$$

$$\cos(\alpha) = -\frac{1}{2}$$

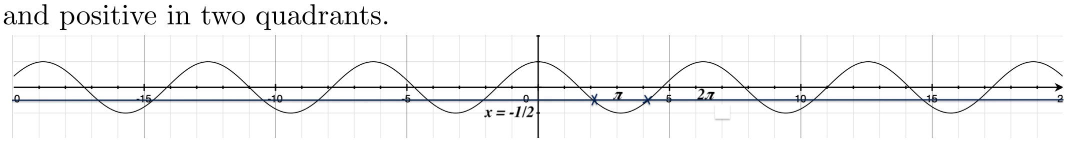
now apply an inverse cosine, realizing that it won't give us every solution

$$\cos^{-1}(\cos(\alpha)) = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\alpha = \frac{2\pi}{3}$$

found from the unit circle...an angle in $[0, \pi]$ whose cosine is $\frac{-1}{2}$

This is just one solution and we know there must be another. Why? For one thing, think of the graph of cosine...the cosine assumes every value in $(-1, 1)$ twice each period, and for another, the cosine is negative in two quadrants and positive in two quadrants.



Knowing that the cosine is negative in Quadrant II and Quandrant III, the angle $\alpha = \frac{2\pi}{3}$ is in Quadrant II with reference angle $\frac{\pi}{3}$, then we look for the angle in Quadrant III with reference angle $\frac{\pi}{3}$. This is $\pi + \frac{\pi}{3}$. So our solutions are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Example 20.2: Find all values of α, $0 \le \alpha < 2\pi$ for which $2\sin(\alpha) + 1 = 0$. (Category A)

This is almost identical to the problem above, but it has a different twist at the end, so we have it here.

At the end we will have: $\sin(\alpha) = -\frac{1}{2}$, and applying the inverse sine function to both sides, we will get
$\alpha = \sin^{-1}\left(-\frac{1}{2}\right)$.

This would give us (after an unwarranted use of the calculator: $-60°$ or $-\frac{\pi}{3}$ radians). But this is neither of our solutions, because our solutions must be in the interval $[0, 2\pi)$. But we note that the sine is negative, which means that the solutions must reside in Quadrant III and Quadrant IV, and the reference angle is $\frac{\pi}{3}$. This gives us our two solutions: $\alpha = \pi + \frac{\pi}{3}$ in Quadrant III and $\alpha = 2\pi - \frac{\pi}{3}$ in Quadrant IV. $\pi = \frac{3\pi}{3}$ and $2\pi = \frac{6\pi}{3}$ and so our two solutions are $\alpha = \frac{4\pi}{3}$ and $\alpha = \frac{5\pi}{3}$.

Instead of using the calculator, if we were to glance at the unit circle for the answer to $\alpha = \sin^{-1}\left(-\frac{1}{2}\right)$, we would immediately see two angles whose terminal side intersects the unit circle at a point with y-coordinate $-\frac{1}{2}$: $\alpha = \frac{4\pi}{3}$ and $\alpha = \frac{5\pi}{3}$ and we would have found our answers. However, $\alpha = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{3}$. So be aware that the value that we first get for the inverse trigonometric function might not be one of our solutions, however, its reference angle is a major key to finding our solutions.

Example 20.3: Find all values of x in $[0, 2\pi)$ for which $7(\tan(x) - 1) = 3 \cdot \tan(x) - 3$ (Category A)

$7 \cdot \tan(x) - 7 = 3 \cdot \tan(x) - 3$	expanding the parentheses
$7 \cdot \tan(x) - 7 - 3 \cdot \tan(x) = 3 \cdot \tan(x) - 3 - 3 \cdot \tan(x)$	
	moving the $+3 \cdot \tan(x)$ by adding $-3 \cdot \tan(x)$ to both sides
$4 \cdot \tan(x) - 7 = -3$	simplify
$4 \cdot \tan(x) - 7 + 7 = -3 + 7$	moving the -48 by adding $+48$ to both sides
$4 \cdot \tan(x) = 4$	simplify
$\frac{4 \cdot \tan(x)}{4} = \frac{4}{4}$	moving the 3 which is multiplied times x by dividing both sides by 3
$\tan(x) = 1$	simplify

We need to apply the inverse tangent function to both sides of the equation, but we understand that applying this inverse function will only give us the single answer in the restricted domain. We will have to find the actual solutions from that.

$\tan(x) = 1$	
$\tan^{-1}(\tan(x)) = \tan^{-1}(1)$	apply the inverse function to both sides
$x = \frac{\pi}{4}$	simplify

This is the solution which lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and it is also in $[0, 2\pi)$. But we know there is another. This solution has reference angle $\frac{\pi}{4}$ and so will the other solution. The other solution must also be in a quadrant where the tangent function is positive, and this would be Quadrant III (since we already have the solution in Quadrant I). So the other answer is $\pi + \frac{\pi}{4}$, which is $\frac{4\pi}{4} + \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Thus, our two solutions are:
$x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Example 20.4: Find all x in $[0, 2\pi)$ for which : $\sin^2(x) - \sin(x) = \cos^2(x)$ (Category B)

Clearly there is more than one variable trigonometric term, and short of a trigonometric identity revolution, we're not going to be able to isolate x as we did in the last two problems. So we move everything to one side. To aid in the factoring, we will use a Pythagorean identity to get everything in terms of a single trigonometric function.

$\sin^2(x) - \sin(x) = \cos^2(x)$	Substitute $\cos^2(x) = 1 - \sin^2(x)$
$\sin^2(x) - \sin(x) = 1 - \sin^2(x)$	seeing all the squares, we move everything to one side (thinking PZP)
$\sin^2(x) - \sin(x) + \sin^2(x) = 1 - \sin^2(x) + \sin^2(x)$	

$2\sin^2(x) - \sin(x) = 1$
$2\sin^2(x) - \sin(x) - 1 = 1 - 1$ simplify
$2\sin^2(x) - \sin(x) - 1 = 0$ simplify
$(2\sin(x) + 1)(\sin(x) - 1) = 0$ factor and use PZP
$2\sin(x) + 1 = 0 \quad \text{or} \quad \sin(x) - 1 = 0$
$2\sin(x) = -1 \quad \text{or } \sin(x) = 1$
$\sin(x) = -\frac{1}{2} \quad \text{or } \sin(x) = 1$

We will apply the inverse sine function to both sides of each and get initial values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which might not be one of our solutions, but which will have the correct reference angle. Thus, we continue

$x = \sin^{-1}\left(-\frac{1}{2}\right) \quad \text{or } x = \sin^{-1}(1)$
$x = -\frac{\pi}{6} \quad \text{or } x = \frac{\pi}{2}$

Looking at $x = \frac{\pi}{2}$, that is one of our solutions because it is in $[0, 2\pi)$. Unlike values strictly between -1 and 1, the sine (and cosine) do not reach $y = 1$ more than one time in $[0, 2\pi)$, and so there is no second solution to go with this one.

Looking at $x = -\frac{\pi}{6}$, this is not one of our solutions because it is not between 0 and 2π, but it has the right reference angle, which is $\frac{\pi}{6}$. So we are looking for angles for which the sine is negative (which is in Quadrant III and Quadrant IV) and with reference angle $\frac{\pi}{6}$. This would be $\pi + \frac{\pi}{6}$ and $2\pi - \frac{\pi}{6}$. That is, $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Thus, our solutions are $x = \frac{\pi}{2}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

Example 20.5 Find all values of θ in $[0, 2\pi)$ for which $\sin(2\theta) + \sqrt{3} \cdot \cos(2\theta) = 0$. (Category A).

There are two variable trigonometric terms, but we are just a division away from having one.

$$\begin{aligned}
\sin(2\theta) + \sqrt{3} \cdot \cos(2\theta) &= 0 \\
\sin(2\theta) + \sqrt{3} \cdot \cos(2\theta) - \sqrt{3} \cdot \cos(2\theta) &= 0 - \sqrt{3} \cdot \cos(2\theta) \\
\sin(2\theta) &= -\sqrt{3}\cos(\theta) \\
\frac{\sin(2\theta)}{\cos(2\theta)} &= \frac{-\sqrt{3}\cos(2\theta)}{\cos(2\theta)} \\
\tan(2\theta) &= -\sqrt{3}
\end{aligned}$$

Typically we are very shy of dividing by a variable term because of the concern for when it might be zero. However, the factor $\cos(2\theta) \neq 0$ for any solution to this equation. Why? For if $\sin(2\theta) + \sqrt{3} \cdot \cos(2\theta) = 0$ and $\cos(2\theta) = 0$ as well, then $\sin(2\theta) = 0$ would have to be true also. The sine and cosine are never both zero at the same time. Thus:

$\tan(2\theta) = -\sqrt{3}$ and so $2\theta = \tan^{-1}(-\sqrt{3})$

Scanning the unit circle, we see that while $2\theta = -\frac{\pi}{3}$ and so (dividing both sides by 2, to isolate θ) $\theta = -\frac{\pi}{6}$, this is not one of the solutions that we are looking for. We are to find those in $[0, 2\pi)$.

What we do gather from $2\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ is that tangent is negative (as it is in Quadrants II and IV) and the reference angle is $\frac{\pi}{3}$. That leads us to $2\theta = \frac{2\pi}{3}$ in Quadrant II, and $2\theta = \frac{5\pi}{3}$ in Quadrant IV. Ordinarily, this where we would stop. However, note that we're looking for $0 \leq \theta < 2\pi$, and therefore, $0 \leq 2\theta < 4\pi$ (multiplying through by 2). Therefore, we go around another 2π to each.

$2\theta = \frac{2\pi}{3}$, $2\theta = \frac{5\pi}{3}$, $2\theta = \frac{2\pi}{3} + 2\pi$, $2\theta = \frac{5\pi}{3} + 2\pi$

That is,

$2\theta = \frac{2\pi}{3}$, $2\theta = \frac{5\pi}{3}$, $2\theta = \frac{8\pi}{3}$, $2\theta = \frac{11\pi}{3}$

Thus, our solutions are: (dividing both sides by 2, to find θ)

$\theta = \frac{\pi}{3}$, $\theta = \frac{5\pi}{6}$, $\theta = \frac{4\pi}{3}$, $2\theta = \frac{11\pi}{6}$

Exercise Set 20: For each of the following equations, find all values of α in $[0, 2\pi)$ which are solutions to the give equation.

1. $\cos(\alpha) + 1 = 0$

2. $\cos(\alpha) = -\frac{\sqrt{3}}{2}$

3. $\tan(\alpha) = -\sqrt{3}$

4. $\sin(\alpha) = -\frac{\sqrt{2}}{2}$

These two will require use of the calculator and decimal radian values.

5. $\cos(\alpha) = 0.28$

6. $\tan(\alpha) = 0.9$

Most of the rest of these involve special angles.

7. $\sqrt{2}\cos(\alpha) - 1 = 0$

8. $9\tan^2(\alpha) - 1 = 0$.

9. $4\sin^2(\alpha) - 4\sin(\alpha) = -1$

10. $3sin^2(\alpha) - 7\sin(\alpha) + 2 = 0$

11. $\tan(\alpha)\sin(\alpha) + \sin(\alpha) = 0$

12. $3\tan^3(\alpha) = \tan(\alpha)$

13. $2\cos(3\alpha) = 1$

14. $3\sin(2\alpha) - 2\sin(\alpha = 0$ **(think** $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$**)**

15. $2\sin^2(\alpha) = 2 + \cos(2\alpha)$

16. $\csc(3\alpha) = 5\sin(3\alpha)$