Mathematical Reasoning — Exam 1
MAT 312, Spring 2023 — D. Ivanšić

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Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) The sun rises in the east and sets in the south.

$\underbrace{\text{The sun rises in the east}}_{T}$ and $\underbrace{\text{sets in the south}}_{F}$

a statement T and F is false

2. (2pts) If an integer is divisible by two, then it is divisible by four.

Statement, False: 6 is divisible by 2, but not by 4

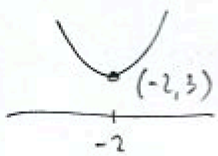
3. (3pts) (universal set=**R**) $2x - 7 > 0$ or $6 \div 3 = 2$

open sentence, truth set = $\mathbb{R}$ because statement $2x-7>0$ OR $\underbrace{6 \div 3 = 2}_{\text{true}}$ is true no matter what x is

4. (4pts) For every $x \in \mathbf{R}$, $x^2 + 4x + 7 > 0$

Statement. Graph of x^2+4x+7: $-\frac{b}{2a} = -\frac{4}{2} = -2$

$(-2)^2+(-8)+7=3$



always above x-axis, so true

5. (3pts) (universal set=**Q**) $x^2 + 3x = 10$

Open sentence.

$x^2+3x-10=0$

$(x+5)(x-2)=0$

$x=-5,2$

Truth set: $\{-5, 2\}$

Negate the following statements.

6. (3pts) You felt like dropping in and you did not expect me to be free.

You didn't feel like dropping in or you expected me to be free.

7. (3pts) If Putin says it, then it is true.

Putin says it and it is not true.

8. (8pts) Use a truth table to prove that $(P \implies Q) \vee Q \equiv \neg P \vee Q$. (Use however many columns you need.)

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \vee Q$	$\neg P$	$\neg P \vee Q$				
T	T	T	T	F	T				
T	F	F	F	F	F				
F	T	T	T	T	T				
F	F	T	T	T	T				

same

9. (12pts) Use previously proven logical equivalences to prove the equivalence $(P \implies Q) \vee (Q \implies P) \equiv T$, where T is "true." Do not use a truth table.

$$(P \Rightarrow Q) \vee (Q \Rightarrow P) \equiv (\neg P \vee Q) \vee (\neg Q \vee P) \equiv (\neg P \vee P) \vee (Q \vee \neg Q)$$

$$\equiv T \vee T \equiv T$$

10. (4pts) Write the converse and contrapositive of the statement: if I am not at home, then the door is locked.

Converse: If the door is locked, then I am not at home

Contrapositive: If the door is not locked, then I am at home.

11. (8pts) Suppose the following statements are true:
If the coin toss is tails, I will jump into the lake. $P \Rightarrow Q$ P = coin toss is tails
The coin toss is tails or pigs can fly. $P \vee F$ Q = I will jump into the lake

Determine truth value of the following statement and justify: I will jump into the lake.

Since $P \vee F$ is true, P is true. Now, because $P \Rightarrow Q$ is true and P is true, then Q must be true. Thus, "I will jump into the lake" is true.

12. (4pts) Use set builder notation to write the set of all rational numbers whose cube is less than the number itself.

$\{x \in \mathbb{Q} \mid x^3 < x\}$

13. (10pts) An integer $p \neq 1$ is *prime* if for all integers a, b, if $ab = p$, then $a = p$ or $b = p$.
a) Write the definition using symbols for quantifiers.
b) Negate the definition using symbols for quantifiers.
c) Finish the sentence (in English): "An integer $p \neq 1$ is not prime if ..."

a) $p \neq 1$ is prime if $(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})(ab = p \Rightarrow (a=p) \vee (b=p))$

b) $p \neq 1$ is not prime if $(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})((ab = p) \wedge ((a \neq p) \wedge (b \neq p))$

c) ... there exist integers a, b such that $ab = p$ and $a \neq p$ and $b \neq p$.

14. (7pts) Prove: if m is an odd integer, and n is an even integer, then $3m - n$ is an odd integer.

Suppose m is an odd integer and n is an even integer. Then $m = 2k+1$, $n = 2\ell$ for some $k, \ell \in \mathbb{Z}$. Then

$3m - n = 3(2k+1) - 2\ell = 6k + 3 - 2\ell = 6k - 2\ell + 2 + 1 = 2(3k - \ell + 1) + 1$

Since $3k - \ell + 1$ is an integer, $3m - n$ is an odd integer.

15. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\forall y \in \mathbf{R})(x^2 + y^2 > 1)$$

a) If $x = 0$, is the statement true?
b) If $x = \sqrt{2}$, is the statement true?
c) Find the truth set (the x's) of the above statement.

a) $x = 0$ gives $(\forall y \in \mathbb{R})(y^2 > 1)$
False, since for $y = 0$, $0^2 > 1$ is false

b) $x = \sqrt{2}$ gives $(\forall y \in \mathbb{R})(2 + y^2 > 1)$
Since $2 + y^2 > 2 > 1$, regardless of y, statement is true.

c) Need x s.t. $x^2 + y^2 > 1$ is true for every y.
$y^2 > 1 - x^2$ is true for every y iff and only if $1 - x^2 < 0$
(otherwise, $y = \sqrt{\frac{1-x^2}{2}}$ is a counterexample)

$1 - x^2 < 0$
$1 < x^2$ iff
$1 < |x|$

Truth set: $(-\infty, -1) \cup (1, \infty)$

16. (15pts) We will call an integer n type-0, type-1, type-2 or type-3 integer if it can be written in the form $n = 4k$, $n = 4k+1$, $n = 4k+2$ or $n = 4k+3$, respectively, for some integer k. Prove that if m is a type-1 integer and n is a type-3 integer, then $3mn - n^2$ is a type-0 integer. Start with a know-show table if you find it helpful.

Suppose m is a type-1 integer and n is a type-3 integer.

Then $m = 4k+1$ and $n = 4l+3$ for some $k, l \in \mathbb{Z}$.

$$3mn - n^2 = 3(4k+1)(4l+3) - (4l+3)^2$$
$$= 3(16kl + 4l + 12k + 3) - (16l^2 + 24l + 9)$$
$$= 48kl + 12l + 36k + 9 - 16l^2 - 24l - 9$$
$$= 48kl - 12l + 36k = 4(12kl - 3l + 9k)$$

Since $12kl - 3l + 9k$ is an integer, $3mn - n^2$ is a type-0 integer.

Bonus. (10pts) A function $f : \mathbf{R} \to \mathbf{R}$ is *continuous at* c if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x \in \mathbf{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

a) Write the definition using symbols for quantifiers.

b) Negate the definition using symbols for quantifiers.

c) Finish the sentence (in English): "A function $f : \mathbf{R} \to \mathbf{R}$ is not continuous at c if ..."

a) f is continuous at c if $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon)$

b) f is not cont. at c if $(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})(|x-c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon)$

c) f is not cont. at c if there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there is an $x \in \mathbb{R}$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon$