Mathematical Reasoning — Final Exam MAT 312, Spring 2023 — D. Ivanšić

Name: Saul Ocean

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) If $\underbrace{0+1=1}_{\text{true}}$, then $\underbrace{2\cdot 3^2>0 \text{ and } 1+3=5}_{\text{true and false = false}}$.

$T \Rightarrow F$ is <u>false</u>

2. (4pts) (universal set=**R**) $x^3-x^2>0$.

$x^3-x^2>0$ Must have $x-1>0$ $x>1$ and $x\neq 0$

$\underbrace{x^2}_{\geq 0}(x-1)>0$ and $x\neq 0$

Truth set $(1,\infty)$

3. (2pts) There exists an $x\in\mathbf{R}$ such that $-x^2+1>0$

True. For example $x=\pm\frac{1}{2}, 0$ $\quad -\left(\frac{1}{2}\right)^2+1=\frac{3}{4}>0$

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If the sum of two numbers is more than 3, then at least one is greater than 1.

Sum of two numbers is more than 3 and both are less than equal to 1

$\neg$(at least one P) $\equiv$ all are not P

$\neg$(If sum $>3 \Rightarrow$ at least one is >1) $\equiv$ (sum >3) $\wedge$ (both are ≤ 1)

5. (4pts) There exists a function $f: X\to Y$, such that the range of f equals Y and f is injective.

For every function $f: X\to Y$, the range of f does not equal Y or f is not injective

$\neg(\exists f: X\to Y)(\text{range}=Y \wedge f \text{ injective}) \equiv (\forall f: X\to Y)(\text{range}\neq Y \vee f \text{ not injective})$

6. (12pts) Use previously proven logical equivalences to prove the equivalence $P\wedge(Q\Longrightarrow R)\equiv\neg(P\Longrightarrow Q)\vee(P\wedge R)$. Do not use a truth table.

$$\neg(P\Rightarrow Q)\vee(P\wedge R)\equiv(P\wedge\neg Q)\vee(P\wedge R)$$
$$\equiv P\wedge(\neg Q\vee R)$$
$$\equiv P\wedge(Q\Rightarrow R)$$

7. (12pts) Consider the statement: if x is irrational, then $x + \sqrt{2}$ or $x - \sqrt{2}$ is irrational.

a) State the converse and prove or disprove it:

If $x+\sqrt{2}$ and $x-\sqrt{2}$ are irrational, then x is irrational

False: $x=0$ is a counterexample.
$0+\sqrt{2}$ and $0-\sqrt{2}$ are irrational, but 0 is rational

b) State the contrapositive and prove or disprove it:

If $x+\sqrt{2}$ and $x-\sqrt{2}$ are rational, then x is rational.

True, proof: let $x+\sqrt{2}$ and $x-\sqrt{2}$ be rational. Then $x+\sqrt{2}+x-\sqrt{2}=2x$ is rational, from which it follows that $\frac{2x}{2}=x$ is rational.

8. (12pts) Let **R** be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(x^2 + y^2 = 10)$$

a) If $x = -2$, is the statement true?
b) If $x = 4$, is the statement true?
c) Find the truth set (the x's) of the above statement.

a) $x=-2$

$(\exists y \in \mathbb{R})(4+y^2=10)$

$4+y^2=10$

$y^2=6$

$y=\pm\sqrt{6}$

so statement is true

b) $x=4$

$(\exists y \in \mathbb{R})(16+y^2=10)$

$16+y^2=10$

$y^2=-6$

has no real solution, so False

c) For which x does $x^2+y^2=10$ have a solution for y?

$y^2=10-x^2$

$y=\pm\sqrt{10-x^2}$

so eq. has a solution for y if and only if $10-x^2 \geq 0$

$x^2 \leq 10$ $\quad |x| \leq \sqrt{10}$

$x \in [-\sqrt{10}, \sqrt{10}]$ = Truth set

9. (14pts) Prove using induction: for every natural number n,

$$7 + 12 + \cdots + (5n+2) = \frac{n(5n+9)}{2}$$

1) For $n=1$ $\quad 7 = \dfrac{1\cdot(5+9)}{2}$, which is true

2) Suppose statement is true for $n=k$

$$7+12+\dots+(5k+2) = \frac{k(5k+9)}{2} \quad \Big| + 5(k+1)+2$$

$$7+12+\dots+(5k+2)+(5(k+1)+2) = \frac{k(5k+9)}{2} + 5(k+1)+2$$

$$= \frac{5k^2+9k+2(5(k+1)+2)}{2}$$

$$= \frac{5k^2+5k+10k+14}{2} = \frac{5k^2+19k+14}{2}$$

$$= \frac{(k+1)(5k+14)}{2} = \frac{(k+1)(5(k+1)+9)}{2}$$

which is the statement for $n=k+1$

10. (12pts) Prove that for every real number x, if $x > 2$, then $x - 4 + \dfrac{1}{x-2} \geq 0$.

Investigation:

$$x-4+\frac{1}{x-2} \geq 0 \quad \Big| \cdot (x-2)$$

$$(x-4)(x-2)+1 \geq 0$$

$$x^2-6x+8+1 \geq 0$$

$$x^2-6x+9 \geq 0$$

$$(x-3)^2 \geq 0$$

true

Proof: Suppose $x>2$. Then $x-2>0$

$(x-3)^2 \geq 0$ is true for any x

$$x^2-6x+9 \geq 0$$

$$x^2-6x+8+1 \geq 0$$

$$(x-2)(x-4)+1 \geq 0 \quad \Big| \div (x-2) > 0$$

$$x-4+\frac{1}{x-2} \geq 0$$

11. (16pts) Consider the statement: for all $a \in \mathbf{Z}$, $5 \mid a$ if and only if $5 \mid a^2$.
a) Write the statement as a conjunction of two conditional statements.
b) Prove each of the conditional statements.

a) For all $a \in \mathbb{Z}$, if $5|a$, then $5|a^2$ and if $5|a^2$, then $5|a$.

b)

$a \equiv \square \pmod 5$	a^2	$a^2 \equiv \square \pmod 5$
0	0	0
1	1	1
2	4	4
3	9	4
4	16	1

From table, we see:

if $5|a$, i.e. $a \equiv 0 \pmod 5$, then $a^2 \equiv 0 \pmod 5$
so $5|a^2$

if $5|a^2$ i.e. $a^2 \equiv 0 \pmod 5$, which is only the top entry in the last column, then $a \equiv 0 \pmod 5$.

12. (14pts) Use the statement in problem 11 to show $\sqrt{30}$ is irrational.

Suppose $\sqrt{30}$ is rational, $\sqrt{30} = \frac{p}{q}$, where $\frac{p}{q}$ is reduced (p, q have no common factors)

Then $30 = \frac{p^2}{q^2}$, $30q^2 = p^2$, so $p^2 = 5 \cdot (6q^2)$. Therefore, $5|p^2$, so $5|p$,

and we can write $p = 5k$ for some $k \in \mathbb{Z}$.

Then $(5k)^2 = 5 \cdot 6q^2$, $25k^2 = 5 \cdot 6q^2$ so $5k^2 = 6q^2$. From here we

conclude* that $5|q^2$ so $5|q$. But then p and q have a common factor 5, a contradiction.

* actually, we need the statement $5|ab \Rightarrow 5|a$ or $5|b$, which is how (11.) should have been phrased, so this is an error on the exam.

13. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 2 \pmod 4\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv 6 \pmod{12}\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

$A = \{\dots, -10, -6, -2, 2, 6, 10, 14, 18, \dots\}$ $B = \{\dots, -18, -6, 6, 18, 30, 42, \dots\}$

a) $A \not\subseteq B$, For example $2 \equiv 2 \pmod 4$ so $2 \in A$
$2 \not\equiv 6 \pmod{12}$ so $2 \notin B$

b) $B \subseteq A$
Let $x \in B$. Then $x = 12k + 6$ for some $k \in \mathbb{Z}$, so
$x = 12k + 4 + 2 = 4(3k+1) + 2$, thus $4 \mid x - 2$
so $x \equiv 2 \pmod 6$, thus $x \in A$

14. (14pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z}_5 \to \mathbf{Z}_5$, $f(x) = 2x^2 + 1 \pmod 5$.
a) Write the table of values of f.
b) Determine the set of preimages of 4 and 1.
c) Is f injective? Justify.
d) Is f surjective? Justify.

a)

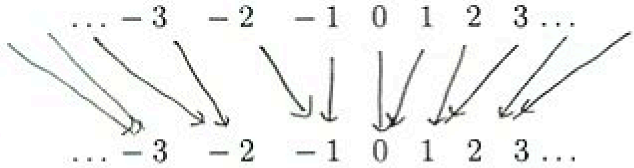
x	$2x^2+1$	$2x^2+1 \equiv \square \pmod 5$
0	1	1
1	3	3
2	9	4
3	19	4
4	33	3

c) f is not injective:
$f(1) = f(4)$ but $1 \neq 4$

d) f is not surjective,
Range $f = \{1, 3, 4\} \neq \mathbb{Z}_5$ codomain,

b) preimages of $4 = \{2, 3\}$
preimages of $1 = \{0\}$

15. (5pts) Draw an arrow diagram between the provided two copies of $\mathbf{Z}$ that illustrates a function $f : \mathbf{Z} \to \mathbf{Z}$ that is is a surjection and the set of preimages of every element in the codomain has two elements (pattern needs to be obvious).



16. (12pts) Let A, B be subsets of a universal set U. Prove that $A \cap B = \emptyset$ if and only if $A - B = A$. (Intuitively it is "clear," but you need to provide a proper proof.)

$\Rightarrow$) Suppose $A \cap B = \phi$.

$$A = A \cap U = A \cap (B \cup B^c) = \underbrace{(A \cap B)}_{=\phi} \cup (A \cap B^c) = A \cap B^c = A - B$$

$\Leftarrow$) Suppose $A - B = A$. By way of contradiction, suppose $A \cap B \neq \emptyset$ and $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $x \in A$, then $x \in A - B$, so $x \in A$ and $x \notin B$. We get that $x \in B$ and $x \notin B$, a contradiction.

Bonus. (10pts) Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^3 + x + 8$. Prove f is injective algebraically. *(Hint: you will find the formula $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ useful.)*

Let $f(x_1) = f(x_2)$

$x_1^3 + x_1 + 8 = x_2^3 + x_2 + 8$

$x_1^3 - x_2^3 + x_1 - x_2 = 0$

$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) + (x_1 - x_2) = 0$

$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 1) = 0$

Suppose $x_1^2 + x_1x_2 + x_2^2 + 1 = 0$

Case 1: $x_1x_2 \geq 0$. We get

$\underbrace{\underbrace{x_1^2 + x_1x_2 + x_2^2}_{\geq 0} + 1}_{\geq 1} = 0$ which is not possible

Case 2 $x_1x_2 < 0$. Adding x_1x_2 to both sides:

$x_1^2 + 2x_1x_2 + x_2^2 + 1 = x_1x_2$

$\underbrace{(x_1 + x_2)^2 + 1}_{\geq 1} = \underbrace{x_1x_2}_{<0}$ which is not possible

Therefore, the other factor $x_1 - x_2$ must be zero, so $x_1 = x_2$