Hyperbolic structure on a complement of tori in the 4-sphere

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Abstract

It is well known that many noncompact hyperbolic 3-manifolds are topologically complements of links in the 3-sphere. We extend this phenomenon to dimension 4 by exhibiting an example of a noncompact hyperbolic 4-manifold that is topologically the complement of 5 tori in the 4-sphere. We also exhibit examples of hyperbolic manifolds that are complements of $5n$ tori in a simply-connected 4-manifold with Euler characteristic $2n$. All the examples are based on a construction of Ratcliffe and Tschantz, who produced 1171 noncompact hyperbolic manifolds with Euler characteristic 1. Our examples are finite covers of the Ratcliffe-Tschantz manifold with the biggest symmetry group.

1 Introduction

Let $\mathbf{H}^n$ be the n-dimensional hyperbolic space and let G be a discrete subgroup of Isom $\mathbf{H}^n$, the isometries of $\mathbf{H}^n$. If G is torsion-free, then $M = \mathbf{H}^n/G$ is a hyperbolic manifold of dimension n. In this paper, the term "hyperbolic manifold" will always be used for a manifold that is also complete, noncompact and has finite volume. Such a manifold M is the interior of a compact manifold with boundary $\overline{M}$ (see, for example, [1]). Every boundary component of $\overline{M}$ is a compact flat (Euclidean) manifold, i.e. a manifold of the form $\mathbf{R}^{n-1}/K$, where K is a discrete subgroup of Isom $\mathbf{R}^{n-1}$, the isometries of $\mathbf{R}^{n-1}$.

We say that M is a (codimension-2) complement in a closed n-manifold N if $M = N - A$, where A is a closed $(n-2)$-submanifold of N that has a tubular neighborhood and has as many components as $\partial\overline{M}$. Typically one would like N to be a familiar manifold, such as S^n.

It is a well-known fact (see [9, 11]) that many hyperbolic 3-manifolds are complements of links in the 3-sphere. The main purpose of this paper is to generalize this phenomenon

to dimension 4, that is, to provide an example of a hyperbolic 4-manifold that is a complement inside the 4-sphere and to provide some examples where the hyperbolic manifolds are complements in other simply-connected 4-manifolds.

In [5] we considered the general problem of when M may be thought of as a complement. Let $M = N - A$ be a complement. Then every component of $\partial\overline{M}$ must be an S^1-bundle since it will be the boundary of a tubular neighborhood of a component of A, and every component of A must be a flat $(n-2)$-manifold, because flat manifolds are S^1-bundles only over flat manifolds (see [5]). In the 3-dimensional case, all components of $\partial\overline{M}$ are either tori or Klein bottles, both of which are S^1-bundles over S^1, so every hyperbolic 3-manifold is a complement. A slight problem arises for $n \geq 4$ where there exist $(n-1)$-dimensional flat manifolds that are not S^1-bundles. It was shown by Nimershiem in [6] that every closed, connected flat 3-manifold is a boundary component of some hyperbolic 4-manifold , so not every hyperbolic 4-manifold is a complement.

We now focus on the case $n = 4$. Again, let $M = N - A$. Then components of A can only be tori or Klein bottles. Furthermore, we proved in [5] that for a fixed N there exist at most finitely many nonisometric hyperbolic manifolds that are complements in N. This is a consequence of the following three facts: that necessarily $\chi(M) = \chi(N)$, that $\mathrm{Vol}(M) = 4\pi^2/3 \cdot \chi(M)$ (Gauss-Bonnet theorem for hyperbolic 4-manifolds), and that there are finitely many nonisometric hyperbolic 4-manifolds with volume less than a given number (Wang's theorem, [10]). Therefore, there exist only finitely many hyperbolic manifolds that are complements in S^4 and they must have Euler characteristic 2.

A good part for the reason that some hyperbolic 4-manifolds have not already been identified as complements is that there is a small number of examples in the literature that are given in such a way that their topology can be handled more or less easily. One way to give a hyperbolic manifold that allows for relatively easy computation of topological invariants is to represent it as a hyperbolic polyhedron whose faces have been identified via hyperbolic isometries. Such constructions have been done by Ratcliffe and Tschantz [8], Nimershiem [6] and the author [4] and we looked for complement candidates among them, primarily among examples with low Euler characteristic.

Ratcliffe and Tschantz used a computer to find different ways of pairing the sides of a 24-faced hyperbolic polyhedron that yield a hyperbolic manifold. They found 1171 nonisometric examples with Euler characteristic 1, most of them unorientable. In our search for complement examples we tested some suitable Ratcliffe-Tschantz manifolds (their boundary components had to be S^1-bundles) and found an example whose double cover is a complement in S^4. The main result of this paper is the following:

Theorem 4.3 *Let M be the Ratcliffe-Tschantz manifold no. 1011. Then M is a complement of 5 tori in a 4-manifold N with $\pi_1 N = \mathbf{Z}_2$. The orientable double cover of M is the complement of 5 tori in S^4. Furthermore, the orientable double cover has cyclic covers of any index n that are complements of $4n + 1$ tori in a simply-connected closed manifold N with $\chi(N) = 2n$.*

There are two examples by the author that have Euler characteristic 2, P/Φ_1 and P/Φ_2, in the notation of [4]. It turns out that P/Φ_2 has boundary components that are not S^1-bundles. We also showed that P/Φ_1 is a complement in a 4-manifold with fundamental group $\mathbf{Z}_2$ and hence has a double cover that is a complement in a simply-connected 4-manifold. Nimershiem's examples are complements when all their ends are S^1-bundles, however, they have higher Euler characteristics and have not been studied by the author from the point of view of being a complement in a familiar 4-manifold.

The large number (1171) of Ratcliffe-Tschantz manifolds may give rise to other examples of complements in the 4-sphere. However, a criterion in §5 rules out most of these manifolds, leaving about 346 (hand count) of them eligible for having a double cover that is a complement in the 4-sphere. At this point, we can claim only one example with that property, the mentioned manifold no. 1011.

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2　Hyperbolic manifolds as complements

The following theorem summarizes some results from [5] that we will use in this paper.

Theorem 2.1 *Let $E = \mathbf{R}^{n-1}/K$ be a compact flat $(n-1)$-manifold, where K is a discrete subgroup of $\mathrm{Isom}\,\mathbf{R}^{n-1} = \{Ax + a | A \in O(n),\ a \in \mathbf{R}^{n-1}\}$. Then E is an S^1-bundle over a manifold B if and only if there exists a translation $t \in K$ such that $\langle t \rangle$ is a normal subgroup of K and t is not a power of any element of K other than $t^{\pm 1}$. If the translation is given by $x \mapsto x + v$ the normality of $\langle t \rangle$ can be expressed as $Av = \pm v$ for every A such that $Ax + a$ is an element of K. Furthermore if $n - 1 \neq 4, 5$ (we are interested in $n - 1 = 3$ in this paper) the manifold B above is a flat manifold.* □

Let $M = \mathbf{H}^n/G$ be a hyperbolic n-manifold that is a complement in some closed manifold N, that is $M = N - A$ as in §1. Let $\partial \overline{M} = E_1 \cup \cdots \cup E_m$, where each E_i is a flat $(n-1)$-manifold. For simplicity, we often imprecisely say the E_i's are boundary components of M

as well. We recall that $\pi_1 E_i \xrightarrow{\text{inclusion}} \pi_1 M$ is injective. Every boundary component of $\overline{M}$ corresponds to a conjugacy class of parabolic subgroups in G, that is, groups consisting only of parabolic hyperbolic isometries that fix a point v in $\partial \mathbf{H}^n$. Parabolic isometries also fix every horosphere centered at v. Inside the horosphere, whose inherited metric is isometric to $\mathbf{R}^{n-1}$, they act as Euclidean isometries. Each E_i is an S^1-bundle over a component B_i of A. Let t_i denote the element of $\pi_1 E_i \subset \pi_1 M$ that represents the fiber of the S^1-bundle E_i. Then, by Theorem 2.1, each t_i must be a translation in this group and $\langle t_i \rangle$ is a normal subgroup of $\pi_1 E_i$. We call such an element of $\pi_1 E_i$ a *normal translation*. Furthermore, if t_i is not a power of any other element of K other than $t_i^{\pm 1}$ we call t_i *primitive in* $\pi_1 E_i$. We have

Proposition 2.2 *Let M be a complement in N and let $\partial \overline{M} = E_1 \cup \cdots \cup E_m$. If $t_1, \ldots, t_m \in \pi_1 M$ represent fibers of the S^1-bundles $E_1, \ldots, E_m$ then $\pi_1 N = \pi_1 M / \langle\langle t_1, \ldots, t_m \rangle\rangle$, where $\langle\langle A \rangle\rangle$ denotes the normal closure of a subset $A \subset \pi_1 M$. In other words, if we have a presentation for M, the presentation for N is obtained by adding relators $t_1 = 1, \ldots, t_m = 1$.*

Proof. We may assume $\partial \overline{M}$ has only one component E — if there are more, the claim follows by induction. Let t represent the S^1-fiber in E and let B be the base manifold in the S^1-bundle. Let D be the 2-disc bundle over B obtained by filling out every S^1-fiber by a disc so that $\partial D = E$ and $N = \overline{M} \cup_E D$ and let $p : D \to B$ be the fiber projection. By van Kampen's theorem

$$\pi_1 N = \pi_1 M * \pi_1 D / \left\langle\left\langle i_{1*}(e) i_{2*}(e)^{-1}, e \in \pi_1 E \right\rangle\right\rangle,$$

where $i_1 : E \to M$ and $i_2 : E \to D$ are inclusions. However $p_* : \pi_1 D \to \pi_1 B$ is an isomorphism and we may consider $\pi_1 E$ as a subgroup of $\pi_1 M$ so the above group is $\pi_1 M * \pi_1 B / \langle\langle e p_*(e)^{-1}, e \in \pi_1 E \rangle\rangle$. The inclusion map $\pi_1 M \to \pi_1 M * \pi_1 B$ induces a map $\pi_1 M / \langle\langle t \rangle\rangle \xrightarrow{\phi} \pi_1 M * \pi_1 B / \langle\langle e p_*(e)^{-1}, e \in \pi_1 E \rangle\rangle$. We show that ϕ is an isomorphism. It is surjective, since for every element $b \in \pi_1 B$ we have $b = p_*(e)$, hence $[b] = [e p_*(e)^{-1} p_*(e)] = [e] \in \operatorname{im} \phi$.

Now suppose that $\phi([g]) = 1$ for some $g \in \pi_1 M$. Then $g = \prod x_i (e_i p_*(e_i)^{-1})^{\pm 1} x_i^{-1}$ where $e_i \in \pi_1 E$ and $x_i \in \pi_1 M * \pi_1 B$ is a product of elements in $\pi_1 M$ and $\pi_1 B$. However, note that a conjugate by $b \in \pi_1 B$, where $b = p_*(e)$, is $byb^{-1} = e(e^{-1} p_*(e)) y (e^{-1} p_*(e))^{-1} e^{-1}$, so if y has the form $\prod g_j (e_j p_*(e_j)^{-1})^{\pm 1} g_j^{-1}$, $g_j \in \pi_1 M$ then

$$byb^{-1} = \prod b g_j (e_j p_*(e_j)^{-1})^{\pm 1} g_j^{-1} b^{-1} = \prod e(e^{-1} p_*(e)) g_j (e_j p_*(e_j)^{-1})^{\pm 1} g_j^{-1} (e^{-1} p_*(e))^{-1} e^{-1} =$$
$$\prod e(e^{-1} p_*(e)) e^{-1} (e g_j) (e_j p_*(e_j)^{-1})^{\pm 1} (e g_j)^{-1} e(e^{-1} p_*(e))^{-1} e^{-1}$$

which has the same form as y since $e \in \pi_1 M$. By repeatedly applying this observation we can transform the representation of g to $g = \prod g_i(e_i p_*(e_i)^{-1})^{\pm 1} g_i^{-1}$, $g_i \in \pi_1 M$. This equation is in $\pi_1 M * \pi_1 B$: in order to satisfy it, all the elements of $\pi_1 B$, namely $p_*(e_i)^{-1}$, must cancel. Then either $p_*(e_i) = 1$ for all i which implies that $e_i = t^{k_i}$, so $g = \prod g_i t^{\pm k_i} g_i^{-1}$ in which case $[g] = 1$ in $\pi_1 M / \langle\langle t \rangle\rangle$, or there exists an i so that the term between $p_*(e_i)^{\pm 1}$ and $p_*(e_{i+1})^{\pm 1}$ cancels. It is not hard to see (there are four cases depending on the exponents ± 1) that the product then reduces to a product of the same form with one fewer term, so the argument is finished by induction. $\square$

In order to simplify computations showing that a proposed M is a complement, we need the following propositions.

Proposition 2.3 *Let $\tilde{X} \xrightarrow{p} X$ be the covering space of a manifold X corresponding to a normal subgroup H of $G = \pi_1 X$. If E is a path-connected subset of X (e.g. a submanifold), and $K = i_*(\pi_1(E))$, where $E \xrightarrow{i} X$ is the inclusion, then the number of path-components of $p^{-1}(E)$ is equal to the index of KH in G. Note that this is the same as the index in G/H of the image of K under the quotient map $G \to G/H$.*

Proof. Choose $x_0 \in E$ and let $Y = p^{-1}(x_0)$. If we choose $y_0 \in Y$ there is a standard correspondence $G/H \leftrightarrow Y$ given by $gH \leftrightarrow$ (the endpoint of the lift of g starting at y_0) that we can compose to a map $G/H \xrightarrow{\phi} Y/\sim$, where $y_1 \sim y_2$ if both are in the same path-component of $p^{-1}(E)$. Suppose $\phi(g_1 H) = \phi(g_2 H)$. Then loops g_1 and g_2 lift to paths starting with y_0 and ending with y_1 and y_2 respectively and there is a path $\tilde{k}$ from y_1 to y_2 in $p^{-1}(E)$. Taking $k = p \circ \tilde{k}$ gives a loop in E whose homotopy class is in K and the lift of $g_1 k$ has the same endpoint as g_2, thus $g_1 k H = g_2 H$, thus $g_2^{-1} g_1 = hk^{-1}$ for some $h \in H$. Now $hk^{-1} = k^{-1}khk^{-1}h$, so $g_2^{-1} g_1 = k^{-1} h_1$, for some $h_1 \in H$ because H is normal. Hence g_1 and g_2 are in the same class of KH. (Note that, due to normality of H, we can write any element of KH as kh for some $k \in K$ and $h \in H$.) Conversely, if $g_1 KH = g_2 KH$ then $g_2 = g_1 kh$ for some $k \in K$, $h \in H$. Then the lifts of $g_1 k$ and g_2 starting at y_0 have the same endpoint y_2, but since we may assume that the loop k lies in E then the endpoint y_1 of g_1 is connected to y_2 by a lift of k starting at y_1, but this path is in $p^{-1}(E)$, so $\phi(g_1 H) = \phi(g_2 H)$. Therefore, ϕ induces a bijection between the set of path components of $p^{-1}(E)$ (same as $Y/\sim$) and the set of left cosets of KH.

Note also that the number of preimages of x_0 that are in the same path-component of $p^{-1}(E)$ is equal to the index of H in KH, which, by the standard isomorphism theorem $KH/H \cong K/(K \cap H)$ is equal to the index of $K \cap H$ in K. $\square$

Proposition 2.4 *Let M be a hyperbolic manifold with $\partial \overline{M} = E_1 \cup \cdots \cup E_m$ and let $t_1 \in \pi_1 E_1, \ldots, t_m \in \pi_1 E_m$ be normal translations. Suppose that $\pi_1 M / \langle\langle t_1, \ldots, t_m \rangle\rangle$ is a finite group of order l and that t_i is primitive in $\pi_1 E_i \cap \langle\langle t_1, \ldots, t_m \rangle\rangle$ for every $i = 1, \ldots, m$.*

Then M has an l-fold normal cover that is a complement inside a simply-connected closed manifold N.

Proof. Set $G = \pi_1 M$, $H = \langle\langle t_1, \ldots, t_m \rangle\rangle$, $K_i = \pi_1 E_i$, $i = 1, \ldots, m$ and let $t_1, \ldots, t_m$ be normal translations satisfying conditions of the theorem and let $\Phi = G/H$, where $q : G \to \Phi$ is the quotient map. For every i, we may choose a finite set $R_i \subset K_i$ that is a transversal (i.e. a set of coset representatives) of $K_i \cap H$ in K_i and a finite set $S_i \subset G$ that is a transversal of $K_i H$ in G. Any element $g \in G$ may be written as $r_i s_i h$ for some $r_i \in R_i$, $s_i \in S_i$, $h \in H$. The conjugate $g^{-1} t_i g$ can then be written as $h^{-1} s_i^{-1} r_i^{-1} t_i r_i s_i h = h^{-1} s_i^{-1} t_i^{\pm 1} s_i h$ since t_i is a normal translation. The elements of $\langle\langle t_1, \ldots, t_m \rangle\rangle$ are products of form $\prod g_j^{-1} t_{i_j}^{l_j} g_j$ which can be written as $\prod h_j^{-1} s_j^{-1} r_j^{-1} t_{i_j}^{l_j} r_j s_j h_j = \prod h_j^{-1} s_j^{-1} t_{i_j}^{\pm l_j} s_j h_j$ where $h_j \in H$, $s_j \in S_{i_j}$ and $r_j \in R_{i_j}$. Therefore, if $\langle\langle A \rangle\rangle_H$ denotes the normal closure in H of a subset A of H, then $H = \langle\langle t_1, \ldots, t_m \rangle\rangle = \langle\langle s_i^{-1} t_i s_i \mid s_i \in S_i, i = 1, \ldots, m \rangle\rangle_H$.

By Proposition 2.3, $p^{-1}(E_i)$ has $[\Phi : q(K_i)] = |S_i|$ components, $i = 1, \ldots, m$. Restricting p to each of the components we get a covering of E_i corresponding to the subgroup $K_i \cap H \subset K_i$. Normality of translations $t_1, \ldots, t_m$ and the fact that each of them is primitive in $K_i \cap H$ ensure that each component of $p^{-1}(E_i)$ is a fiber bundle and that we may choose as a fiber any lift of a loop representing the element t_i that is contained in $p^{-1}(E_i)$. Therefore, the cover of M corresponding to $H \subset \pi_1 M$ is going to be a complement in some closed manifold N. What is $\pi_1 N$? As seen above, components of $p^{-1}(E_i)$ are in 1-1 correspondence with elements of S_i and a fiber in each component represents the element $s_i^{-1} t_i s_i$, where s_i ranges through S_i. Inserting 2-disc-bundles that close off components of $p^{-1}(E_i)$ gives us N, whose fundamental group, according to Proposition 2.2, is $H / \langle\langle s_i^{-1} t_i s_i \mid s_i \in S_i, i = 1, \ldots m \rangle\rangle_H = H/H = 1$. $\square$

3   Moving around a tiling of $\mathbf{H}^n$

Let G be a discrete subgroup of Isom $\mathbf{H}^n$ that is generated by side-pairings of its finite-sided fundamental polyhedron P (see [7] or [4]). If S is a side of P, let s be the side-pairing that sends S to its pair S' (thus, $sS = S'$). Then s^{-1} will pair S' to S. For every point x in P let $[x]$, the cycle of x, be the set of all points in P that are obtained by sending x around P via side-pairing transformations. As in [7], for every $x \in P$ let $\omega(x)$ be the measure of the "spatial angle" that P subtends at x, that is $\omega(x) = \mathrm{Vol}(B(x, r) \cap P) / \mathrm{Vol}\, B(x, r)$, where $B(x, r)$ is a ball around x with radius r small enough so that $B(x, r)$ intersects only the sides

on which x lies. Set $\omega[x] = \sum_{y \in [x]} \omega(y)$. Then G will be a torsion free group if and only if $\omega[x] = 1$ for every $x \in P$ (see [7]). Note that for a polyheron P with ideal vertices $[x]$ is well-defined for an ideal vertex as well.

Since P is a fundamental polyhedron, $\mathbf{H}^n = \cup_{g \in G} gP$ where $g_1(\text{int } P) \cap g_2(\text{int } P) = \emptyset$ for $g_1 \neq g_2$ and any two polyhedra $g_1 P$ and $g_2 P$ either do not intersect or intersect along one of their k-faces. Imagine a piecewise-linear path whose every vertex is in $\text{int } gP$ for some $g \in G$ and whose edges join vertices in adjacent translates of P so that the edge between two consecutive vertices passes through the interiors of the sides that the adjacent translates share. If the translates are $g_1 P$ and $g_2 P$ and we exited through side $g_1 S$ of the polyhedron $g_1 P$, then we entered $g_2 P$ through side $g_2 S'$, where S' is the side paired to S under the side-pairing of P. If s is the transformation that pairs S to S', then $g_2 = g_1 s^{-1}$. Thus, if we start in translate gP and the path exits, sequentially, through the translates of sides $S_1, \ldots, S_m$ of P, then the vertices of the path are in translates gP, $gs_1^{-1}P$, $gs_1^{-1}s_2^{-1}P$,...,$gs_1^{-1} \ldots s_m^{-1}P$.

Let H be a finite-index subgroup of G with transversal X, so $G = \cup_{x \in X} Hx$. Then $Q = \cup_{x \in X} xP$ is a fundamental polyhedron for H (elements of X can be chosen so that Q is connected). How are sides of Q paired in order to generate H? If R is a side of Q, then $R = xS$ for some side S of P. Let f be the transformation that pairs R to a side R' of Q. Adjacent to Q on the other side of R' is fQ and $R' = yS'$ for some $y \in X$. Now in the tiling by P, translates fxP and yP meet along paired sides of P and one gets from yP to fxP by exiting through yS'. By the above we get $fx = ys$, which implies $ysx^{-1} = f \in H$ making y the coset representative for Hxs^{-1}. Therefore, side xS gets paired to side yS' via $f = ysx^{-1}$, where S' is the pair of S under side-pairing s of P and y is the element of the transversal of H that represents the coset Hxs^{-1}.

We now turn to the special case where G is generated by reflections in the sides of a polyhedron P. The following proposition is inspired by and generalizes a claim from [8] that was specific to the polyhedra P and Q Ratcliffe and Tschantz were considering.

Proposition 3.1 *Let G be generated by reflections in the sides of a polyhedron P and let H be a finite-index subgroup of G so that its transversal is a finite group K. (In other words, $G = HK$ and $H \cap K = 1$.) Then the fundamental polyhedron for H is $Q = \cup_{k \in K} kP$ which is invariant under K. Furthermore, any side R of Q is paired to a side R' of Q by a transformation of the form kr, where $k \in K$ is such that $k(R) = R'$ and r is a reflection in R. Alternatively, R is paired to R' by a transformation of the form $r'k$, where r' is a reflection in R' and $k \in K$ such that $k(R) = R'$.*

Proof. Applying the above discussion to this situation, we see that side $R = k_1 S$ is paired to side $R' = k_2 S'$ via $f = k_2 s k_1^{-1}$, where s is a reflection in S. Then $f = (k_2 k_1^{-1})(k_1 s k_1^{-1})$, but $r = k_1 s k_1^{-1}$ is a reflection in $k_1 S = R$ and $k = k_2 k_1^{-1}$ is an element of K that sends

$R = k_1 S$ to $R' = k_2 S$, so $f = kr$. Now we note that $kr = krk^{-1}k$ and $r' = krk^{-1}$ is the reflection in $kR = R'$. $\square$

In this paper we will be interested in finding generators of parabolic subgroups of G that correspond to fundamental groups of boundary components of the manifold. Let v be an ideal vertex of P and let X_v be the finite set of transformations in G that send vertices in the cycle $[v]$ to v. Thus for $w \in [v]$ there is a unique $x \in X_v$ so that $xw = v$. The subgroup $G_v = \mathrm{stab}_v\, G = \{g \in G \mid gv = v\}$ is the fundamental group of the boundary component of $M = \mathbf{H}^n/G$ that corresponds to the ideal vertex v. Since the manifold is complete, G_v consists only of parabolic elements, so it will fix every horosphere centered at v. Choose a sufficiently small horosphere C centered at v that intersects only those sides of P in whose closure v is. Then G_v is a group of Euclidean isometries of $C \cong \mathbf{R}^{n-1}$. At other vertices in $[v]$ we choose horospheres $x^{-1}C$, where $x \in X_v$. The set $(x^{-1}C) \cap P$ is a Euclidean polyhedron in the horosphere $x^{-1}C$ whose sides are intersections of sides of P with the horosphere $x^{-1}C$.

The fundamental polyhedron for G_v and its side-pairing transformations are found in much the same way as the fundamental polyhedron of a finite-index subgroup of G, above. First of all, the fundamental polyhedron for G_v is $P_v = (\cup_{x \in X_v} xP) \cap C$ where we can choose elements of X_v so that the above union is connected. Let $x_1 S$ be a side of $x_1 P \cap C$, $v_1 \in [v]$ so $x_1 v_1 = v$; let s be the side-pairing transformation that sends S to S', and let $v_2 = sv_1$. (Note that v_1 is on side S and v_2 is on side S'.) Then $x_1 S \cap C$ will be paired to $x_2 S' \cap C$ and the side-pairing transformation is $x_2 s x_1^{-1}$ — note that this transformation is in G_v.

We will make use of the following proposition. For simplicity of notation, when we discuss the fundamental polyhedron of G_v, we are assuming the intersection of the polyhedron P and its sides with the horosphere C so we drop the "$\cap C$".

Proposition 3.2 *Let G, P, K, H and Q be as in Proposition 3.1, v an ideal vertex of P, $G_v = \mathrm{stab}_v G$, and let X_v be as above. Then $Q_v = \cup_{x \in X_v} xQ$ is the fundamental polyhedron for G_v. Let side xS of Q_v be paired to yS' via $f = ysx^{-1}$ where $x, y \in X_v$. Consider a piecewise linear path (as above) that starts in xQ, goes to yQ and exits through side yS' of yQ. Suppose the path from xQ to yQ passes through different translates $x_1 Q = xQ, x_2 Q, \ldots, x_m Q, yQ$ of Q, exiting every time through side $x_i S_i$ of $x_i Q$ and finally exiting through yS'. If $x = y$ set $m = 0$. This means that $y = x s_1^{-1} s_2^{-1} \ldots s_m^{-1}$, where s_i is the transformation that pairs S_i.*

Then the side-pairing f can be written as $f = (qq_m \ldots q_1) x (k_1^{-1} \ldots k_m^{-1} k^{-1}) x^{-1}$ where q_i is the reflection in the plane $x_i S_i$, q is the reflection in plane yS', and k_i and k are elements of K in the decompositions of side pairings s_i and s^{-1}, respectively, into form kr as in Proposition 3.1.

In other words, as we go from xP to ysP we pass through a sequence of hyperplanes containing v. The side-pairing that sends xS to yS' is the product of reflections in those

hyperplanes and the conjugate of products of elements of K that correspond to the side-pairings of sides whose translates we exit through.

Proof. Let $x = 1$. We first prove the claim $s_1^{-1} \ldots s_m^{-1} = (q_m \ldots q_1)(k_1^{-1} \ldots k_m^{-1})$ by induction on m. Note that $x_2 = s_1^{-1}$, $x_3 = s_1^{-1}s_2^{-1}$ and so on. For $m = 1$ we have $s_1^{-1} = (k_1 r_1)^{-1} = r_1 k_1^{-1}$, and r_1 is a reflection in $x_1 S_1 = S_1$. Suppose the claim holds for $m - 1$. Then $s_1^{-1} \ldots s_{m-1}^{-1} s_m^{-1} = (s_1^{-1} \ldots s_{m-1}^{-1}) r_m^{-1} (s_{m-1} \ldots s_1)(s_1^{-1} \ldots s_{m-1}^{-1}) k_m^{-1} = q_m (s_1^{-1} \ldots s_{m-1}^{-1}) k_m^{-1} = q_m (q_{m-1} \ldots q_1 k_1^{-1} \ldots k_{m-1}^{-1}) k_m^{-1}$. Here we used that $(s_1^{-1} \ldots s_{m-1}^{-1}) r_m^{-1} (s_{m-1} \ldots s_1)$ is a reflection in the plane $s_1^{-1} \ldots s_{m-1}^{-1} S_m = x_m S_m$. Of course now $s_1^{-1} \ldots s_m^{-1} s = s_1^{-1} \ldots s_m^{-1} (s^{-1})^{-1}$ and the claim follows from what we just proved since $s^{-1} = kr$.

For a general x notice that the element of G sending x to $ys = xs_1^{-1} \ldots s_m^{-1} s$ is given by $xs_1^{-1} \ldots x_m^{-1} sx^{-1} = (xqq_m \ldots q_1 x^{-1})(xk_1^{-1} \ldots k_m^{-1} k^{-1} x^{-1})$. However, if q_i is a reflection in $s_1^{-1} \ldots s_{i-1}^{-1} S_i$ then $xq_i x^{-1}$ is a reflection in $xs_1^{-1} \ldots s_{i-1}^{-1} S_i = x_i S_i$ and the claim is proved. $\square$

Remark 3.3 Note that $k_1^{-1} \ldots k_m^{-1} k^{-1}$ is an element of K that preserves Q, the vertex $x^{-1} v$ and $x^{-1} C$, hence $x(k_1^{-1} \ldots k_m^{-1} k^{-1}) x^{-1}$ preserves v, C and xQ, so the side-pairing is a transformation that keeps xQ fixed followed by reflections in the above hyperplanes which contain v.

4 Example of a complement in the 4-sphere

We now recall Ratcliffe and Tschantz's 24-sided polyhedron (see [8]) and a side-pairing that yields an unorientable manifold with Euler characteristic 1.

We use the ball model of hyperbolic 4-space $\mathbf{H}^4$, so $\mathbf{H}^4 = B^4$, the unit 4-ball, with the differential metric $ds^2 = \frac{4|dx|^2}{(1-|x|^2)^2}$. Hyperplanes in $\mathbf{H}^4$ are round 3-spheres in $\mathbf{R}^4$ that are perpendicular to ∂B^4.

Consider the 24 spheres of radius 1 that are centered at points whose two coordinates are ± 1 and the other two are 0. Each of those spheres is perpendicular to $\partial \mathbf{H}^4$ and any two of them either intersect at a right angle, are tangential with tangent point on $\partial \mathbf{H}^4$ or are disjoint. The spheres thus determine 24 hyperplanes that intersect at right angles. Let Q be the polyhedron that is the intersection of the 24 half-spaces determined by the hyperplanes, where we always take the half-space that contains the origin. We will denote the sides of Q and the hyperplanes that they lie on by S_{****}, where "$****$" is a string with two zeroes and two +'s or –'s that determines the center of the sphere on which the hyperplane lies. For example, S_{+0-0} is the side lying on the sphere centered at $(1, 0, -1, 0)$.

The polyhedron Q has ideal vertices $v_{*000} = (\pm 1, 0, 0, 0)$, $v_{0*00} = (0, \pm 1, 0, 0)$, $v_{00*0} = (0, 0, \pm 1, 0)$, $v_{000*} = (0, 0, 0, \pm 1)$ and $v_{****} = (\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$, where the "$*$" in the subscript is always "+" or "−". The polyhedron has no real vertices.

Note that two sides intersect when the strings describing them have equal nonzero entries in one position and the positions of the remaining nonzero entries are different. Two sides touch at $\partial \mathbf{H}^4$ when they have equal nonzero entries in one position and they have opposite nonzero entries in another position or if the positions where they have nonzero entries are complementary. Furthermore, an ideal vertex will be on a side if its euclidean distance from the center of the sphere defining the side is 1. (For example, S_{0-+0} and S_{+-00} intersect, S_{0-+0} and S_{0+0+} are disjoint, S_{0-+0} and S_{0--0} touch at v_{0-00}, and S_{0-+0} and S_{-00-} touch at v_{--+-}).

The intersection of the horosphere centered at an ideal vertex v with the hyperplanes that contain v looks like a cube because this horosphere intersects six sides that intersect either at right angles or meet only at v, hence their intersection with the horosphere are pairs of parallel planes that intersect at right angles. Due to symmetry of Q, the intersection of Q with C will be a cube rather than a general rectangular box.

Let P denote the intersection of Q with the positive hexadecant and let K be the order-16 group generated by reflections in hyperplanes $x_i = 0$, $i = 1, \ldots, 4$. Elements of K we denote as k_{****} where every "$*$" is a "$+$" or a "$-$". It is clear that $Q = KP = \{kP \mid k \in K\}$.

Ratcliffe and Tschantz investigated $\Gamma_2^4 \subset \operatorname{Isom} \mathbf{H}^n$, the congruence two group of integral Lorentzian 5×5 matrices (see [8] for definition) and found that it is isomorphic to the group generated by reflections in the sides of the polyhedron P. Thus, $\Gamma_2^4 = \langle k_{-+++}, k_{+-++}, k_{++-+}, k_{+++-}, r_{++00}, r_{+0+0}, r_{0++0}, r_{+00+}, r_{0+0+}, r_{00++} \rangle$, where r_{****} is a reflection in the side S_{****}. If G is a finite-index torsion-free subgroup of Γ_2^4, then K acts on the set Γ_2^4/G freely so K must divide $[\Gamma_2^4 : G]$. If G has minimal index 16, then K is a set of coset representatives so Q will be a fundamental polyhedron for G, and G will be generated by side-pairings of Q. Ratcliffe and Tschantz showed that any side-pairing transformation of Q that takes side R to side R' must be of form kr or $r'k$, where $k \in K$ takes R to R' and r and r' are reflections in R and R', respectively. (This claim follows from Proposition 3.1.) Thus, a side-pairing transformation will be specified by giving an element of K.

Ratcliffe and Tschantz used a computer to find all the side-pairings of Q for which $\omega[x] = 1$ for every $x \in \partial Q$. They are the side-pairings that yield a hyperbolic manifold. The list of 1171 nonisometric manifolds that they obtained, along with their side-pairings is in their paper [8]. They also found that all sides of Q labeled by a string with the two zeroes in the same place have the same corresponding k.

A specialization of Proposition 3.2 will be useful:

Proposition 4.1 *Let G, Q, and K be as in proposition 3.2, specialized to any of the Ratcliffe-Tschantz examples. In this case $K = \mathbf{Z}_2^4$, in particular, it is abelian. Then, with the set-up from Proposition 3.2, a side-pairing of Q_v can be written in form $f = (qq_m \ldots q_1)x(k_1 \ldots k_m)x^{-1} = (qq_m \ldots q_1)(p_j \ldots p_1)(k_1 \ldots k_m k)(p_j \ldots p_1)^{-1}$ where $p_1, \ldots, p_j$ are reflections in the hyperplanes that one passes through by going from Q to xQ.*

Proof. We only need to rewrite $x(k_1^{-1} \ldots k_m^{-1} k^{-1})x^{-1}$. From Proposition 3.2 we can write $x = (p_j \ldots p_1)(h_1^{-1} \ldots h_j^{-1})$ by following a path from Q to xQ, where $h_1, \ldots, h_j$ are appropriate elements of K. Commutativity of K now reduces $x(k_1^{-1} \ldots k_m^{-1} k^{-1})x^{-1}$ to $(p_j \ldots p_1)(k_1^{-1} \ldots k_m^{-1} k^{-1})(p_j \ldots p_1)^{-1}$ and the fact that every element of K has order 2 allows us to write k_i instead of k_i^{-1}. $\square$

Remark 4.2 Note that $k_1 \ldots k_m k$ preserves v since it preserves $w = h_1^{-1} \ldots h_j^{-1} v$ and the two transformations commute. Each p_i is a reflection in a hyperplane containing v which will make $(p_j \ldots p_1)(k_1 \ldots k_m k)(p_j \ldots p_1)^{-1}$ easy to visualize: it is nothing but the transformation $k_1 \ldots k_m k$ moved to $(p_j \ldots p_1)Q$.

We now show that the orientable double cover of manifold no. 1011 (call it M) in Ratcliffe and Tschantz's list is a complement in the 4-sphere and that it has covers of every even index that are complements in some other simply-connected 4-manifold. Of the 1171 Ratcliffe and Tschantz's examples, we chose to try this one because it has the largest symmetry group, which slightly simplifies the calculations.

First of all, the side-pairing for M is given by transformations that we name $a, b, \ldots, k, l$ as follows (r is always the reflection in the originating side):

$$
\begin{array}{llll}
S_{++00} \xrightarrow{a} S_{-+00} & S_{+-00} \xrightarrow{b} S_{--00} & a, b = k_{-+++}r \\
S_{+0+0} \xrightarrow{c} S_{+0-0} & S_{-0+0} \xrightarrow{d} S_{-0-0} & c, d = k_{++-+}r \\
S_{0++0} \xrightarrow{e} S_{0--0} & S_{0+-0} \xrightarrow{f} S_{0-+0} & e, f = k_{----}r \\
S_{+00+} \xrightarrow{g} S_{-00-} & S_{+00-} \xrightarrow{h} S_{-00+} & g, h = k_{----}r \\
S_{0+0+} \xrightarrow{i} S_{0-0+} & S_{0+0-} \xrightarrow{j} S_{0-0-} & i, j = k_{+-++}r \\
S_{00++} \xrightarrow{k} S_{00+-} & S_{00-+} \xrightarrow{l} S_{00--} & k, l = k_{+++-}r
\end{array}
$$

The presentation for the group G generated by these side-pairings is obtained by performing an edge-chase (see e.g. [4]). Since there are no real vertices, every 2-face of Q (an "edge", or a codimension-2 face) has an ideal vertex. We will use fundamental polyhedra Q_v, as above, to find generators and normal translations of $\pi_1 E_i$, where $\partial \overline{M} = E_1 \cup \cdots \cup E_5$. These polyhedra will also help with the edge chase, since each of their codimension-2 faces

is the intersection of a codimension-2 face of Q with a horosphere C and can be used to represent an edge in the cycle. All the edges in a cycle have a vertex v that is an element of the cycle of some vertex, so all the edges of a cycle will be represented by a codimension-2 face of some polyhedron Q_v. Furthermore, since every 2-face of Q has an ideal vertex that is among v_{*000}, v_{0*00}, v_{00*0} and v_{000*}, the codimension-2 faces of polyhedra $Q_{v_{+000}}$, $Q_{v_{0+00}}$, $Q_{v_{00+0}}$ and $Q_{v_{000+}}$ will represent all the edges from all cycles.

Figure 1 contains pictures of $Q_{v_{+000}}$, $Q_{v_{0+00}}$, $Q_{v_{00+0}}$, $Q_{v_{000+}}$. Using Proposition 4.1 we can determine the side-pairings of each Q_v and the generating translations of the translation subgroup of G_v as well as which ones among them are normal.

We explain the method on the example $Q_{v_{+000}}$. The only other vertex in the cycle of the vertex v_{+000} is $v_{-000} = av_{+000}$. Hence $Q_{v_{+000}} = (Q \cup a^{-1}Q) \cap C$ will be a union of two side-by-side cubes, it is the top rectangular box depicted in Fig. 1. The strings labeling the sides of $Q_{v_{+000}}$ come from the labeling of sides of Q (sides on the right cube are actually translates of the sides of Q with the indicated label). The symbol v_{-000} above the right cube denotes that the right cube is a translate (by a^{-1}) of Q intersected with a horosphere centered at v_{-000}.

From Fig. 1 we can read off the cycles of edges. For example, look at $S_{+0-0} \cap S_{+00+}$. We see that

$$S_{+00+} \cap S_{+0+0} \xrightarrow{c} S_{+0-0} \cap S_{+00+} \xrightarrow{g} S_{-00-} \cap S_{-0+0} \xrightarrow{d} S_{-0-0} \cap S_{-00-} \xrightarrow{g^{-1}} S_{+00+} \cap S_{+0+0}$$

and this edge cycle contributes the relation $g^{-1}dgc = 1$ to the presentation of G.

Now for the side-pairing transformations of $Q_{v_{+000}}$. Note that under the natural identification $C \cong \mathbf{R}^3$ the origin of $\mathbf{R}^{n-1}$ may be taken to be the center of the cube $Q \cap C$. In Fig. 1 we indicate the directions of the coordinate axes at left — they apply to only the left cubes in the diagram, as the orientation of the right cubes is governed by the side-pairing.

Side S_{+0+0} is paired to side S_{+0-0} via c. Since $cv_{+000} = v_{+000}$ we see that $c \in G_{v_{+000}}$. By Proposition 4.1 c is the composite of the reflection in S_{+0-0} and k_{++-+}^{-1}, which is a reflection in the plane $x_3 = 0$, making c a translation in the x_3-direction (but with opposite orientation). We use similar reasoning to see that $a^{-1}S_{-0+0}$ is paired to $a^{-1}S_{-0-0}$ via a translation in the x_3-direction.

Side S_{+-00} is paired to side S_{--00} via b. Since $bv_{+000} = v_{-000}$ then S_{+-00} will be sent to $a^{-1}S_{--00}$ by an element of $G_{v_{+000}}$. This transformation will be the composite of reflections in S_{++00} and $a^{-1}S_{--00}$ (respectively, the center dividing plane and the right side of the top box in Fig. 1, respectively) and of $k_{-+++}k_{-+++} = 1$, so will thus be a translation in the x_2 direction.

Side S_{+00+} is paired to side S_{-00-} via g. Since $gv_{+000} = v_{-000}$ there is an element of $G_{v_{+000}}$ that sends S_{+00+} to $a^{-1}S_{-00-}$. This transformation is a composite of reflections in S_{++00}
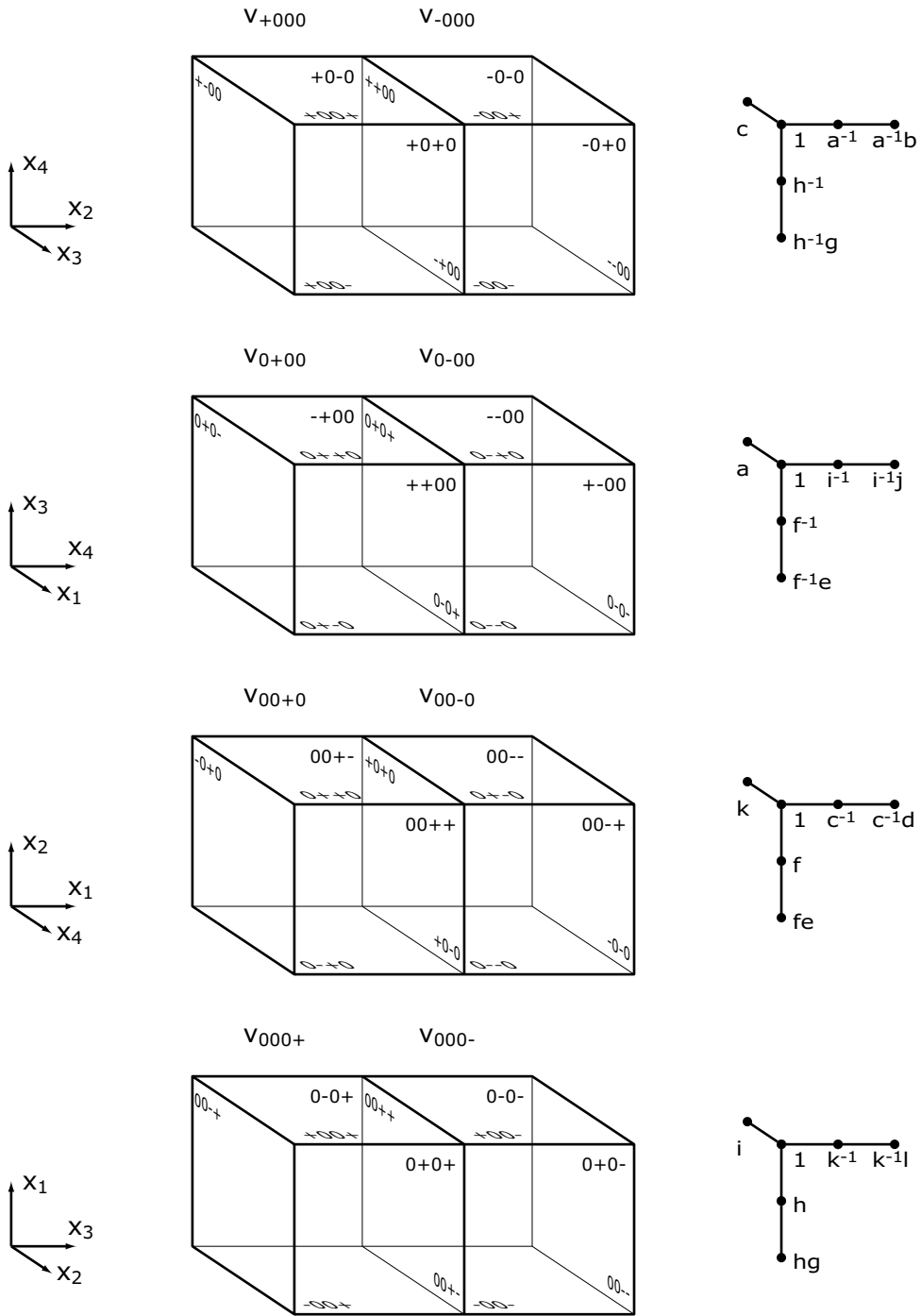
Figure 1: Fundamental polyhedra for $G_{v_{+000}}$, $G_{v_{0+00}}$, $G_{v_{00+0}}$, $G_{v_{000+}}$ and generators of their translation subgroups.
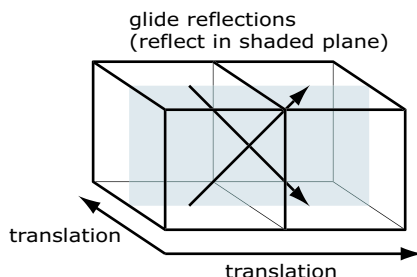
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Figure 2: Side-pairing pattern for fundamental polyhedra $Q_{v_{+000}}$, $Q_{v_{0+00}}$, $Q_{v_{00+0}}$, $Q_{v_{000+}}$

and $a^{-1}S_{-00-}$ and $k_{-+++}k_{----} = k_{+---}$. Hence, its rotational part will be the composite of reflections in $x_2 = 0$, $x_4 = 0$, which are parallel to S_{++00} and $a^{-1}S_{-00-}$, respectively, and $k_{+---}|_C$. The composite is thus a reflection in $x_3 = 0$. It is now clear that S_{++00} is sent to $a^{-1}S_{-00-}$ by a glide reflection, that is, a translation composed with a reflection in a plane containing the translation. The plane is $x_3 = 0$, the plane going through the middle of the two cubes and parallel to the front side. We use similar reasoning to see that S_{+00-} is paired to $a^{-1}S_{-00+}$ via a glide reflection, where the reflecting is again performed in the plane $x_3 = 0$.

The pattern of side-pairings, which is the same for each of the rectangular boxes $Q_{v_{+000}}$, $Q_{v_{0+00}}$, $Q_{v_{00+0}}$ and $Q_{v_{000+}}$ is illustrated on Fig. 2 — arrows indicate translations and it is noted if a translation is composed with any reflections. We can see that $G_{v_{+000}}$ will be generated by three elements: the translation c that pairs the front and back sides, the translation $a^{-1}b$ that pairs the left and right sides and the glide reflection $a^{-1}g$ that pairs half of the top side with half of the bottom side. (The second glide reflection is simply the composite of the inverse of the glide reflection mentioned and the second translation.)

The actual elements that generate the group are obtained by the method discussed at the beginning of § 3. A graph is convenient for this purpose: each vertex represents a translate of Q and the label on the vertex indicates by which element Q was moved. for example, the horizontal edges in the top graph show that if we pass through S_{++00}, we arrive in $a^{-1}Q$; passing further through $a^{-1}S_{--00}$ gets us in $a^{-1}bQ$.

Since the group of rotational parts of $G_{v_{+000}}$ has order 2, the subgroup of translations has index 2 in $G_{v_{+000}}$, which means that $Q_{v_{+000}} \cup (a^{-1}g)Q_{v_{+000}}$ is a fundamental polyhedron for this subgroup. It is visually clear that this polyhedron will fill out $\mathbf{R}^3$ if it is moved around by composites of translations $t_1 = c$, $t_2 = a^{-1}b$ and $t_3 = h^{-1}g$, hence those three translations generate the subgroup of translations ($h^{-1}g$ is a translation using the same explanation as $a^{-1}b$). The normal translations among them are those whose vectors are ± 1 eigenvectors of
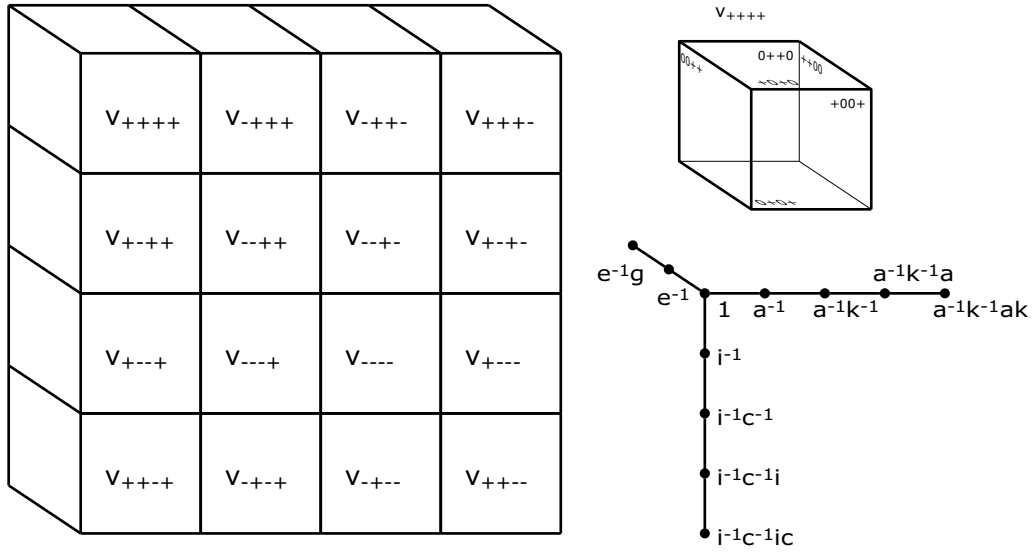
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Figure 3: Fundamental polyhedron for $G_{v_{++++}}$ and generators of its translation subgroup

the reflection in $x_3 = 0$, which are multiples of t_1 and any combination of t_2 and t_3.

Fig. 3 shows the fundamental polyhedron $Q_{v_{++++}}$. The smaller cube illustrates how the faces are labeled on the portion of $Q_{v_{++++}}$ labeled v_{++++} ; this labeling implies the labeling on every other cube that makes up $Q_{v_{++++}}$. For example, the front side on cube v_{----} is labeled $-00-$, we simply insert zeroes in the appropriate places into the string labeling v_{****}. Using techniques as above (or see §5) it is easy to see that the left and right sides of the rectangular box, as well as the bottom and top sides are paired by translations. The side pairing pattern for the back side is illustrated in Fig. 4 and the pattern for the front side is identical. Note that shaded rectangles are paired by a glide reflection. To see this, note that S_{0++0} is paired to S_{0--0} via e. Since $ev_{++++} = v_{----}$ the back side of the cube labeled v_{++++} is paired to the back side of the cube v_{----}. As in 4.1, take a path from the cube labeled v_{++++} that goes two steps to the right, then two steps down and then exits out the back of the cube labeled v_{----}. The composite of the reflections in the first four planes that we pass through gives a translation in the direction of arrow in Fig. 4, composing with a reflection in the back plane makes it a glide translation. We also have to compose this with a conjugate of a certain element of K which turns out to be 1 because the only element of K that sends v_{++++} to itself is 1. The translation subgroup of $G_{v_{++++}}$ is obtained in the same way as for $G_{v_{+000}}$ above.

We obtain all the relations and all the generators of the translation subgroups for each
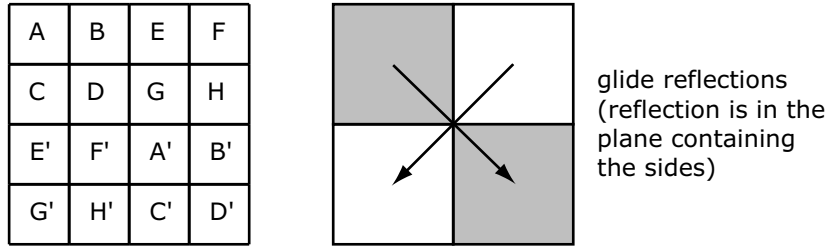
15

A	B	E	F
C	D	G	H
E'	F'	A'	B'
G'	H'	C'	D'

glide reflections
(reflection is in the
plane containing
the sides)

Figure 4: Side-pairing pattern of front and back sides of $Q_{v_{++++}}$

vertex cycle in the way described above. Here are the relations:

$$\left.\begin{array}{c} b^{-1}d^{-1}bc = 1 \\ a^{-1}d^{-1}ac = 1 \\ b^{-1}ha^{-1}g = 1 \\ b^{-1}ga^{-1}h = 1 \\ g^{-1}dgc = 1 \\ h^{-1}dhc = 1 \end{array}\right\} \text{from } Q_{v_{+000}} \qquad \left.\begin{array}{c} j^{-1}b^{-1}ja = 1 \\ i^{-1}b^{-1}ia = 1 \\ j^{-1}fi^{-1}e = 1 \\ j^{-1}ei^{-1}f = 1 \\ e^{-1}bea = 1 \\ f^{-1}bfa = 1 \end{array}\right\} \text{from } Q_{v_{0+00}}$$

$$\left.\begin{array}{c} d^{-1}l^{-1}dk = 1 \\ c^{-1}l^{-1}ck = 1 \\ d^{-1}f^{-1}c^{-1}e = 1 \\ d^{-1}ec^{-1}f^{-1} = 1 \\ e^{-1}lek = 1 \\ flf^{-1}k = 1 \end{array}\right\} \text{from } Q_{v_{00+0}} \qquad \left.\begin{array}{c} l^{-1}j^{-1}li = 1 \\ k^{-1}j^{-1}ki = 1 \\ l^{-1}h^{-1}k^{-1}g = 1 \\ l^{-1}gk^{-1}h^{-1} = 1 \\ g^{-1}jgi = 1 \\ hjh^{-1}i = 1 \end{array}\right\} \text{from } Q_{v_{000+}}$$

Recall that every edge of Q has an ideal vertex that is among v_{*000}, v_{0*00}, v_{00*0} and v_{000*}, hence every cycle of edges of Q is represented by a cycle of edges of one of $Q_{v_{+000}}$, $Q_{v_{0+00}}$, $Q_{v_{00+0}}$ and $Q_{v_{000+}}$. Cycles of edges of $Q_{v_{++++}}$ will not add any new relations, so they can be omitted.

Now we list all the generating translations for each parabolic subgroup.

subgroup	generating translations	normal translations
G_{v+000}	$t_1 = c$ $t_2 = a^{-1}b$ $t_3 = h^{-1}g$	nt_1 $mt_2 + nt_3$
G_{v0+00}	$t_1 = a$ $t_2 = i^{-1}j$ $t_3 = f^{-1}e$	nt_1 $mt_2 + nt_3$
G_{v00+0}	$t_1 = k$ $t_2 = c^{-1}d$ $t_3 = fe$	nt_1 $mt_2 + nt_3$
G_{v000+}	$t_1 = i$ $t_2 = k^{-1}l$ $t_3 = hg$	nt_1 $mt_2 + nt_3$
G_{v++++}	$t_1 = e^{-1}g$ $t_2 = a^{-1}k^{-1}ak$ $t_3 = i^{-1}c^{-1}ic$	nt_1 $mt_2 + nt_3$

We now choose translations in each boundary component that we want to represent the fiber of the S^1-bundle. We simply go for the easiest expression in order to simplify computation. Choose c^n, a, k, i and $e^{-1}g$ and compute $\pi_1 M / \langle\langle c^n, a, k, i, e^{-1}g \rangle\rangle$. Substituting $a, k, i = 1$ into the four groups of relators we get the following four respective groups:

$$\left.\begin{aligned} b^{-1}d^{-1}bc &= 1 \\ d &= c \\ b^{-1}hg &= 1 \\ b^{-1}gh &= 1 \\ g^{-1}dgc &= 1 \\ h^{-1}dhc &= 1 \end{aligned}\right\} \qquad \left.\begin{aligned} b &= 1 \\ j^{-1}fe &= 1 \\ j^{-1}ef &= 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} l &= 1 \\ d^{-1}f^{-1}c^{-1}e &= 1 \\ d^{-1}ec^{-1}f^{-1} &= 1 \end{aligned}\right\} \qquad \left.\begin{aligned} j &= 1 \\ l^{-1}h^{-1}g &= 1 \\ l^{-1}gh^{-1} &= 1 \end{aligned}\right\}$$

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Taking into account now that $l, b, j, e^{-1}g = 1$ and $c = d$, we get the relations:

$$h = g^{-1}, \ h = g, \ g = e, \ f = e^{-1}, \ g^{-1}cgc = 1, \ c^{-1}ec^{-1}e = 1$$

which immediately implies that the group is generated by c and e, and that $e^2 = 1$, $e^{-1}cec = 1$ and $c^{-1}ec^{-1}e = 1$. The last two equations are both equivalent to $e^{-1}ce = c^{-1}$ so the presentation of the group is $\langle c, e \mid c^n = 1, e^2 = 1, e^{-1}ce = c^{-1} \rangle = \langle c \mid c^n = 1 \rangle \odot \langle e \mid e^2 = 1 \rangle = \mathbf{Z}_n \odot \mathbf{Z}_2$, where $\mathbf{Z}_2$ acts on $\mathbf{Z}_n$ by inversion in the semidirect product $\mathbf{Z}_n \odot \mathbf{Z}_2$. Letting $H = \langle\langle c^n, a, k, i, e^{-1}g \rangle\rangle$, we note that c^n is primitive in $H \cap G_{v_{+000}}$, a is primitive in $H \cap G_{v_{0+00}}$ and so on. Proposition 2.4 now tells us that M has a $2n$-fold cover $\tilde{M}$ that is a complement in a simply-connected manifold N.

When $n = 1$ the two-fold cover of M is actually the orientable double cover of M, since via the homomorphism $\pi_1 M \to \pi_1 M / \langle\langle c, a, k, i, g^{-1}e \rangle\rangle \cong \mathbf{Z}_2$ the orientation reversing generators e, f, g, h of G are sent to the nonidentity element of $\mathbf{Z}_2$ while the orientation preserving generators a, b, c, d, i, j, k, l are sent to the identity element. Furthermore, $\chi(N) = \chi(\text{double cover of } M) = 2$. Since $H_4 N = H_0 N = \mathbf{Z}$ and $H_1 N = H_3 N = 0$ this means $H_2 N = 0$. Now due to Freedman's (see [2, 3]) classification of simply-connected 4-manifolds, the only simply-connected 4 manifold N with $H_2 N = 0$ is the 4-sphere.

When $n > 2$, let $\pi_1 M \xrightarrow{q} \mathbf{Z}_n \odot \mathbf{Z}_2$ be the quotient homomorphism. The orientation-reversing generators of $\pi_1 M$ are sent to the generator of $\mathbf{Z}_2$, so the subgroup of orientation-preserving transformations of $\pi_1 M$ is $q^{-1}(\mathbf{Z}_n)$. Since $\ker q \subset q^{-1}(\mathbf{Z}_n)$, the $2n$-fold cover corresponding to $\ker q$ covers the orientable double cover which corresponds to $q^{-1}(\mathbf{Z}_n)$. The group of deck transformations is $q^{-1}(\mathbf{Z}_n) / \ker q \cong \mathbf{Z}_n$ so $\tilde{M}$ is a cyclic cover of the double orientable cover.

We use Proposition 2.3 to find the number of components of $\tilde{M}$. The subgroup $G_{v_{+000}}$ is generated by translations c and $a^{-1}b$ and an orientation-reversing glide reflection. Those transformations are respectively sent to $c, 1, e \in \mathbf{Z}_n \odot \mathbf{Z}_2$, hence the index of $q(G_{v_{+000}})$ in $\mathbf{Z}_n \odot \mathbf{Z}_2$ is 1. Similarly, we see that $q(G_{v_{0+00}}) = q(G_{v_{00+0}}) = q(G_{v_{000+}}) = q(G_{v_{++++}}) = \langle e \rangle$ so the index of each of those subgroups of $\mathbf{Z}_n \odot \mathbf{Z}_2$ is n. Therefore, $\tilde{M}$ has $4n + 1$ boundary components.

Thus we have proved

Theorem 4.3 *Let M be the Ratcliffe-Tschantz manifold no. 1011. Then M is a complement of 5 tori in a 4-manifold N with $\pi_1 N = \mathbf{Z}_2$. The orientable double cover of M is the complement of 5 tori in S^4. Furthermore, the orientable double cover has cyclic covers of any index n that are complements of $4n + 1$ tori in a simply-connected closed manifold N with $\chi(N) = 2n$. $\square$*

5   A necessary condition

With a wealth of Ratcliffe-Tschantz examples available, one would like to know whether many have a double cover that is a complement in S^4 or if this is a rare occurence. We give a criterion that rules out an example from having such a double cover that (unfortunately) applies to most Ratcliffe-Tschantz examples.

Proposition 5.1 *Let M be a hyperbolic manifold that has a double cover which is a complement in a simply-connected closed manifold. Then there exist translations $t_1, \ldots, t_m$ so that $\pi_1 M / \langle\langle t_1, \ldots, t_m \rangle\rangle = \mathbf{Z}_2$.*

Proof. Let $H \subset \pi_1 M$ be the subgroup of $\pi_1 M$ corresponding to the double cover in question. Since the double cover is a complement there exist translations normal in H so that $H / \langle\langle t_1, \ldots, t_m \rangle\rangle_H = 1$ (see proposition 2.2). However, H is normal in $\pi_1 M$ (it has index 2) so $H = \langle\langle t_1 \ldots t_m \rangle\rangle_H \subset \langle\langle t_1 \ldots t_m \rangle\rangle \subset H$ which immediately gives $\langle\langle t_1 \ldots t_m \rangle\rangle = H$ and the conclusion of the proposition.

Note that the double cover of M may have more boundary components than M, hence, in general, m is greater of equal to the number of boundary components of M. □

Now we consider a Ratcliffe-Tschantz manifold M and use the notation of §4. Let $G = \pi_1 M$ and let $\mathbf{Z}_2^6$ be the abelian group whose summands are generated by the 6 order-2 elements r_{**00}, r_{*0*0}, r_{0**0}, r_{*00*}, r_{0*0*}, r_{00**}. Note that only here "$*$" is a symbol and not a slot allowing other symbols. As we have mentioned in §4, $G \subset \Gamma_2^4$, where Γ_2^4 is generated by reflections in the sides of the polyhedron P whose angles are all $\pi/2$. There is a homomorphism $\phi : \Gamma_2^4 \to \mathbf{Z}_2^6$ given by $k_{****} \mapsto 0$, $r_{++00} \mapsto r_{**00}$, $r_{+0+0} \mapsto r_{*0*0}$, etc. We restrict ϕ to G and note that $\phi : G \to \mathbf{Z}_2^6$ is surjective: the side-pairing for side S_{++00} maps to r_{**00}, the side-pairing for S_{+0+0} maps to r_{*0*0} and so on.

Referring to the 24-sided polyhedron Q from §4, let v be any of the ideal vertices that has only one positive nonzero position. If C is a horosphere centered at v, then $C \cap Q$ is a cube and translates of Q that intersect give a tiling of $C \cong \mathbf{R}^3$ by cubes. Note that the parallel sides of the cube $Q \cap C$ are sides of the polyhedron Q that have nonzero entries in the same position, so their side-pairings have the same k in the kr decomposition of proposition 3.1.

In what follows, like in proposition 4.1, we drop "$\cap C$". If we start a path in the cube Q that exits through side S, goes through the adjacent cube xQ and then exits through the parallel side opposite the entry side, finishing in translate gQ, the transformation g is equal to $q_2 q_1 k_1^{-1} k_2^{-1}$ according to proposition 4.1. However, since side-pairings for parallel sides have the same K-part, $k_1 = k_2$, so we get $g = q_2 q_1$, where q_1 and q_2 are reflections in planes that we passed through. It then follows that g is a translation. Also, $\phi(g) = 0$

since both q_1 and q_2 map to the same generator of $\mathbf{Z}_2^6$ (every reflection q_i is a conjugate of $r_{++00}, r_{0+0+}, \ldots, r_{00++}$ by elements of K).

Therefore, translations by two lengths of the side of Q in each of the three directions parallel to the edges of Q are contained in the translation subgroup of G_v. Let T_v denote the group of translations they generate. Since the fundamental polyhedron for G_v is two side-by-side cubes and the fundamental polyhedron for T_v is eight cubes, we get $[G_v : T_v] = 4$. Note also that $\phi(T_v) = 0$.

Similarly, if v is the ideal vertex v_{++++} and C a horosphere centered at v, then $Q \cap C$ is again a cube whose parallel sides have nonzero entries in complementary positions. A path that goes perpendicularly through a side of the cube and continues in a straight line until it has crossed 4 sides finishes in the translate gQ, where (proposition 4.1 again) $g = q_4 q_3 q_2 q_1 k_1^{-1} k_2^{-1} k_3^{-1} k_4^{-1}$. However, since the K-part of a side-pairing is the same for every side that has nonzero entries in fixed positions, we have $k_1 = k_3$ and $k_2 = k_4$, so $g = q_4 q_3 q_2 q_1$ owing to commutativity of $K = \mathbf{Z}_2^4$ and the fact that every element of K has order 2. This makes g a translation by four lengths of a side of Q as it is the composite of four reflections in parallel planes. Also, $\phi(g) = 0$ because $\Phi(k_i) = 0$, $\Phi(q_1) = \Phi(q_3)$ and $\Phi(q_2) = \Phi(q_4)$. Again let T_v be the subgroup of translations generated by translations by four cube widths in the three directions parallel to the edges of the cube Q. The fundamental polyhedron of T_v comprises 64 cubes, while the fundamental polyhedron of G_v contains either 16 cubes, if $||[v]|| = 16$, or 8 cubes if $||[v]|| = 8$. Thus, $[G_v : T_v] = 4$ or 8, respectively. Again note that $\phi(T_v) = 0$.

Theorem 5.2 *Let M be a Ratcliffe-Tschantz manifold and let T denote the set of all parabolic translations in $G = \pi_1 M$. Viewing $\mathbf{Z}_2^6$ as a $\mathbf{Z}_2$-vector space, if the dimension of the subspace generated by $\phi(T)$ is less than 5, then M does not have a double cover that is a complement in S^4.*

In particular, let M be a 5-cusped Ratcliffe-Tschantz manifold, let n_X denote the number of boundary components (corresponding to cusps) of M of type X, $X = A, B, F, G, H, I, J$ and let $\alpha(M) = 2n_A + 2n_B + n_G + n_H$. If $\alpha(M) < 5$ then M does not have a double cover that is a complement in S^4.

For a 6-cusped Ratcliffe-Tschantz manifold M, let n'_X denote the number of boundary components of type X that correspond to the two cycles of the ideal vertices $v_{\pm,\pm,\pm,\pm}$ (the last two letters in Ratcliffe and Tschantz's table of 6-cusped manifolds are the ones that identify boundary component types corresponding to those vertices) and let n_X denote the number of boundary components of type X that correspond to the 4 cycles of the remaining ideal vertices. Set $\alpha'(M) = 2n_A + 2n_B + n_G + n_H + 2n'_A + 2n'_B + n'_F + n'_G + n'_H + n'_I + n'_J$. If $\alpha'(M) < 5$ then M does not have a double cover that is a complement in S^4.

Proof. We prove the contrapositive. Suppose M has a double cover that is a complement in a simply-connected manifold and let $G = \pi_1 M$. According to Proposition 5.1 there exist translations $t_1, \ldots, t_m$ so that $\pi_1 M / \langle\langle t_1, \ldots, t_m \rangle\rangle = \mathbf{Z}_2$. The homorphism $\phi : G \to \mathbf{Z}_2^6$ defined above induces a surjective homomorphism $G / \langle\langle t_1 \ldots t_m \rangle\rangle \to \mathbf{Z}_2^6 / \langle \phi(t_1) \ldots \phi(t_m) \rangle \to \mathbf{Z}_2^6 / \langle \phi(T) \rangle$ so $\dim \mathbf{Z}_2^6 / \langle \phi(T) \rangle \leq 1$ which gives $\dim \langle \phi(T) \rangle \geq 5$.

For the remaining two parts, we notice first that if the type of the boundary component is F, I or J and G_v is its corresponding parabolic subgroup, then the index of the translation subgroup of G_v is 4. Hence, if the index of T_v in G_v is 4, then the translation subgroup of G_v is equal to T_v.

Now suppose that M is a 5-cusped Ratcliffe-Tschantz manifold that has a double cover $\tilde{M}$ that is a complement in S^4. By Proposition 5.1 there exist $t_1, \ldots, t_m$ so that $G / \langle\langle t_1, \ldots, t_m \rangle\rangle = \mathbf{Z}_2$. Any translation t_i that is in a parabolic subgroup corresponding to a boundary component of type F, I or J maps to 0 via ϕ by the above discussion, since the index of T_v in G_v is always 4 in the 5-cusped case. Boundary components of type G and H are unorientable, so their lifts in $\tilde{M}$ are orientable and thus have only one component; this means that only one translation in the collection $t_1, \ldots, t_m$ is in the parabolic subgroup corresponding to a boundary component of type G or H. Boundary components of type A or B could have two components in their lifts and each component could have a different choice for fiber of the S^1-fiber bundle, therefore, at most two translations in the collection $t_1, \ldots, t_m$ are in the parabolic subgroup corresponding to each boundary component of type A or B. Therefore, the number of nonzero elements of $\phi(t_1), \ldots, \phi(t_m)$ is at most $2n_A + 2n_B + n_G + n_H$. Again, surjectivity of the induced map $G / \langle\langle t_1 \ldots t_m \rangle\rangle \to \mathbf{Z}_2^6 / \langle \phi(t_1) \ldots \phi(t_m) \rangle$ gives us that $2n_A + 2n_B + n_G + n_H \geq \dim \langle \phi(t_1), \ldots, \phi(t_m) \rangle \geq 5$ proving the contrapositive of the second claim.

If M is a 6-cusped Ratcliffe-Tschantz manifold, we reason in the same fashion. For a vertex in a cycle of $v_{\pm,\pm,\pm\pm}$ the index of T_v in G_v is 8, hence T_v has index 2 in the translation subgroup of G_v. Therefore, even if the boundary component type is F, I or J, there could be a nonzero element corresponding to it in the collection $\phi(t_1), \ldots, \phi(t_m)$ and, as above, there could be two if the boundary component is orientable. Thus, the number of nonzero elements in $\phi(t_1), \ldots, \phi(t_m)$ is at most $2n_A + 2n_B + n_G + n_H + 2n'_A + 2n'_B + n'_F + n'_G + n'_H + n'_I + n'_J$ and the proof is finished as in the 5-cusped case. (Note that, since the boundary component types corresponding to the two cycles of $v_{\pm,\pm,\pm,\pm}$ are always equal, only one of the numbers n'_A, n'_B, n'_F, n'_G, n'_H, n'_I, n'_J is nonzero.) $\square$

Example 5.3 Proposition 5.2 rules out easily many Ratcliffe-Tschantz examples to get that at most about 346 (of 1171) can have double cover complements in S^4.

First of all, any nonorientable example that contains a boundary component of type F cannot have a double cover complement in S^4, since the boundary component will lift to two copies of boundary components of type F which are not S^1-bundles. A hand count that rules out all nonorientable examples with a boundary component of type F and rules out all manifolds that satisfy the second or the third part of Proposition 5.2 leaves 18 (of 22) 5-cusped orientable manifolds, about 287 (of 1068) 5-cusped nonorientable and 41 (of 81) 6-cusped nonorientable manifolds eligible to be complements inside S^4. For example, if M_k denotes the Ratcliffe-Tschantz manifold(s) ranging through k's, we have $\alpha(M_{19-22}) = 4$, $\alpha(M_{811-865}) \leq 2$, $\alpha'(M_{1101-1104}) = 4$.

Example 5.4 There are many Ratcliffe-Tschantz manifolds that can be ruled out using the first, stronger, part of Proposition 5.2 by computing $\dim \phi(T)$. For example, while $\alpha(M_{51}) = 7$, it is not hard to compute that $\dim \phi(T) = 4$. Similarly, $\alpha(M_{545}) = 6$, and $\alpha'(M_{1162}) = 10$, yet $\dim \phi(T)$ is 3 and 4, respectively.

We have used the first part Proposition 5.2 to investigate many potential candidates for complements in S^4 and have been able to rule most of them out. Some examples remain that cannot be excluded in this way and they are possible candidates for complements. However, we feel that only a few Ratcliffe-Tschantz examples will turn out to have double cover complements in S^4 and have confirmed only the one from Theorem 4.3.

6 Remark

In the proof of Theorem 4.3 we used Freedman's theory which applies to the topological category, that is, we can only claim that the double cover of M from Theorem 4.3 embeds in a space N that is homeomorphic to S^4. At this time it is still unknown whether a manifold that is homeomorphic to S^4 is also diffeomorphic to the S^4 with the standard differentiable structure (the "differentiable Poincaré conjecture in dimension 4"). If we had a handle decomposition for N, diffeomorphism with S^4 can be verified by using Kirby calculus to transform the Kirby diagram of N to the Kirby diagram of the standard S^4. It is not too hard to produce a Kirby diagram of the manifold if it is given by a side-pairing of a polyhedron (spherical, euclidean or hyperbolic). Then one can obtain a Kirby diagram for N, which turns out to be rather complicated (it has 23 1-handles, 53 2-handles, 34 3-handles and 5 4-handles). A computation using Kirby moves shows that N really is diffeomorphic to S^4 — we plan to write the details in a future paper.

Hence, we have produced a complicated Kirby diagram of S^4 obtained in a seemingly nontrivial, top-down fashion (as opposed to a bottom-up method where one would start with

a simple Kirby diagram of the standard S^4 and make it more complicated via Kirby moves). While of course this does not give a counterexample to the differentiable Poincaré conjecture in dimension 4, it seems to indicate that a possible counterexample given by a Kirby diagram may indeed be very complicated.

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