Mathematical Reasoning — Exam 2
MAT 312, Fall 2020 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (14pts) Prove using induction: for every natural number n,
$1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

2. (14pts) Prove: for any integer n, the number $\dfrac{2}{3}n^3 + \dfrac{4}{3}n$ is an integer.

3. (16pts) We know that $\sqrt{2}$ is irrational. Are the following statements true? Justify with a counterexample or a proof.

a) There exists a real number x such that $\dfrac{1}{x+\sqrt{2}}$ is rational.

b) If x is rational, then $\dfrac{1}{x+\sqrt{2}}$ is irrational.

4. (18pts) Consider the statement: for all $a, b \in \mathbf{Z}$, $5 \mid a^2 + 2b^2$ if and only if $5 \mid a$ and $5 \mid b$.

a) Write the statement as a conjunction of two conditional statements.

b) Determine whether each of the conditional statements is true, and write a proof, if so.

c) Is the original statement true?

5. (14pts) We have shown a similar statement on homework: for every integer n, if $7 \mid n^2$, then $7 \mid n$. Use this proposition to show that $\sqrt{7}$ is irrational.

6. (10pts) Sketch all points (x, y) in the plane that satisfy $|y - x| < 3$. (Hint: what inequalities without absolute value is the inequality $|u| < a$ equivalent to?)

7. (14pts) Prove both statements for all real numbers x (one is easy):
a) if $x < -1$, then $x + \dfrac{1}{x+1} < 0$; b) if $x > -1$, then $x + \dfrac{1}{x+1} \geq 1$.

Bonus. (10pts) Let $0 \leq a_0, a_1, \ldots, a_n \leq 9$ be integers.
a) Use (mod 3) calculus to show that

$$a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 \pmod 3$$

b) Use a) to show that a natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.