Mathematical Reasoning — Exam 3
MAT 312, Spring 2023 — D. Ivanšić

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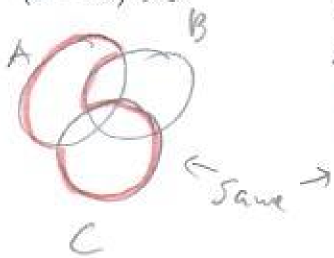
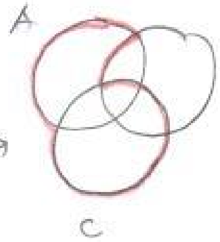
Show all your work!

1. (14pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the four sets, two are equal. Use set algebra to show they are equal.

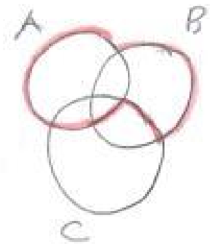
$(A-B)\cup C$



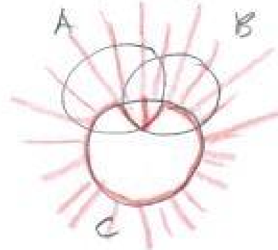
$(A\cup C)-(B-C)$



$A\cup(B-C)$



$(A\cap B)\cup C^c$



$$(A\cup C)-(B-C) = (A\cup C)\cap(B-C)^c = (A\cup C)\cap(B\cap C^c)^c$$
$$= (A\cup C)\cap(B^c\cup C) = (A\cap B^c)\cup C = (A-B)\cup C$$

2. (12pts) Let $U=\{0,1,2,3,4,5,6,7,8,9\}$. Consider the sets $A=\{0,1,3,5,8\}$, $B=\{3,6,9\}$, $C=\{6,7,8,9\}$ and write the following subsets using the roster method.

$A\cap(B\cup C) = A\cap\{3,6,9,7,8\} = \{3,8\}$

$A\cup(B\cap C) = A\cup\{6,9\} = \{0,1,3,5,6,8,9\}$

$B^c = \{0,1,2,4,5,7,8\}$

$C-A = \{6,7,9\}$

$A^c\cup B^c = (A\cap B)^c = \{3\}^c = \{0,1,2,4,5,6,7,8,9\}$

$(C-B^c)\cap A = (C\cap B)\cap A = \{6,9\}\cap A = \emptyset$

3. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 2 \pmod 4\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv 6 \pmod{12}\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

$A = \{-10, -6, -2, 2, 6, 10, 14, 18, \ldots\}$

a) $A \not\subseteq B$

$2 \in A$ but $2 \not\equiv 6 \pmod{12}$ so $2 \notin B$

$B = \{\ldots, -18, -6, 6, 18, 30, \ldots\}$

b) Let $x \in B$. Then $x = 12q + 6$ for some $q \in \mathbb{Z}$

$x = 12q + 4 + 2 = 4(3q+1) + 2$ so $4 \mid x - 2$, thus $x \equiv 2 \pmod 4$

so $x \in A$,

$B \subseteq A$.

4. (12pts) Let $\mathbf{Z}_4 = \{0, 1, 2, 3\}$, and let $f : \mathbf{Z}_4 \to \mathbf{Z}_4$, $f(x) = x^2 \pmod 4$.
a) Write the table of values of f.
b) Determine the set of preimages of 2 and 1.
c) Determine the range of the function.
d) Let $g(x) : \mathbf{Z} \to \mathbf{Z}_4$ be given by $g(x) = x^2 \pmod 4$. Is $f = g$?

a)

x	$f(x)$
0	0
1	1
2	0
3	1

b) preimages of 2: $\emptyset$

preimages of 1: $\{1, 3\}$

c) range $= \{0, 1\}$

d) $f \neq g$ because their domains are different, $\mathbb{Z}_4$ and $\mathbb{Z}$.

5. (16pts) Let $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$, $f(m,n) = mn$.
a) Is f injective? Justify.
b) Is f surjective? Justify.
c) Determine the set of preimages of 6 by listing all its elements.

a) $f(2,1) = 2 = f(1,2)$ and $(2,1) \neq (1,2)$
so f is not injective

b) Let $k \in \mathbb{Z}$. Then $f(1,k) = k$ so f is surjective

c) $\{(m,n) \mid mn = 6\} = \{(1,6), (6,1), (2,3), (3,2), (-1,-6), (-6,-1), (-2,-3), (-3,-2)\}$
set of preimages of 6.

6. (12pts) Let $f(x) = x + \dfrac{1}{x}$ and assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) What is the range of this function? Justify your answer.

a) Can't have $x = 0$

$(-\infty, 0) \cup (0, \infty)$

b) $x + \frac{1}{x} = y \quad | \cdot x$

$x^2 + 1 = yx$

$x^2 - yx + 1 = 0$

$x = \dfrac{-(-y) \pm \sqrt{(-y)^2 - 4 \cdot 1 \cdot 1}}{2}$

$= \dfrac{y \pm \sqrt{y^2 - 4}}{2}$

For a solution to be real, we must have $y^2 - 4 \geq 0$

$y^2 \geq 4$

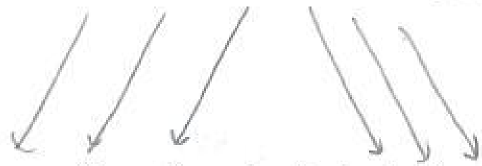
$|y| \geq 2$

Range $= (-\infty, -2) \cup (2, \infty)$

7. (10pts) Draw arrow diagrams between two copies of **Z** below that illustrate a function $f : \mathbf{Z} \to \mathbf{Z}$ that:

a) has range equal to $\{-1, 0, 1\}^c$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$



$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

b) is an injection that is not a surjection

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

8. (12pts) Let A, B be subsets of a universal set U. Prove that $A \cap B = \emptyset$ if and only if $A - B = A$. (Intuitively it is "clear," but you need to provide a proper proof.)

$\Rightarrow$) Suppose $A \cap B = \emptyset$.

Then $A = A \cap U = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c) = \emptyset \cup (A - B) = A - B$

$\Leftarrow$) Suppose $A - B = A$

Assume $A \cap B \neq \emptyset$, so there is an $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $A = A - B$, $x \in A - B$, which means $x \in A$ and $x \notin B$. We get the contradiction that $x \in B$ and $x \notin B$.

Bonus. (10pts) Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^3 + x + 8$. Prove f is injective algebraically. *(Hint: you will find the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ useful.)*

Suppose $f(x_1) = f(x_2)$

$x_1^3 + x_1 + 8 = x_2^3 + x_2 + 8$

$x_1^3 - x_2^3 + x_1 - x_2 = 0$

$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) + (x_1 - x_2) = 0$

$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 1) = 0$

Suppose $x_1^2 + x_1x_2 + x_2^2 + 1 = 0$

Case 1) $x_1x_2 \geq 0$. This is not possible, since

$x_1^2 + x_1x_2 + x_2^2 + 1 \geq 1$, so can't be 0

Case 2) $x_1x_2 < 0$. Adding x_1x_2 to both sides:

$x_1^2 + 2x_1x_2 + x_2^2 + 1 = x_1x_2$

$(x_1 + x_2)^2 + 1 = x_1x_2$ which is also not possible since left side is positive, right side is negative.

Therefore, other factor $x_1 - x_2$ is 0,

so $x_1 = x_2$