Mathematical Reasoning — Final Exam
MAT 312, Fall 2020 — D. Ivanšić

Name: ______________________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) If $x \in \mathbf{R}$ and $x^2 < -3$, then $x^2 < -1$.

2. (4pts) (universal set=$\mathbf{R}$) $x^2 - 9x + 14 < 0$.

3. (2pts) For every $x \in \mathbf{R}$, $3x + 7 \geq 0$

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If the product of two real numbers is positive, then both numbers are positive.

5. (4pts) There exists an $x \in A$, such that x is even and $x \equiv 1 \pmod 3$.

6. (10pts) Use previously proven logical equivalences to prove the equivalence $(P \implies Q) \vee (R \implies Q) \equiv (P \wedge R) \implies Q$. Do not use a truth table.

7. (12pts) Consider the statement: if x is irrational, then $4 + \frac{1}{x}$ is irrational.

a) State the converse and prove or disprove it:

b) State the contrapositive and prove or disprove it:

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\forall y \in \mathbf{R})(y^2 > x)$$

a) If $x = -3$, is the statement true?
b) If $x = 4$, is the statement true?
c) Find the truth set (the x's) of the above statement.

9. (14pts) Prove using induction: for every natural number n,

$$\frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \cdots + \frac{n}{2} = \frac{n(n+1)}{4}$$

10. (12pts) Prove that for all real numbers x and y, $x^2 + 9y^2 \geq 6xy$.

11. (16pts) Consider the statement: for all $a, b \in \mathbf{Z}$, $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$.
a) Write the statement as a conjunction of two conditional statements.
b) Prove each of the conditional statements.

12. (14pts) Use the statement in problem 11 to show $\sqrt{15}$ is irrational. (Note that a square is a product of two numbers.)

13. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the three sets, two are equal. Use set algebra to show they are equal.

$A \cap B \cap C$ $\qquad\qquad$ $(C - (C - A)) \cap B$ $\qquad\qquad$ $(A - B) \cap C$

14. (16pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z} \to \mathbf{Z}_5$, $f(x) = x^2 + 2x \pmod 5$.
a) Write the table of values of f for $x = 0, 1, 2, 3, 4$.
b) Is f injective? Justify.
c) Is f surjective? Justify.
d) Note the domain is $\mathbf{Z}$ (not $\mathbf{Z}_5$). Determine the set of preimages of 4. List at least three elements of this set and describe the set. The table from a) tells you everything you need to know.

15. (5pts) Draw an arrow diagram between the provided two copies of $\mathbf{Z}$ that illustrates a function $f : \mathbf{Z} \to \mathbf{Z}$ that is is not a surjection and the set of preimages of every element in the codomain has either two or zero elements (pattern needs to be obvious).

$$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$$

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16. (12pts) Let A, B be subsets of a universal set U. Prove that $A = B$ if and only if $A \cup B = A \cap B$.

Bonus. (10pts) Let $0 \leq a_0, a_1, \ldots, a_n \leq 9$ be integers.

a) Use $\pmod 3$ calculus to show that

$$a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 \pmod 3$$

b) Use a) to show that a natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.