Mathematical Reasoning — Exam 1
MAT 312, Spring 2023 — D. Ivanšić

Name: ____________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) The sun rises in the east and sets in the south.

2. (2pts) If an integer is divisible by two, then it is divisible by four.

3. (3pts) (universal set=$\mathbf{R}$) $2x - 7 > 0$ or $6 \div 3 = 2$

4. (4pts) For every $x \in \mathbf{R}$, $x^2 + 4x + 7 > 0$

5. (3pts) (universal set=$\mathbf{Q}$) $x^2 + 3x = 10$

Negate the following statements.

6. (3pts) You felt like dropping in and you did not expect me to be free.

7. (3pts) If Putin says it, then it is true.

8. (8pts) Use a truth table to prove that $(P \implies Q) \vee Q \equiv \neg P \vee Q$. (Use however many columns you need.)

P	Q								
T	T								
T	F								
F	T								
F	F								

9. (12pts) Use previously proven logical equivalences to prove the equivalence $(P \implies Q) \vee (Q \implies P) \equiv T$, where T is "true." Do not use a truth table.

10. (4pts) Write the converse and contrapositive of the statement: if I am not at home, then the door is locked.

Converse:

Contrapositive:

11. (8pts) Suppose the following statements are true:
If the coin toss is tails, I will jump into the lake.
The coin toss is tails or pigs can fly.

Determine truth value of the following statement and justify: I will jump into the lake.

12. (4pts) Use set builder notation to write the set of all rational numbers whose cube is less than the number itself.

13. (10pts) An integer $p \neq 1$ is *prime* if for all integers a, b, if $ab = p$, then $a = p$ or $b = p$.
a) Write the definition using symbols for quantifiers.
b) Negate the definition using symbols for quantifiers.
c) Finish the sentence (in English): "An integer $p \neq 1$ is not prime if ..."

14. (7pts) Prove: if m is an odd integer, and n is an even integer, then $3m - n$ is an odd integer.

15. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\forall y \in \mathbf{R})(x^2 + y^2 > 1)$$

a) If $x = 0$, is the statement true?
b) If $x = \sqrt{2}$, is the statement true?
c) Find the truth set (the x's) of the above statement.

16. (15pts) We will call an integer n type-0, type-1, type-2 or type-3 integer if it can be written in the form $n = 4k$, $n = 4k + 1$, $n = 4k + 2$ or $n = 4k + 3$, respectively, for some integer k. Prove that if m is a type-1 integer and n is a type-3 integer, then $3mn - n^2$ is a type-0 integer. Start with a know-show table if you find it helpful.

Bonus. (10pts) A function $f : \mathbf{R} \to \mathbf{R}$ is *continuous at* c if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x \in \mathbf{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.
a) Write the definition using symbols for quantifiers.
b) Negate the definition using symbols for quantifiers.
c) Finish the sentence (in English): "A function $f : \mathbf{R} \to \mathbf{R}$ is not continuous at c if ..."

Mathematical Reasoning — Exam 2
MAT 312, Spring 2023 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (14pts) Prove: the sum of squares of any two consecutive integers always gives remainder 1 when divided by 4.

2. (16pts) Prove using induction: for every natural number $n \geq 9$, $2^n > 5n^2$. (You do not have to evaluate these expressions for $n < 9$.)

3. (14pts) We know that $\sqrt{2}$ is irrational. Show that for any rational numbers p and q, the number $\dfrac{p}{q+\sqrt{2}}$ is irrational.

4. (18pts) Consider the statement: for all $n \in \mathbf{Z}$, n is divisible by 5 if and only if $n^2 + 3n$ is divisible by 5.
a) Write the statement as a conjunction of two conditional statements.
b) Determine whether each of the conditional statements is true, and write a proof, if so.
c) Is the original statement true?

5. (14pts) We have shown a similar statement on homework: for every integer n, if $3 \mid n^3$, then $3 \mid n$. Use this proposition to show that $\sqrt[3]{3}$ is irrational.

6. (10pts) Let $\min\{a, b\}$ denote the smaller of real numbers a and b, or either one if they are equal. Prove the equation below. (Considering cases is easiest.)

$$\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$$

7. (14pts) Prove that for every real number $x \neq 0$, $x^2 \geq 2 - \dfrac{1}{x^2}$.

Bonus. (10pts) Let p and q be rational numbers and x an irrational number. Show that the number $\dfrac{p-x}{q+x}$ is irrational.

Mathematical Reasoning — Exam 3
MAT 312, Spring 2023 — D. Ivanšić

Name: ____________________

Show all your work!

1. (14pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the four sets, two are equal. Use set algebra to show they are equal.

$(A - B) \cup C$ $\qquad$ $(A \cup C) - (B - C)$ $\qquad$ $A \cup (B - C)$ $\qquad$ $(A \cap B) \cup C^c$

2. (12pts) Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Consider the sets $A = \{0, 1, 3, 5, 8\}$, $B = \{3, 6, 9\}$, $C = \{6, 7, 8, 9\}$ and write the following subsets using the roster method.

$A \cap (B \cup C) =$

$A \cup (B \cap C) =$

$B^c =$

$C - A =$

$A^c \cup B^c =$

$(C - B^c) \cap A =$

3. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 2 \pmod 4\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv 6 \pmod{12}\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

4. (12pts) Let $\mathbf{Z}_4 = \{0, 1, 2, 3\}$, and let $f : \mathbf{Z}_4 \to \mathbf{Z}_4$, $f(x) = x^2 \pmod 4$.
a) Write the table of values of f.
b) Determine the set of preimages of 2 and 1.
c) Determine the range of the function.
d) Let $g(x) : \mathbf{Z} \to \mathbf{Z}_4$ be given by $g(x) = x^2 \pmod 4$. Is $f = g$?

5. (16pts) Let $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$, $f(m, n) = mn$.
a) Is f injective? Justify.
b) Is f surjective? Justify.
c) Determine the set of preimages of 6 by listing all its elements.

6. (12pts) Let $f(x) = x + \dfrac{1}{x}$ and assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) What is the range of this function? Justify your answer.

7. (10pts) Draw arrow diagrams between two copies of $\mathbf{Z}$ below that illustrate a function $f : \mathbf{Z} \to \mathbf{Z}$ that:

a) has range equal to $\{-1, 0, 1\}^c$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

b) is an injection that is not a surjection

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

8. (12pts) Let A, B be subsets of a universal set U. Prove that $A \cap B = \emptyset$ if and only if $A - B = A$. (Intuitively it is "clear," but you need to provide a proper proof.)

Bonus. (10pts) Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^3 + x + 8$. Prove f is injective algebraically. *(Hint: you will find the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ useful.)*

Mathematical Reasoning — Final Exam
MAT 312, Spring 2023 — D. Ivanšić

Name: ______________________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) If $0 + 1 = 1$, then $2 \cdot 3^2 > 0$ and $1 + 3 = 5$.

2. (4pts) (universal set=$\mathbf{R}$) $x^3 - x^2 > 0$.

3. (2pts) There exists an $x \in \mathbf{R}$ such that $-x^2 + 1 > 0$

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If the sum of two numbers is more than 3, then at least one is greater than 1.

5. (4pts) There exists a function $f : X \to Y$, such that the range of f equals Y and f is injective.

6. (12pts) Use previously proven logical equivalences to prove the equivalence
$P \wedge (Q \implies R) \equiv \neg(P \implies Q) \vee (P \wedge R)$. Do not use a truth table.

7. (12pts) Consider the statement: if x is irrational, then $x + \sqrt{2}$ or $x - \sqrt{2}$ is irrational.

a) State the converse and prove or disprove it:

b) State the contrapositive and prove or disprove it:

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(x^2 + y^2 = 10)$$

a) If $x = -2$, is the statement true?
b) If $x = 4$, is the statement true?
c) Find the truth set (the x's) of the above statement.

9. (14pts) Prove using induction: for every natural number n,

$$7 + 12 + \cdots + (5n + 2) = \frac{n(5n + 9)}{2}$$

10. (12pts) Prove that for every real number x, if $x > 2$, then $x - 4 + \dfrac{1}{x - 2} \geq 0$.

11. (16pts) Consider the statement: for all $a \in \mathbf{Z}$, $5 \mid a$ if and only if $5 \mid a^2$.
a) Write the statement as a conjunction of two conditional statements.
b) Prove each of the conditional statements.

12. (14pts) Use the statement in problem 11 to show $\sqrt{30}$ is irrational.

13. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 2 \pmod 4\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv 6 \pmod{12}\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

14. (14pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z}_5 \to \mathbf{Z}_5$, $f(x) = 2x^2 + 1 \pmod 5$.
a) Write the table of values of f.
b) Determine the set of preimages of 4 and 1.
c) Is f injective? Justify.
d) Is f surjective? Justify.

15. (5pts) Draw an arrow diagram between the provided two copies of $\mathbf{Z}$ that illustrates a function $f : \mathbf{Z} \to \mathbf{Z}$ that is is a surjection and the set of preimages of every element in the codomain has two elements (pattern needs to be obvious).

$$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$$

$$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$$

16. (12pts) Let A, B be subsets of a universal set U. Prove that $A \cap B = \emptyset$ if and only if $A - B = A$. (Intuitively it is "clear," but you need to provide a proper proof.)

Bonus. (10pts) Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^3 + x + 8$. Prove f is injective algebraically. *(Hint: you will find the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ useful.)*