Mathematical Reasoning — Exam 2
MAT 312, Spring 2023 — D. Ivanšić

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Show all your work!

1. (14pts) Prove: the sum of squares of any two consecutive integers always gives remainder 1 when divided by 4.

Let $k \in \mathbb{Z}$. $k^2 + (k+1)^2 = k^2 + k^2 + 2k + 1 = 2k^2 + 2k + 1$

$= 2k(k+1) + 1$

Since one of $k, k+1$ is even $k(k+1)$ is even, so

$k(k+1) = 2l$

Thus $k^2 + (k+1)^2 = 2 \cdot 2l + 1 = 4l + 1$

so remainder in division by 4 is 1.

2. (16pts) Prove using induction: for every natural number $n \geq 9$, $2^n > 5n^2$. (You do not have to evaluate these expressions for $n < 9$.)

1) $n = 9$ $2^9 = 512$

$5 \cdot 9^2 = 5 \cdot 81 = 405$

$512 > 405$

2) Suppose true for $n = k$

$2^k > 5k^2 \quad | \cdot 2$

$2^{k+1} > 10k^2$

If we can show $10k^2 > 5(k+1)^2$, done

$2k^2 > k^2 + 2k + 1$

$k^2 - 2k > 1$

$k(k-2) > 1$

Since $k \geq 9$ $k \geq 9$, $k - 2 \geq 7$

so $k(k-1) \geq 9 \cdot 9 > 1$

$k^2 - 2k > 1 \quad | + k^2 + 2k$

$2k^2 > k^2 + 2k + 1$

$2k^2 > (k+1)^2$

$10k^2 > 5(k+1)^2$

since $10k^2 > 5(k+1)^2$

we get $2^{k+1} > 5(k+1)^2$,

the claim for $n = k+1$

3. (14pts) We know that $\sqrt{2}$ is irrational. Show that for any rational numbers p and q, the number $\dfrac{p}{q+\sqrt{2}}$ is irrational.

Suppose $\dfrac{p}{q+\sqrt{2}} = u$ and u is rational

$$\frac{q+\sqrt{2}}{p} = \frac{1}{u}$$

$$q+\sqrt{2} = \frac{p}{u}$$

$$\sqrt{2} = \frac{p}{u} - q$$

Since $\frac{p}{u} - q$ is rational, this would mean $\sqrt{2}$ is rational, a contradiction

4. (18pts) Consider the statement: for all $n \in \mathbf{Z}$, n is divisible by 5 if and only if $n^2 + 3n$ is divisible by 5.

a) Write the statement as a conjunction of two conditional statements.

b) Determine whether each of the conditional statements is true, and write a proof, if so.

c) Is the original statement true?

a) If n is divisible by 5, then n^2+3n is divisible by 5 (A)

and

If n^2+3n is divisible by 5, then n is divisible by 5 (B)

b)

$n \equiv \square$	$n^2+3n \equiv \square$ (mod 5)
0	0
1	4
2	$10 \equiv 0$
3	$18 \equiv 3$
4	$28 \equiv 3$

Table shows:

If $5 \mid n$ ($n \equiv 0 \pmod 5$) then $5 \mid n^2+3n$ ($n^2+3n \equiv 0 \pmod 5$) proving A)

$5 \mid 2^2+3\cdot 2$ but $5 \nmid 2$, so (B) is false

c) Since one statement in a) is false, the original statement is false,

5. (14pts) We have shown a similar statement on homework: for every integer n, if $3 \mid n^3$, then $3 \mid n$. Use this proposition to show that $\sqrt[3]{3}$ is irrational.

Suppose $\sqrt[3]{3}$ is rational, $\sqrt[3]{3} = \frac{m}{n}$ for $m, n \in \mathbb{Z}$, with no common factors. Then $3 = \left(\frac{m^3}{n^3}\right.$

$3n^3 = m^3$ which means $3 \mid m^3$

so $m = 3k$ for some $k \in \mathbb{Z}$

$3n^3 = (3k)^3$

$3n^3 = 27k^3$

$n^3 = 9k^3 = 3 \cdot (3k^3)$ which implies $3 \mid n^3$

so $n = 3l$ for some $l \in \mathbb{Z}$

But then m, n have a common factor 3, a contradiction.

6. (10pts) Let $\min\{a, b\}$ denote the smaller of real numbers a and b, or either one if they are equal. Prove the equation below.

$$\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$$

Cases: 1) $a \geq b$, then $\min\{a,b\} = b$

$a - b \geq 0$, so $\frac{a+b}{2} - \frac{|a-b|}{2} = \frac{a+b}{2} - \frac{a-b}{2} = \frac{2b}{2} = b$ } equal

2) $a < b$, then $\min\{a,b\} = a$

$a - b < 0$ so $\frac{a+b}{2} - \frac{|a-b|}{2} = \frac{a+b}{2} - \frac{b-a}{2} = \frac{2a}{2}$ $2a$ } equal

7. (14pts) Prove that for every real number $x \neq 0$, $x^2 \geq 2 - \dfrac{1}{x^2}$.

$$\left.\begin{array}{c} x^2 \geq 2 - \frac{1}{x^2} \\ x^2 - 2 + \frac{1}{x^2} \geq 0 \\ \left(x + \frac{1}{x}\right)^2 \geq 0 \end{array}\right\} \text{investigation}$$

true

Let $x \neq 0$. Then

$$\left(x - \frac{1}{x}\right)^2 \geq 0$$

$$x^2 - 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2} \geq 0$$

$$x^2 - 2 + \frac{1}{x^2} \geq 0$$

$$x^2 \geq 2 - \frac{1}{x^2}$$

Bonus. (10pts) Let p and q be rational numbers and x an irrational number. Show that the number $\dfrac{p-x}{q+x}$ is irrational.

Suppose $\dfrac{p-x}{q+x} = u$, where u is rational

$$p - x = u(q+x)$$

$$p - x = uq + ux$$

$$p - uq = x + ux$$

$$p - uq = (1+u)x$$

$x = \dfrac{p-uq}{1+u}$, which is a rational number, a contradiction.